On Hamiltonian Virasoro spaces

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(based on work with Anton Alekseev)

University of Auckland
Moduli spaces and Vertex algebras
February 21, 2023

Based on:

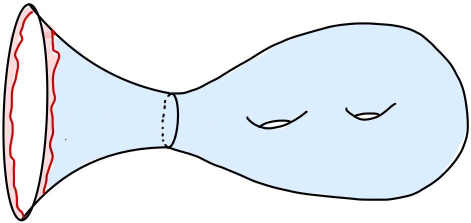
A. Alekseev, E.M.: On the coadjoint Virasoro action
(Preprint, arXiv:2211.06216)

A. Alekseev, E.M.: Hamiltonian Virasoro spaces
(in preparation)

1. Motivation

Motivation from physics: JT gravity.

Moduli spaces of Riemann surfaces with wiggly boundary



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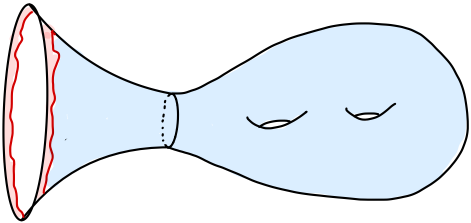

arising in JT gravity (Saad-Shenker-Stanford 2019, Stanford-Witten 2019, and others).

$\rightsquigarrow$ Schwarzian derivative, Virasoro algebra, DH measures, Mirzakhani recursion formulas etc.

Motivation from physics: JT gravity.

We claim:

There is an ∞-dimensional Teichmüller space

$$\mathcal{T}(\Sigma) = \frac{\text{hyperbolic 0-metrics on } \Sigma}{\mathrm{Diff}_0(\Sigma, \partial\Sigma)}$$

which is a Hamiltonian space, with momentum map taking values in $\mathrm{vir}_1^*(\partial\Sigma)$.

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Also expect more general spaces from hyperbolic metrics with singularities, and so on.

Motivation from mathematics: loop group spaces

Let G connected, with invariant metric $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$.

Definition

A Hamiltonian LG-space is given by

- an LG-manifold $\mathcal{M}$,
- an invariant symplectic form $\omega_{\mathcal{M}}$,
- an equivariant momentum map $\Phi_{\mathcal{M}}\colon \mathcal{M}\to \widehat{L\mathfrak{g}}_1^*$ satisfying

$$\iota(\xi_{\mathcal{M}})\omega_{\mathcal{M}} = -\mathrm{d}\langle \Phi_{\mathcal{M}},\xi\rangle, \quad \xi\in L\mathfrak{g}.$$

Equivariance is with respect to gauge action

$$g\cdot\mu = \mathrm{Ad}_g\,\mu - g^*\theta^R,$$

for $\mu\in\widehat{L\mathfrak{g}}_1^*\cong\Omega^1(S^1,\mathfrak{g})$. ($G$-connections on S^1.)

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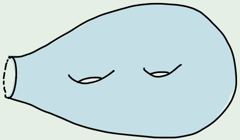
Example

Σ oriented surface with boundary



$$\mathcal{M}_G(\Sigma) = \frac{\text{flat } G\text{-connections on } \Sigma}{\text{gauge transformations with } g|_{\partial\Sigma} = 1}$$

with momentum map taking values in

$$G\text{-connections on } \partial\Sigma \cong \widehat{L\mathfrak{g}}_1^*.$$

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Hamiltonian LG-spaces $\leftrightarrow$ quasi-Hamiltonian G-spaces

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- etc. (??)

2. Virasoro Lie algebra

Central extensions

Principle: Given an affine action

$$K \circlearrowright E$$

with underlying linear action $\mathrm{Ad}\colon K \circlearrowright \mathfrak{k}^*$, get central extension

$$0 \to \mathbb{R} \to \widehat{\mathfrak{k}} \to \mathfrak{k} \to 0$$

such that $E \cong \widehat{\mathfrak{k}}^*_1$, the affine hyperplane at level 1.

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In fact, we take

$$\widehat{\mathfrak{k}} = \mathrm{Hom}_{\mathrm{aff}}(E, \mathbb{R}), \qquad [\widehat{X}, \widehat{Y}](\mu) = \langle X,\ Y \cdot \mu \rangle.$$

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Example (Gauge algebra)

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Special case: $P \cong S^1 \times G \ \rightsquigarrow\ \mathrm{Gau}(P) \cong LG$.

