Trialities of $\mathcal{W}$-algebras

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1. Vertex operator algebras

Vertex operator algebras (VOAs) were studied by physicists in the 1980s and axiomatized by Borcherds (1986).

A VOA $\mathcal{V}$ is a vector space which is linearly isomorphic to an algebra of formal power series in $\mathrm{End}(\mathcal{V})[[z, z^{-1}]]$.

$$a \leftrightarrow a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}, \qquad a(n) \in \mathrm{End}(\mathcal{V}).$$

$\mathcal{V}$ has Wick product $:ab:$, generally nonassociative, noncommutative.

Unit 1, derivation $\partial = \frac{d}{dz}$.

Conformal weight grading $\mathcal{V} = \bigoplus_{n \geq 0} \mathcal{V}[n]$, $n \in \mathbb{Z}$ or $\frac{1}{2}\mathbb{Z}$

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2. Operator product expansion

Let $\mathcal{V}$ be a VOA, $a, b \in \mathcal{V}$. Then

$$a(z)b(w) = \sum_{n \geq 0} (a_{(n)}b)(w)(z-w)^{-n-1} + :a(z)b(w):.$$

Expansion of meromorphic function with poles along $z = w$, where

1. $:a(z)b(w):$ is regular part.
2. $(a_{(n)}b)(w)$ is polar part of order $n+1$.

Defines bilinear products $(-_{(n)}-) : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$, where $(a, b) \mapsto a_{(n)}b$.

Also $:a(z)b(w):|_{z=w}$ coincides with Wick product.

Often write

$$a(z)b(w) \sim \sum_{n \geq 0} (a_{(n)}b)(w)(z-w)^{-n-1},$$

where $\sim$ means equal modulo regular part.

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3. Operator product expansion

Often, a VOA is **presented** by giving generators and OPE relations.

Ex: Affine VOA $V^k(\mathfrak{g})$, for $\mathfrak{g}$ a simple Lie algebra with basis $\xi_1, \dots, \xi_n$.

$V^k(\mathfrak{g})$ is generated by fields X^{ξ_i}, $i = 1, \dots, n$, satisfying

$$X^{\xi_i}(z) X^{\xi_j}(w) \sim k(\xi_i|\xi_j)(z-w)^{-2} + X^{[\xi_i,\xi_j]}(w)(z-w)^{-1}.$$

Fact: $V^k(\mathfrak{g})$ has a PBW basis consisting of monomials

$$:\partial^{k_1^1} X^{\xi_1} \cdots \partial^{k_{r_1}^1} X^{\xi_1} \cdots \partial^{k_1^n} X^{\xi_n} \cdots \partial^{k_{r_n}^n} X^{\xi_n}:,$$

$$k_1^1 \geq k_2^1 \geq \cdots \geq k_{r_1}^1, \qquad k_1^n \geq k_2^n \geq \cdots \geq k_{r_n}^n.$$

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4. Strong and free generations

We say that a VOA $\mathcal{V}$ is **strongly generated** by a set $\{\alpha_i |\ i \in I\}$ if $\mathcal{V}$ is spanned by monomials

$$\{: \partial^{k_1}\alpha_{i_1} \cdots \partial^{i_r}\alpha_{i_r} : \mid k_j \geq 0, i_j \in I\}.$$

Suppose $\{\alpha_1, \alpha_2, \dots\}$ is an **ordered** strong generating set for $\mathcal{V}$.

We say $\mathcal{V}$ is **freely generated** by $\{\alpha_1, \alpha_2, \dots\}$ if

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Equivalently, $\mathcal{V}$ is linearly isomorphic to polynomial algebra on $\partial^k \alpha_i$ for $i = 1, 2, \dots,$ and $k \geq 0$.

Ex: $V^k(\mathfrak{g})$ is freely generated by X^{ξ_i}.

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5. Conformal structure

The **Virasoro Lie algebra** is a central extension of the (complexified) Lie algebra of vector fields on the circle.

