Quantization of Poisson brackets and its relation to Lie theory

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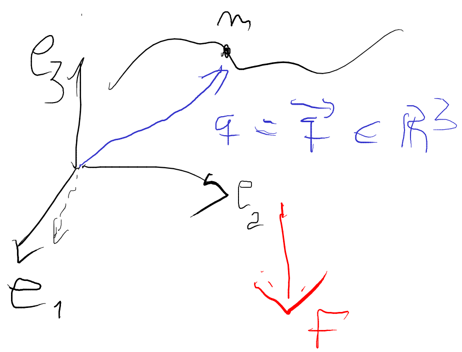
Plan:

classical vs. quantum
"quantization"
relation to Lie theory

Classical mechanics

$\rightsquigarrow$ Poisson brackets



$$m \frac{d^2}{dt^2} q = F = -\nabla V \quad \Longleftrightarrow \quad (H) \begin{cases} p = m \dfrac{d}{dt} q \\ \dfrac{d}{dt} p = -\nabla V \end{cases}$$

"Hamilton's eqs"

$V : \mathbb{R}^3 \to \mathbb{R}$

unknown $t \mapsto (q(t), p(t)) \in P$

Phase space $(P \simeq \mathbb{R}^3 \times \mathbb{R}^3)$

$f : P \to \mathbb{R}$, $f = f(q,p)$ $\nearrow q^j$, $\searrow p_k$ $\quad j,k = 1,2,3$

Poisson brackets $\{\,,\,\} : f, g \longmapsto \{f,g\} := \sum_{j=1}^{3} \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i}$

Prop (Exercise) if $\gamma(t) = (q(t), p(t))$ solution of (H), $f : P \to \mathbb{R}$ any,

$$\frac{d}{dt}\Big(f(\gamma(t))\Big) = \{f, H\}(\gamma(t))$$

where $H(q,p) = \underbrace{\frac{1}{2m} \|p\|^2}_{\text{kinetic}} + \underbrace{V(q)}_{\text{potential}}$ Hamilt. function "Energy"

Moral

$(P, \{,\}, H) \rightsquigarrow$ Hamiltonian system

Def a Poisson bracket is an operation

$$f, g \longmapsto \{f, g\}$$

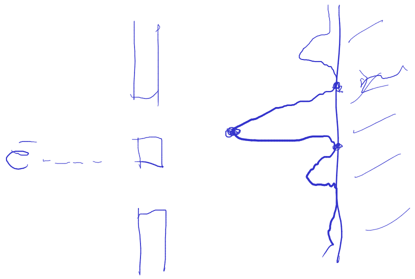
- $\mathbb{R}$ - bilinear
- $\{f, g\} = -\{g, f\}$
- $\{f, gh\} = \{f, g\}h + g\{f, h\}$

Jacobi $\to$
- $\{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\}\} = 0$

Ex:
- $\{f, g\} = 0 \quad \forall f, g$
- $\{,\}$ on $P = \mathbb{R}^3 \times \mathbb{R}^3$ "symplectic" $\nearrow$ $\{f, g\} = 0 \ \forall g \Rightarrow df = 0$
- $P = \mathbb{R}^3$, $\{f, g\}_x = (\nabla f \times \nabla g) \cdot x$

Quantum mechanics

inreference



Classical	Quantum
P	$\mathcal{H}$ Hilbert space over $\mathbb{C}$
$H, f \ldots$	$A : \mathcal{H} \to \mathcal{H}$ linear
$\{ , \}$	?

unknown

$t \mapsto \psi_t \in \mathcal{H}$, $\quad i\hbar\, \partial_t \psi_t = \hat{H} \psi_t$

Hamiltonian operator

Schrodinger equation

A observable $\xrightarrow{\text{evolution}}$ $A_t = U_t^{-1} A U_t$, $\quad U_t : \mathcal{H} \to \mathcal{H}$ evolution operator

$\left(U_t = e^{\frac{t}{i\hbar} \hat{H}} \right)$

Prop $\quad \partial_t A_t = \frac{1}{i\hbar} [A_t, \hat{H}]$

commutators

$[A, B] = AB - BA$

(analogous to $\frac{d}{dt}[f(\gamma(t))] = \{f, H\}$)

canonical quantization

$$(P, \{,\}, H) \xrightarrow{\text{"quantization"}} \left(\mathcal{H}, \tfrac{1}{i\hbar}[\,,\,], \hat{H}\right)$$
$$f \qquad\qquad\qquad \hat{f}$$

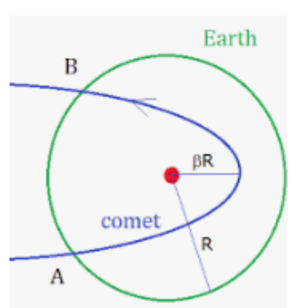
Classical	Quantum
$P = \mathbb{R}^{2n} \ni (q,p)$	$\psi \equiv \psi(q)$ $\psi \in L^2(\mathbb{R}^n)$
$f = q^i, p_i$	$(\hat{q}^i \psi)(q) = q^i \psi(q)$ $\hat{p}_j \psi = -i\hbar\, \partial_{q^j} \psi$
$f(q,p)$ function	$Q_\hbar(f)$ operator