Virasoro Lie algebra

Let C = oriented circle. To define

$$\mathsf{vir}(\mathsf{C}) = \widehat{\mathsf{Vect}}(\mathsf{C}),$$

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In particular, $\mathrm{Vect}(\mathsf{C})^* = |\Omega|^2_{\mathsf{C}}$.

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Definition

A Hill operator is a 2nd order linear differential operator

$$L\colon |\Omega|_{\mathbb{C}}^{-\frac{1}{2}} \to |\Omega|_{\mathbb{C}}^{\frac{3}{2}}$$

such that $L^* = L, \quad \sigma(L) = 1.$

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The resulting central extension $\mathrm{vir}(\mathsf{C}) = \widehat{\mathrm{Vect}}(\mathsf{C})$ is the Virasoro Lie algebra. Its action on

$$\mathrm{vir}_1^*(\mathsf{C}) = \mathrm{Hill}(\mathsf{C}).$$

is called the coadjoint Virasoro action.

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Here

$$\mathcal{S}(\mathsf{F}) = \frac{\mathsf{F}'''}{\mathsf{F}'} - \frac{3}{2}\Big(\frac{\mathsf{F}''}{\mathsf{F}'}\Big)^2.$$

is the Schwarzian derivative.

3. Classification of orbits

Gauge orbits

$P \to C$ principal G bundle over oriented circle.

Theorem (well-known)

The coadjoint $\mathrm{Gau}(P)$*-orbits in* $\widehat{\mathfrak{gau}}_1^*(P)$ *are classified by* G*-conjugacy classes:*

$$\widehat{\mathfrak{gau}}_1^*(P)/\,\mathrm{Gau}(P) = G/G.$$

Classification is given by monodromy.

In detail:

Gauge orbits

Write $\mathsf{C} = \widetilde{\mathsf{C}}/\mathbb{Z}$, let $\kappa \in \mathrm{Diff}(\widetilde{\mathsf{C}})$ generator of $\mathbb{Z}$-action.

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Let $\widetilde{P} \to \widetilde{\mathsf{C}}$ pullback, and

$$\mathcal{S}(P) = \{\tau \in \Gamma(\widetilde{P}) |\ \exists h \in G \colon \kappa^* \tau = h \cdot \tau\}$$

quasi-periodic sections.

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Actually, have Morita equivalence of actions ($\Rightarrow$ same stabilizers, same normal representations).

Gauge orbits

$\rightsquigarrow$ Morita equivalence of action groupoids:

$$\begin{array}{ccccc} \mathrm{Gau}(P) \ltimes \widehat{\mathfrak{gau}}_1^*(P) & & \mathcal{S}(P) & & G \ltimes G \\ \downdownarrows & \overset{p}{\swarrow} & & \overset{q}{\searrow} & \downdownarrows \\ \widehat{\mathfrak{gau}}_1^*(P) & & & & G \end{array}$$

Virasoro orbits

Theorem (Goldman, Segal)

The coadjoint Virasoro orbits are classified by conjugacy classes in some open subset of $\widetilde{SL}(2,\mathbb{R})$:

$$\mathrm{vir}_1^*(\mathsf{C})/\operatorname{Diff}_+(\mathsf{C}) = \widetilde{\mathrm{SL}}(2,\mathbb{R})_+/\operatorname{PSL}(2,\mathbb{R}).$$

Other versions: Lazutkin-Pankratova, Kirillov, Witten, Hitchin.

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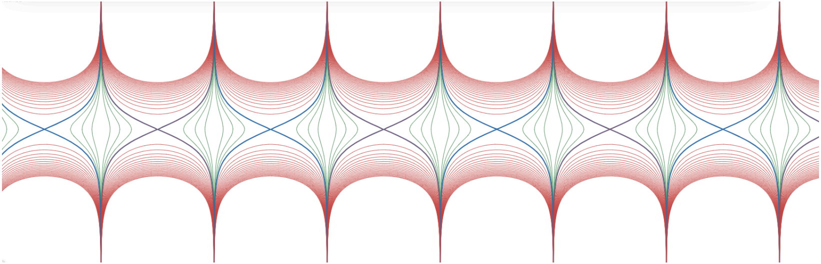
Picture of $\widetilde{SL}(2,\mathbb{R})$:



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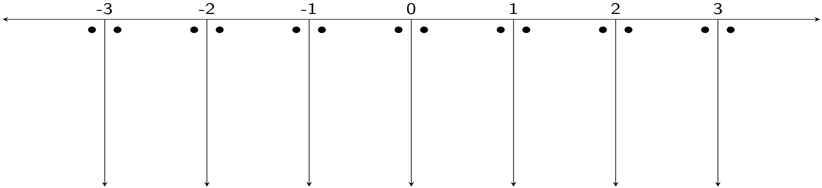

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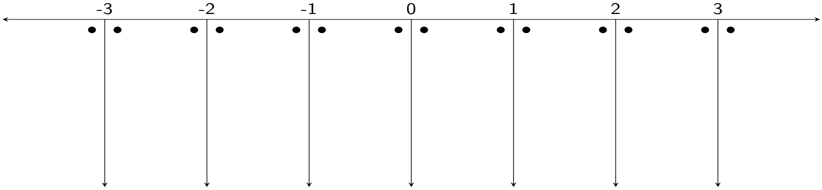

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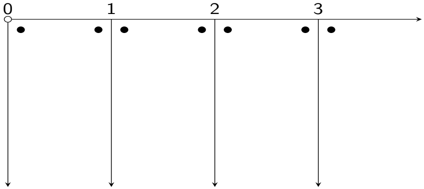

Virasoro orbits

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Definition

A developing map is an oriented local diffeomorphism $\gamma\colon \widetilde{\mathsf{C}} \to \mathbb{R}\mathrm{P}(1)$ s.t.

$$\gamma(\kappa(x)) = h \cdot \gamma(x)$$

for some $h \in \mathrm{PSL}(2,\mathbb{R})$.

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Denote

$$\mathcal{D}(\mathsf{C}) = \{\text{ developing maps }\}.$$

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The map p.

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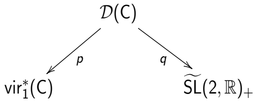

The map p. For $L \in \mathrm{vir}_1^*(\mathsf{C}) = \mathrm{Hill}(\mathsf{C})$, any fundamental system

$$u_1, u_2 \in |\Omega|_{\widetilde{\mathsf{C}}}^{-\frac{1}{2}}, \quad Lu_i = 0, \quad W[u_1, u_2] = -1$$

defines a developing map $\gamma = (u_1 : u_2) \in \mathcal{D}(\mathsf{C})$. Conversely, γ determines $p(\gamma) = L$.

Virasoro orbits

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p is also the quotient map for

$$\mathrm{PSL}(2,\mathbb{R}) \circlearrowright \mathcal{D}(\mathsf{C}), \; (g.\gamma)(x) = g.\gamma(x).$$

$$\mathrm{vir}_1^*(\mathsf{C}) \xleftarrow{\;p\;} \mathcal{D}(\mathsf{C}) \xrightarrow{\;q\;} \widetilde{\mathrm{SL}}(2,\mathbb{R})_+$$

The map q.

Virasoro orbits

$$\mathcal{D}(\mathsf{C}) \xrightarrow{p} \mathsf{vir}_1^*(\mathsf{C}), \qquad \mathcal{D}(\mathsf{C}) \xrightarrow{q} \widetilde{\mathsf{SL}}(2,\mathbb{R})_+$$

The map q. Any developing map map be written $\gamma = (\sin\phi : \cos\phi)$ where $\phi\colon \widetilde{\mathsf{C}} \to \widetilde{\mathbb{R}\mathsf{P}}(1) = \mathbb{R}$ satisfies

$$\phi(\kappa(x)) = \widetilde{h} \cdot \phi(x)$$

where $\widetilde{h} \in \widetilde{\mathsf{SL}}(2,\mathbb{R})_+$. Put $q(\gamma) = \widetilde{h}$.

Virasoro orbits

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\begin{array}{ccc}
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q is also the quotient map for

$$\widetilde{\mathrm{Diff}}_+(\mathsf{C}) = \mathrm{Diff}_{\mathbb{Z}}(\widetilde{\mathsf{C}}) \circlearrowright \mathcal{D}(\mathsf{C}), \quad (\mathsf{F}.\gamma)(x) = \gamma(\mathsf{F}^{-1}(x)).$$

Virasoro orbits

$$\begin{array}{ccc} & \mathcal{D}(\mathsf{C}) & \\ {}^{/\,\mathrm{PSL}(2,\mathbb{R})}\swarrow & & \searrow^{/\,\widetilde{\mathrm{Diff}}_+(\mathsf{C})} \\ \mathrm{vir}_1^*(\mathsf{C}) & & \widetilde{\mathrm{SL}}(2,\mathbb{R})_+ \end{array}$$