Generators $L_n = -t^{n+1}\frac{d}{dt}$, $n \in \mathbb{Z}$, and central element κ,

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n+m,0}\frac{n^3-n}{12}\kappa.$$

A **Virasoro element** of a vertex algebra $\mathcal{V}$ is a field $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \in \mathcal{V}$ satisfying

$$L(z)L(w) \sim \frac{c}{2}(z-w)^{-4} + 2L(w)(z-w)^{-2} + \partial L(w)(z-w)^{-1}.$$

$[L_0, -]$ is required to act diagonalizably and $[L_{-1}, -]$ acts by ∂.

Constant c is called the **central charge**.

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6. Conformal structure, cont'd

Conformal weight grading is eigenspace decomposition under L_0.

If $a \in \mathcal{V}$ has weight d, then

$$L(z)a(w) \sim \cdots + da(w)(z-w)^{-2} + \partial a(w)(z-w)^{-1}.$$

Note that L always has weight 2.

Virasoro VOA Vir^c is freely generated by $L(z)$.

Conformal structure on $\mathcal{V}$ comes from homomorphism $\mathrm{Vir}^c \to \mathcal{V}$.

Ex: $V^k(\mathfrak{g})$ has Virasoro element

$$L^{\mathfrak{g}} = \frac{1}{2(k+h^\vee)} \sum_{i=1}^{n} : X^{\xi_i} X^{\xi_i'} :, \qquad k \neq -h^\vee.$$

Central charge $c = \frac{k\dim(\mathfrak{g})}{k+h^\vee}$ where $h^\vee$ is dual Coxeter number.

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Conformal structure on $\mathcal{V}$ comes from homomorphism $\mathrm{Vir}^c \to \mathcal{V}$.

Ex: $V^k(\mathfrak{g})$ has Virasoro element

$$L^{\mathfrak{g}} = \frac{1}{2(k+h^\vee)} \sum_{i=1}^{n} :X^{\xi_i} X^{\xi'_i}:, \qquad k \neq -h^\vee.$$

Central charge $c = \frac{k \dim(\mathfrak{g})}{k+h^\vee}$ where $h^\vee$ is dual Coxeter number.

7. Rational VOAs

There is a natural notion of **modules** for a VOA $\mathcal{V}$.

$\mathcal{V}$ is called **rational** if its module category is semisimple and has finitely many simple objects.

Example: For $\mathfrak{g}$ simple and $k \in \mathbb{N}$, $V^k(\mathfrak{g})$ is not simple.

Simple quotient $L_k(\mathfrak{g})$ is rational.

Example: Let p, q be coprime positive integers with $2 \leq p < 1$.

For $c = 1 - 6\frac{(p-q)^2}{pq}$, Vir^c is not simple.

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8. $\mathcal{W}$-algebras

Class of VOAs associated to

1. A simple, finite-dimensional Lie (super)algebra $\mathfrak{g}$,
2. A nilpotent element f in the even part of $\mathfrak{g}$.

$\mathcal{W}^k(\mathfrak{g}, f)$ the $\mathcal{W}$-algebra at level k associated to $\mathfrak{g}$ and f via (generalized) Drinfeld-Sokolov reduction.

Freely generated by fields corresponding to lowest weight vectors for irreducible $\mathfrak{sl}_2$-submodules of $\mathfrak{g}$.

For each such module of dimension d, get a field of weight $\frac{d+1}{2}$.

If $f = 0$, $\mathcal{W}^k(\mathfrak{g}, 0) = V^k(\mathfrak{g})$.

For $\mathfrak{g} = \mathfrak{sl}_2$ and $f = f_{\text{prin}}$ principal nilpotent, $\mathcal{W}^k(\mathfrak{sl}_2, f_{\text{prin}})$ is just the Virasoro algebra Vir^c for $\psi = k + 2$ and $c = -\frac{(2\psi-3)(3\psi-2)}{\psi}$.

For $k = -2 + \frac{p}{q}$ an admissible level, simple quotient $\mathcal{W}_k(\mathfrak{sl}_2, f_{\text{prin}}) \cong \text{Vir}_{p,q}$.

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9. Notation and examples

For this talk: We will replace k with the **shifted level** $\psi = k + h^\vee$.