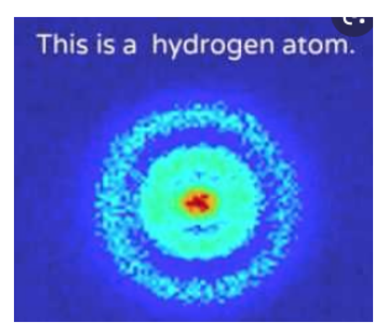
$\hbar \sim 10^{-34} \frac{m^2 kg}{s}$ $\quad \hbar$: scale param. (Planck's const)

$(\hbar \to 0$ classical limit$)$

Ex: $H(q,p) = \frac{1}{2m}\|p\|^2 - \frac{\alpha}{\|q\|}$

Kepler problem





$Q_\hbar(H) = \frac{-\hbar^2}{2m}\Delta - \frac{\alpha}{\|\hat{q}\|}$

Hidrogen atom

$$\{q^i, p_j\} = \delta^i_j \quad \longleftrightarrow \quad \frac{1}{i\hbar}[\hat{q}^i, \hat{p}_j] = \delta^i_j \quad \{\hat{q}^i, \hat{p}_j\}$$

"Corresp. Principle"

$$\boxed{Q_\hbar(f) \circ Q_\hbar(g) = Q_\hbar(f *_\hbar g)}$$

↳ star product

↳ symbol calculus with (pseudo) diff. op's

Formulas and expansions

$x=(q,p)$ $\hbar\in\mathbb{R}$

$$f *_\hbar g\,(x) \overset{\text{Weyl}}{=} \int_{x_1,x_2\in M} (\pi\hbar)^{-2m}\, f(x_1)\, g(x_2)\, e^{\frac{i}{\hbar}\left[\int_{T(x,x_1,x_2)} 4\,\omega_{can}\right]}$$

area

$$= (2\pi\hbar)^{-2m} \int_{\xi_1,\xi_2\in(\mathbb{R}^{2m})^*} \underbrace{\hat{\mathcal{F}}_\hbar(f)(\xi_1)}_{\text{Fourier Transform}}\, \hat{\mathcal{F}}_\hbar(g)(\xi_2)\, e^{\frac{i}{\hbar}\left[(\xi_1+\xi_2)x+\frac{1}{2}\pi(\xi_1,\xi_2)\right]}$$

asymptotic $\hbar\to 0$

$$\overset{\hbar=0+\lambda\ (\text{formal})}{\simeq} e^{-\frac{i\lambda}{2}\Delta}(f\otimes g)\Big|_{x_1=x_2=x}$$

$\partial_{q_i}f\otimes\partial_{p_i}g-\partial_{q_i}g\otimes\partial_{p_i}f$

$$= f *_\lambda g \in C^\infty(\mathbb{R}^{2m})[[\lambda]]$$

formal power series

Axiomatizing star products

$(M,\{,\})$ Poisson manifold

$f,g\longmapsto f *_\hbar g$

$(\hbar\to 0)$ • $f *_\hbar g = fg + O(\hbar)$ deformation of product

• $f *_\hbar g - g *_\hbar f = i\hbar\{f,g\} + O(\hbar^2)$ Correspondence Principle

$(AB)C = A(BC)$

• $f_1 *_\hbar (f_2 *_\hbar f_3) = (f_1 *_\hbar f_2) *_\hbar f_3$ $\forall\ \hbar$ orders

Existence and classification: hard problem!

1998

Solved by Kontsevich in the formal case

Heuristics behind the Lie-theoretic connection

From canonical quantization:

Classical (geometry)	Quantum (algebra)
symplectic manifold (S,ω) Ex: $\mathbb{R}^{2m}$	vector space V Ex $\underbrace{L^2(\mathbb{R}^m)}_{\ni \psi}$
$(S,-\omega)$	V^* dual
$S_1 \times S_2$	$V_1 \otimes V_2$
Lagrangian submanifold L Ex: $\{(q, p = \partial_q S)\} \subseteq \mathbb{R}^{2m}$, $S : \mathbb{R}^m_q \to \mathbb{R}$	element $\psi \in V$ Ex $\psi(q) = a_\hbar(q)\, e^{\frac{i}{\hbar}S(q)}$
? (S,ω) $L_m \hookrightarrow (S,-\omega)\times(S,-\omega)\times(S,\omega)$ $L_1 \hookrightarrow (S,\omega)$	$V = A$ algebra $V\otimes V \xrightarrow{m} V \leftrightarrow \psi_m \in V^*\otimes V^*\otimes V$ $1 \in V$ unit

Symplectic Groupoid!

$(G \rightrightarrows M, \omega)$

compatibility eq.

$m^*\omega = p_1^*\omega + p_2^*\omega$ (think of $f(g_1 g_2) = f(g_1) + f(g_2)$)

$g_1, g_2 \in G,\ f : G \to \mathbb{R}$