This proves the Goldman-Segal result:

$$\mathrm{vir}_1^*(\mathsf{C})/\,\mathrm{Diff}_+(\mathsf{C}) = \widetilde{\mathrm{SL}}(2,\mathbb{R})_+/\,\mathrm{PSL}(2,\mathbb{R}).$$

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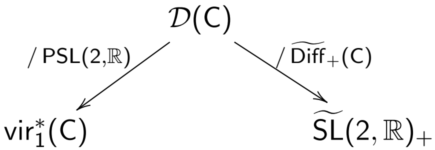

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N.B.: This is not quite a Morita equivalence of actions:

The $\widetilde{\mathsf{Diff}}_+(\mathsf{C})$-action on $\mathcal{D}(\mathsf{C})$ has (discrete) stabilizers.

$$
\begin{array}{ccc}
 & \mathcal{D}(\mathsf{C}) & \\
 \swarrow^{/\,\mathrm{PSL}(2,\mathbb{R})} & & \searrow^{/\,\widetilde{\mathrm{Diff}}_+(\mathsf{C})} \\
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Virasoro orbits

However [AM]: We still get a Morita equivalence of groupoids.

$$\begin{array}{ccccc} \mathcal{G}_1 & & \mathcal{D}(C) & & \mathcal{G}_2 \\ \downdownarrows & \swarrow^{p} & & \searrow^{q} & \downdownarrows \\ \mathrm{vir}_1^*(C) & & & & \widetilde{\mathrm{SL}}(2,\mathbb{R})_+ \end{array}$$

Here

$$\begin{aligned} \mathcal{G}_1 &= (\mathrm{Diff}_+(C) \ltimes \mathrm{vir}_1^*(C))/\sim \\ \mathcal{G}_2 &= \mathrm{PSL}(2,\mathbb{R}) \ltimes \widetilde{\mathrm{SL}}(2,\mathbb{R})_+ \end{aligned}$$

5. Poisson geometric aspects

Morita equivalence for coadjoint LG-action

$$\begin{array}{ccccc} & & \mathcal{S}(P) & & \\ & \swarrow^{p} & & \searrow^{q} & \\ \widehat{\mathfrak{gau}}_1^*(P) & & & & G \end{array}$$

Morita equivalence for coadjoint LG-action

$$\widehat{\mathfrak{gau}}_1^*(P) \xleftarrow{\ p\ } \mathcal{S}(P) \xrightarrow{\ q\ } G$$

Theorem (Alekseev-M 2009)

There is a canonical $G \times \mathrm{Aut}(P)$-invariant 2-form

$$\varpi_{\mathcal{S}(P)} \in \Omega^2(\mathcal{S}(P)),$$

satisfying

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- $\iota(X_{\mathcal{S}(P)})\varpi_{\mathcal{S}(P)} = \frac{1}{2} q^*\langle \theta^L + \theta^R, X\rangle \quad$ *for* $X \in \mathfrak{g}$

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- $\iota(\xi_{\mathcal{S}(P)})\varpi_{\mathcal{S}(P)} = -p^* \int_C \langle dA, \xi\rangle \quad$ *for* $\xi \in \mathfrak{gau}(P)$.

Morita equivalence for coadjoint LG-action

Get Morita equivalence of (quasi)-symplectic groupoids (in sense of Ping Xu):

$$\begin{array}{ccccc}
(\mathrm{Gau}(P) \ltimes \widehat{\mathfrak{gau}}(P)^*_1, \omega_1) & & (\mathcal{S}(P), \varpi_{\mathcal{S}}) & & (G \ltimes G, \omega_2) \\
\downdownarrows & \swarrow & & \searrow & \downdownarrows \\
\widehat{\mathfrak{gau}}(P)^*_1 & & & & G
\end{array}$$

Morita equivalence for coadjoint LG-action

One obtains a 1-1 correspondence between Hamiltonian loop group spaces and quasi-Hamiltonian G-spaces

$$\Big((\mathcal{M},\omega_{\mathcal{M}}) \xrightarrow{\Phi_{\mathcal{M}}} \widehat{\mathfrak{gau}}(P)^*_1\Big) \quad \xleftrightarrow{1-1} \quad \Big((M,\omega_M) \xrightarrow{\Phi_M} G\Big)$$

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For example:

$$\text{coadjoint } \mathrm{Gau}(P)\text{-orbits} \quad \stackrel{1-1}{\longleftrightarrow} \quad G\text{-conjugacy classes}$$