$\mathcal{W}^\psi(\mathfrak{g}, f)$ will always denote $\mathcal{W}^k(\mathfrak{g}, f)$ with $k = \psi - h^\vee$.

If $f = f_{\text{prin}}$ is a principal nilpotent, write $\mathcal{W}^\psi(\mathfrak{g}, f) = \mathcal{W}^\psi(\mathfrak{g})$.

$\mathcal{W}^\psi(\mathfrak{g})$ is freely generated of type $\mathcal{W}(d_1, \dots, d_r)$, where $r = \text{rank}(\mathfrak{g})$, and $d_1, \dots, d_r$ degrees of fundamental invariants of $\mathfrak{g}$.

This means strong generators have conformal weights $d_1, \dots, d_r$.

Thm (Arakawa, 2015): If $\psi = \frac{p}{q}$ is a nondegenerate admissible level, $\mathcal{W}_\psi(\mathfrak{g})$ is rational.

Special case of the **Kac-Wakimoto conjecture**, proven by McRae in 2022.

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10. Feigin-Frenkel duality

Thm: (Feigin, Frenkel, 1991) Let $\mathfrak{g}$ be a simple Lie algebra. Then

$$\mathcal{W}^{\psi}(\mathfrak{g}) \cong \mathcal{W}^{\psi'}({}^L\mathfrak{g}), \qquad r^{\vee}\psi\psi' = 1.$$

Here ${}^L\mathfrak{g}$ is the Langlands dual Lie algebra, and $r^{\vee}$ is the lacity of $\mathfrak{g}$.

In fact, a similar result holds for $\mathfrak{g} = \mathfrak{osp}_{1|2n}$.

Thm: (Creutzig, Genra)

$$\mathcal{W}^{\psi}(\mathfrak{osp}_{1|2n}) \cong \mathcal{W}^{\psi'}(\mathfrak{osp}_{1|2n}), \qquad 4\psi\psi' = 1.$$

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11. Coset construction

Let $\mathcal{V}$ be a VOA and $\mathcal{A} \subseteq \mathcal{V}$ a subVOA

The **coset** $\mathcal{C} = \mathrm{Com}(\mathcal{A}, \mathcal{V})$ is the subVOA of $\mathcal{V}$ which commutes with $\mathcal{A}$, that is,

$$\mathcal{C} = \{v \in \mathcal{V} | \ [a(z), v(w)] = 0, \ \forall a \in \mathcal{A}\}.$$

If $\mathcal{V}$, $\mathcal{A}$ have Virasoro elements $L^{\mathcal{V}}$, $L^{\mathcal{A}}$, then $\mathcal{C}$ has Virasoro element

$$L^{\mathcal{C}} = L^{\mathcal{V}} - L^{\mathcal{A}},$$

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12. Coset construction of principal $\mathcal{W}$-algebras

Thm: (Arakawa, Creutzig, L., 2018) Let $\mathfrak{g}$ be simple and simply-laced. We have diagonal embedding

$$V^{k+1}(\mathfrak{g}) \hookrightarrow V^k(\mathfrak{g}) \otimes L_1(\mathfrak{g}), \qquad u \mapsto u \otimes 1 + 1 \otimes u, \qquad u \in \mathfrak{g}.$$

Set

$$\mathcal{C}^k(\mathfrak{g}) = \mathrm{Com}(V^{k+1}(\mathfrak{g}), V^k(\mathfrak{g}) \otimes L_1(\mathfrak{g})).$$

We have an isomorphism of 1-parameter VOAs

$$C^k(\mathfrak{g}) \cong \mathcal{W}^{\psi}(\mathfrak{g}), \qquad \psi = \frac{k + h^{\vee}}{k + h^{\vee} + 1}.$$

Coset realization for B (and C) is different.

Thm: (Creutzig-L., 2021) We have an isomorphism of 1-parameter VOAs

$$\mathrm{Com}(V^k(\mathfrak{sp}_{2n}), V^k(\mathfrak{osp}_{1|2n})) \cong \mathcal{W}^{\psi}(\mathfrak{so}_{2n+1}), \qquad \psi = \frac{2k + 2n + 1}{2(1 + k + n)}.$$

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13. What are trialities of $\mathcal{W}$-algebras?