Morita for coadjoint Virasoro-action

$$\mathrm{vir}_1^*(C) \xleftarrow{\;p\;} \mathcal{D}(C) \xrightarrow{\;q\;} \widetilde{\mathrm{SL}}(2,\mathbb{R})_+$$

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$$\mathcal{D}(\mathsf{C}) \xrightarrow{p} \mathrm{vir}_1^*(\mathsf{C}), \qquad \mathcal{D}(\mathsf{C}) \xrightarrow{q} \widetilde{\mathrm{SL}}(2,\mathbb{R})_+$$

Theorem (Alekseev-M 2022)

There is a canonical $\mathrm{PSL}(2,\mathbb{R}) \times \widetilde{\mathrm{Diff}}_+(\mathsf{C})$*-invariant 2-form*

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- $\iota(v_{\mathcal{D}(\mathsf{C})})\varpi_{\mathcal{D}(\mathsf{C})} = -p^* \int_{\mathsf{C}} (dL)\, v$ *for* $v \in \mathrm{Vect}(\mathsf{C})$.

In fact, we obtain a Morita equivalence of (quasi-)symplectic groupoids

$$
\begin{array}{ccccc}
(\mathcal{G}_1,\omega_1) & & (\mathcal{D}(\mathsf{C}),\varpi_{\mathcal{D}}) & & (\mathcal{G}_2,\omega_2) \\
\downarrow\downarrow & \swarrow & & \searrow & \downarrow\downarrow \\
\mathrm{vir}_1^*(\mathsf{C}) & & & & \widetilde{\mathrm{SL}}(2,\mathbb{R})_+
\end{array}
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where

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One obtains a 1-1 correspondence between certain Hamiltonian Virasoro spaces and certain quasi-Hamiltonian $\mathrm{PSL}(2,\mathbb{R})$-spaces

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For example:

coadjoint Virasoro-orbits $\stackrel{1-1}{\longleftrightarrow}$ Conjugacy classes in $\widetilde{\mathrm{SL}}(2,\mathbb{R})_+$

6. Coordinate expressions

The 2-form ϖ: Explicit formulas for S^1

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$\mathcal{D}(S^1)$ consists of quasi-periodic maps $\gamma \colon \mathbb{R} \to \mathbb{R}P(1)$;

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Then $p(\gamma) = \dfrac{\partial^2}{\partial x^2} + T$ with Hill potential

$$T(x) = \phi'(x)^2 + \frac{\phi'''(x)}{\phi'(x)} - \frac{3}{2}\Big(\frac{\phi''(x)}{\phi'(x)}\Big)^2$$

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View T as a $|\Omega|^2$-valued function on $\mathcal{D}(S^1)$, and view

$$\Theta = -\frac{\mathrm{d}\phi}{\phi'}$$

as a $|\Omega|^{-1}$-valued 1-form on $\mathcal{D}(S^1)$.

The 2-form ϖ

With these ingredients,

$$T(x) = \phi'(x)^2 + \frac{\phi'''(x)}{\phi'(x)} - \frac{3}{2}\Big(\frac{\phi''(x)}{\phi'(x)}\Big)^2, \quad \Theta = -\frac{\mathrm{d}\phi}{\phi'}$$

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we have:

Theorem (Alekseev-M)

The 2-form $\varpi \in \Omega^2(\mathcal{D}(S^1))$ is given by

$$\varpi = -\int_0^1 dT \wedge \Theta - 2T_0\Theta_0\Theta_1 - \frac{1}{2}(\Theta_1' \wedge \Theta_0' - \Theta_1 \wedge \Theta_0'' - \Theta_1'' \wedge \Theta_0).$$

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Expanding, get complicated expression in terms of $\mathrm{d}\phi,\ \mathrm{d}\phi', \mathrm{d}\phi'', \mathrm{d}\phi'''$.

7. Hamiltonian Virasoro spaces

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Let Σ be a surface of genus $g \geq 1$ with one boundary component.

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Hamiltonian Virasoro spaces

Theorem (Alekseev-M)

The spaces

$$\mathcal{T}(\Sigma) = \frac{\textit{hyperbolic 0-metrics on } \Sigma}{\mathrm{Diff}_0(\Sigma, \partial\Sigma)}$$

$$\mathcal{M}(\Sigma) = \frac{\textit{hyperbolic 0-metrics on } \Sigma}{\mathrm{Diff}^+(\Sigma, \partial\Sigma)}$$

arise in this way.

Concluding remarks

Thanks !