Let $f \in \mathfrak{g}$ be a nilpotent, and complete f to a copy $\{f, h, e\}$ of $\mathfrak{sl}_2$ in $\mathfrak{g}$.

Let $\mathfrak{a} \subseteq \mathfrak{g}$ denote the centralizer of this $\mathfrak{sl}_2$ in $\mathfrak{g}$.

Then $\mathcal{W}^{\psi}(\mathfrak{g}, f)$ has affine subVOA $V^{\psi'}(\mathfrak{a})$, for some level ψ'.

By the **affine coset**, we mean $\mathcal{C}^{\psi}(\mathfrak{g}, f) := \text{Com}(V^{\psi'}(\mathfrak{a}), \mathcal{W}^{\psi}(\mathfrak{g}, f))$.

Sometimes we also take invariants under some group of **outer automorphisms**.

Trialities are isomorphisms between three different affine cosets

$$\mathcal{C}^{\psi}(\mathfrak{g}, f) \cong \mathcal{C}^{\psi'}(\mathfrak{g}', f') \cong \mathcal{C}^{\psi''}(\mathfrak{g}'', f'').$$

These unify and generalize both Feigin-Frenkel duality and the coset realization of principal $\mathcal{W}$-algebras.

13. What are trialities of $\mathcal{W}$-algebras?

Let $f \in \mathfrak{g}$ be a nilpotent, and complete f to a copy $\{f, h, e\}$ of $\mathfrak{sl}_2$ in $\mathfrak{g}$.

Let $\mathfrak{a} \subseteq \mathfrak{g}$ denote the centralizer of this $\mathfrak{sl}_2$ in $\mathfrak{g}$.

Then $\mathcal{W}^{\psi}(\mathfrak{g}, f)$ has affine subVOA $V^{\psi'}(\mathfrak{a})$, for some level ψ'.

By the **affine coset**, we mean $\mathcal{C}^{\psi}(\mathfrak{g}, f) := \text{Com}(V^{\psi'}(\mathfrak{a}), \mathcal{W}^{\psi}(\mathfrak{g}, f))$.

Sometimes we also take invariants under some group of **outer automorphisms**.

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14. Hook-type $\mathcal{W}$-algebras in type A

Recall: For $n \geq 1$, write

$$\mathfrak{sl}_{n+m} = \mathfrak{sl}_n \oplus \mathfrak{gl}_m \oplus \Big(\mathbb{C}^n \otimes (\mathbb{C}^m)^*\Big) \oplus \Big((\mathbb{C}^n)^* \otimes \mathbb{C}^m\Big).$$

Let $f_n \in \mathfrak{sl}_{n+m}$ be the nilpotent which is **principal** in $\mathfrak{sl}_n$ and **trivial** in $\mathfrak{gl}_m$.

Then f_n corresponds to the **hook-type partition** $n + 1 + \cdots + 1$.

Define shifted level $\psi = k + n + m$, and define

$$\mathcal{W}^{\psi}(n, m) := \mathcal{W}^{\psi}(\mathfrak{sl}_{n+m}, f_n),$$

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Principal: For $n \geq 2$, $\mathcal{W}^{\psi}(n, 0) = \mathcal{W}^{\psi}(\mathfrak{sl}_n)$

Subregular: For $n \geq 2$, $\mathcal{W}^{\psi}(n, 1) = \mathcal{W}^{\psi}(\mathfrak{sl}_{n+1}, f_{\text{subreg}})$

Trivial: For $m \geq 1$, $\mathcal{W}^{\psi}(1, m) \cong \mathcal{W}^{\psi}(\mathfrak{sl}_{m+1}, 0) = V^{\psi-m-1}(\mathfrak{sl}_{m+1})$

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16. Features of $\mathcal{W}^{\psi}(n, m)$

For $m \geq 2$, $\mathcal{W}^{\psi}(n, m)$ has affine subalgebra

$$V^{\psi-m-1}(\mathfrak{gl}_m) = \mathcal{H} \otimes V^{\psi-m-1}(\mathfrak{sl}_m).$$

Additional **even** generators are in weights $2, 3, \dots, n$ together with $2m$ **even** fields in weight $\frac{n+1}{2}$ which transform under $\mathfrak{gl}_m$ as $\mathbb{C}^m \oplus (\mathbb{C}^m)^*$.

We define the case $\mathcal{W}^{\psi}(0, m)$ separately as follows.

1. For $m \geq 2$,

$$\mathcal{W}^{\psi}(0, m) = V^{\psi-m}(\mathfrak{sl}_m) \otimes \mathcal{S}(m),$$

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17. Hook-type $\mathcal{W}$-superalgebras of type A

For $n+m \geq 2$ and $n \neq m$, write

$$\mathfrak{sl}_{n|m} = \mathfrak{sl}_n \oplus \mathfrak{gl}_m \oplus \left(\mathbb{C}^n \otimes (\mathbb{C}^m)^*\right) \oplus \left((\mathbb{C}^n)^* \otimes \mathbb{C}^m\right).$$

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For $m \geq 2$, $\mathcal{V}^{\psi}(n, m)$ has affine subalgebra

$$\begin{aligned} &V^{-\psi-m+1}(\mathfrak{gl}_m), \qquad m \neq n, \\ &V^{-\psi-n+1}(\mathfrak{sl}_n), \qquad m = n. \end{aligned}$$

Additional **even** generators in weights $2, 3, \dots, n$, together with $2m$ **odd** fields in weight $\frac{n+1}{2}$ transforming under $\mathfrak{gl}_m$ as $\mathbb{C}^m \oplus (\mathbb{C}^m)^*$.

We define the cases $\mathcal{V}^{\psi}(0, m)$ and $\mathcal{V}^{\psi}(1, 1)$ separately as follows.

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   where $\mathcal{E}(m)$ is the rank m bc-system.
2. $\mathcal{V}^{\psi}(1, 1) = \mathcal{A}(1)$, rank one symplectic fermion algebra.
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20. Trialities in type A

Consider the affine cosets

$$\mathcal{C}^{\psi}(n,m) = \mathrm{Com}(V^{\psi-m-1}(\mathfrak{gl}_m), \mathcal{W}^{\psi}(n,m)),$$

$$\mathcal{D}^{\psi}(n,m) = \mathrm{Com}(V^{-\psi-m+1}(\mathfrak{gl}_m), \mathcal{V}^{\psi}(n,m)), \qquad n \neq m,$$

$$\mathcal{D}^{\psi}(n,n) = \mathrm{Com}(V^{-\psi-n+1}(\mathfrak{sl}_n), \mathcal{V}^{\psi}(n,n))^{U(1)}.$$

Thm: (Creutzig-L., 2020) Let $n \geq m$ be non-negative integers. We have isomorphisms of 1-parameter VOAs

$$\mathcal{D}^{\psi}(n,m) \cong \mathcal{C}^{\psi^{-1}}(n-m,m) \cong \mathcal{D}^{\psi'}(m,n), \qquad \frac{1}{\psi} + \frac{1}{\psi'} = 1.$$

Originally conjectured in physics by Gaiotto and Rapčák (2017).

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$$\mathcal{D}^{\psi}(n,m) = \mathrm{Com}(V^{-\psi-m+1}(\mathfrak{gl}_m), \mathcal{V}^{\psi}(n,m)), \qquad n \neq m,$$

$$\mathcal{D}^{\psi}(n,n) = \mathrm{Com}(V^{-\psi-n+1}(\mathfrak{sl}_n), \mathcal{V}^{\psi}(n,n))^{U(1)}.$$

Thm: (Creutzig-L., 2020) Let $n \geq m$ be non-negative integers. We have isomorphisms of 1-parameter VOAs

$$\mathcal{D}^{\psi}(n,m) \cong \mathcal{C}^{\psi^{-1}}(n-m,m) \cong \mathcal{D}^{\psi'}(m,n), \qquad \frac{1}{\psi} + \frac{1}{\psi'} = 1.$$

Originally conjectured in physics by Gaiotto and Rapčák (2017).

20. **Trialities in type A**

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21. Some special cases

$\mathcal{D}^{\psi}(n,0) \cong \mathcal{C}^{\psi^{-1}}(n,0)$ recovers **Feigin-Frenkel duality** in type A.

Isomorphisms $\mathcal{D}^{\psi}(n,m) \cong \mathcal{C}^{\psi^{-1}}(n-m,m)$ are of **Feigin-Frenkel type**.

$\mathcal{D}^{\psi}(n,0) \cong \mathcal{D}^{\psi'}(0,n)$ recovers the coset realization of $\mathcal{W}^{\psi}(\mathfrak{sl}_n)$.

Isomorphisms $\mathcal{D}^{\psi}(n,m) \cong \mathcal{D}^{\psi'}(m,n)$ are of **coset realization type**.

One more example:

$$\mathcal{D}^{\psi}(n,1) \cong \mathcal{C}^{\psi^{-1}}(n-1,1) \cong \mathcal{D}^{\psi'}(1,n),$$

recovers a duality conjectured by Feigin and Semikhatov and proved in a different way by Creutzig, Genra, and Nakatsuka.

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22. Sketch of proof, cont'd

Step 1: In the $\psi \to \infty$ limit, both $\mathcal{C}^{\psi}(n,m)$ and $\mathcal{D}^{\psi}(n,m)$ become GL_m-orbifolds of certain **free field algebras**.

Using **classical invariant theory**, it is shown that

1. $\mathcal{C}^{\psi}(n,m)$ has generating type $\mathcal{W}(2,3,\dots,(m+1)(m+n+1)-1)$,
2. $\mathcal{D}^{\psi}(n,m)$ has generating type $\mathcal{W}(2,3,\dots,(m+1)(n+1)-1)$.

Step 2: Universal two-parameter $\mathcal{W}_{\infty}$-algebra $\mathcal{W}(c,\lambda)$ serves is a **classifying object** for VOAs of type $\mathcal{W}(2,3,\dots,N)$ for some N.

$\mathcal{W}(c,\lambda)$ is freely generated of type $\mathcal{W}(2,3,\dots)$, and is defined over the polynomial ring $\mathbb{C}[c,\lambda]$.

Weight zero component $\mathcal{W}(c,\lambda)[0] \cong \mathbb{C}[c,\lambda]$.

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23. One-parameter quotients of $\mathcal{W}(c, \lambda)$

Let $I \subseteq \mathbb{C}[c, \lambda]$ be a prime ideal.

Let $I \cdot \mathcal{W}(c, \lambda)$ be the VOA ideal generated by I.

The quotient

$$\mathcal{W}^I(c, \lambda) = \mathcal{W}(c, \lambda)/(I \cdot \mathcal{W}(c, \lambda))$$

is a VOA over $R = \mathbb{C}[c, \lambda]/I$.

$\mathcal{W}^I(c, \lambda)$ is simple for a generic ideal I.

But for certain discrete families of ideals I, $\mathcal{W}^I(c, \lambda)$ is not simple.

Let $\mathcal{W}_I(c, \lambda)$ be simple graded quotient of $\mathcal{W}^I(c, \lambda)$.

In fact, **all** simple, one-parameter VOAs of type $\mathcal{W}(2, 3, \dots, N)$ satisfying mild hypotheses, are of this form.

Variety $V(I) \subseteq \mathbb{C}^2$ is called the **truncation curve**.

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Then $\mathcal{C}^{\psi}(n, m)$ and $\mathcal{D}^{\psi}(n, m)$ are of the form $\mathcal{W}_I(c, \lambda)$ for some I.

Step 3: Explicit truncation curves for $\mathcal{C}^{\psi}(n, m)$ and $\mathcal{D}^{\psi}(n, m)$.

$\mathcal{W}^{\psi}(n, m)$ is an extension $V^{\psi-m+1}(\mathfrak{gl}_m) \otimes \mathcal{W}_I(c, \lambda)$ for some I

Extension is generated by $2m$ fields in weight $\frac{n+1}{2}$ which transform as $\mathbb{C}^m \oplus (\mathbb{C}^m)^*$ under $\mathfrak{gl}_m$.

Existence of such an extension uniquely and explicitly determines I.

Same method works for $\mathcal{V}^{\psi}(n, m)$.

Triality theorem follows from explicit form of I.

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25. Some applications

Let $I_{n,m}$ be ideal corresponding to $\mathcal{C}^{\psi}(n, m)$

Nontrivial isomorphisms $\mathcal{C}_{\psi}(n, m) \cong \mathcal{C}_{\psi'}(n', m')$ correspond to intersection points in $V(I_{n,m}) \cap V(I_{n',m'})$.

All intersections between the curves $V(I_{n,m})$ are rational points.

Recall: $\mathcal{W}_{\psi}(\mathfrak{sl}_n) = \mathcal{W}_{\psi}(n, 0) = \mathcal{C}_{\psi}(n, 0)$.

Thm: For all $2 \leq n < m$,

$$\mathcal{W}_{\psi}(\mathfrak{sl}_n) \cong \mathcal{W}_{\psi'}(\mathfrak{sl}_m), \qquad \psi = \frac{n}{n+m}, \qquad \psi' = \frac{m}{m+n}.$$

Conjectured by Gaberdiel and Gopakumar (2011).

If m, n coprime, $\psi = \frac{n}{m+n}$ and $\psi' = \frac{m}{m+n}$ are **boundary admissible** in the sense of Kac-Wakimoto.

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Recall: $\mathcal{W}_{\psi}(\mathfrak{sl}_n) = \mathcal{W}_{\psi}(n,0) = \mathcal{C}_{\psi}(n,0)$.

Thm: For all $2 \leq n < m$,

$$\mathcal{W}_{\psi}(\mathfrak{sl}_n) \cong \mathcal{W}_{\psi'}(\mathfrak{sl}_m), \qquad \psi = \frac{n}{n+m}, \qquad \psi' = \frac{m}{m+n}.$$

Conjectured by Gaberdiel and Gopakumar (2011).

If m, n coprime, $\psi = \frac{n}{m+n}$ and $\psi' = \frac{m}{m+n}$ are **boundary admissible** in the sense of Kac-Wakimoto.

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26. Some applications

Recall: $\mathcal{W}_\psi(n-1,1) = \mathcal{W}_\psi(\mathfrak{sl}_n, f_{\text{subreg}})$.

Affine subalgebra is just a Heisenberg algebra $\mathcal{H}$, and

$$\mathcal{C}_\psi(n-1,1) \cong \text{Com}(\mathcal{H}, \mathcal{W}_\psi(\mathfrak{sl}_n, f_{\text{subreg}})).$$

By finding intersection points in $V(I_{n-1,1}) \cap V(I_{r,0})$, we can classify isomorphisms $\text{Com}(\mathcal{H}, \mathcal{W}_\psi(\mathfrak{sl}_n, f_{\text{subreg}})) \cong \mathcal{W}_\phi(\mathfrak{sl}_r)$.

Cor: For all $n \geq 2$, if $r+1$ and $r+n$ are coprime, $\mathcal{W}_\psi(\mathfrak{sl}_n, f_{\text{subreg}})$ is a simple current extension of $V_L \otimes \mathcal{W}_\phi(\mathfrak{sl}_r)$, where

$$\psi = \frac{n+r}{n-1}, \qquad \phi = \frac{r+1}{r+n}, \qquad L = \sqrt{nr}\,\mathbb{Z}.$$

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27. Universal objects

More generally, a **universal object** is a VOA $\mathcal{V}$ with the following properties:

1. $\mathcal{V}$ has some prescribed strong generating type $\mathcal{W}(d_1, d_2, \dots)$.
2. $\mathcal{V}$ is defined over a commutative $\mathbb{C}$-algebra R.
3. $\mathcal{V}$ cannot be defined in a nontrivial way over a ring with larger Krull dimension.

Condition (3) means: If $\mathcal{V}'$ is a VOA of type $\mathcal{W}(d_1, d_2, \dots)$ and is defined over a ring S of higher Krull dimension, then we have

$$\mathcal{V}' = \mathcal{V} \otimes_R S,$$

possibly after localizing.

Universal objects have two main applications:

1. Classification of VOAs by strong generating type.
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