Homological Algebra

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Acknowledgements

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Chapter 1

Introduction

Bo's take

This is a short exact sequence (SES):

$$0 \to A \to B \to C \to 0\,.$$

To see why this plays a central role in algebra, suppose that A and C are subspaces of B, then, by going through the definitions of a SES in 3.1, one can notice that this line arrows encodes information about the decomposition of B into A its orthogonal compliment C. If B is a module and A and C are now submodules of B, we would like to be able to describe how A and C can "span" B in a similar way. The usefulness of SES is that given submodules $A, C \subseteq B$, and a surjection $B \to C$, we are able to determine if A and C "span" B by just checking if $\ker(B \to C) = A$. If so, encode this situation in the SES.

Now, notice that in the above description, I was able to move seamlessly between subspaces and submodules for A, B and C. This is captured beautifully by category theory (c.f. §4.1, which aims to capture the underlying structure of the object of study. When we would like to study two mathematical objects with similar structures, we look to functors between the two categories. Since SES is part of the structure of a category, a question is if functors can preserve SESs. The answer turns out to be not always. Those functors that preserve SESs are named exact sequences and for those that are not, we would like to understand what obstructs the functor from being so. And this is where homological algebra comes in.

Chapter 2

Rings and Modules

In this chapter, we will set out the notation and introduce the main characters of homological algebra. Readers are assumed to be familiar with groups and basic algebra. References will be provided for results that are deemed to be basic or finicky. We will present examples that we hope the reader can carry throughout the course and will draw from number theory, algebraic geometry and algebraic topology.

2.1 Rings

Definition. A group, G, is a set with a binary operation $* : G \times G \to G$ such that for all $a, b, c \in G$,

(i) the binary operation is *associative*: $a * (b * c) = (a * b) * c$;

(ii) there exists an *identity* element $e \in G$ such that $e * a = a * e = a$;

(iii) there exists an *inverse* element $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$.

Definition. A *ring*, R, is a set with two operations

$$+ : R \times R \to R, \quad \cdot : R \times R \to R$$

such that for all $x, y, z \in R$,

(i) $(R, +)$ is an abelian group;

(ii) multiplication is associative: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;

(iii) and distributive over addition: $x \cdot (y + z) = x \cdot y + x \cdot z, (y + z) \cdot x = y \cdot x + z \cdot x$;

(iv) there exists a multiplicative inverse $1_R \in R$ such that $1_R \cdot x = x \cdot 1_R = x$ for all $x \in R$.

Furthermore, if $\times$ is commutative, i.e. $x \cdot y = y \cdot x$, then we say that R is a *commutative ring.*

For more properties and examples of commutative rings, one can consult the holy bible of the subject: *Introduction to Commutative Algebra* by Atiyah and Macdonald.

Now, let R be a ring and $I \subset (R, +)$ be a subgroup then since $(R, +)$ is abelian, we have that $I \trianglelefteq R$. So we can consider the quotient group R/I, where the operation is also denoted by $+$. The natural course of action is to turn this set into a ring by giving it a multiplicative operation. A natural option for this is to define multiplication $\frac{R}{I} \times \frac{R}{I} \to \frac{R}{I}$ as

$$(a+I, b+I) \mapsto ab + I.$$

We need to check if our definition is well-defined. Let $a, b, a', b' \in I$, then we need to check that if

$$a + I = a' + I, b + I = b' + I,$$

then we have

$$ab + I = a'b' + I.$$

We can restate the condition that we have to check as

$$a - a', b - b' \in I$$

$$\implies ab - a'b' = (a - a')b + a'(b - b') \in I.$$

So if we have $a' = 0, a \in I$ and $b \in R$, then we need to have $ab \in I$. Let's use this to form our definition of an ideal and we will see later that this really does work.

Definition. An ideal is a subset $I \subseteq R$ such that:

(i) $I \leq (R, +)$;

(ii) If $a \in I, b \in R$, then $ab \in I$.

Then R/I is a ring under

(i) $(a + I) + (b + I) = (a + b) + I$,

(ii) $(a + I) \cdot (b + I) = (ab) + I$.

for all $a, b \in R$.

The introduction of ideals allows us to speak about Ring homomorphisms, isomorphism theorems. Again, one should refer to other sources to find out more about basic ring properties.

2.2 Modules

Notice that ideals are subsets of the ring such that we have a ring action on them. Just as in group theory, where a group action on itself is the most natural thing in the world, ideals form the analogue for ring actions. But the power of group actions lies in the action of groups on a different set, so we will see that this generalisation on rings gives us modules.

Let k be a field, in particular, it is also a ring. Let V be a k-vectorspace, then we can see that k acts on V in the usual MATHS250 way. In particular, vectorspaces over a field generalises to modules over a ring.

Definition. Let R be a ring. A **left module** is an abelian group $(M, +)$ together with a map $R \times M \to M$, $(r, m) \mapsto rm$, such that we have distributivity, associativity and identity:

(i) $(r_1 + r_2)m = r_1 m + r_2 m, \quad \forall r_1, r_2 \in R, m \in M$;

(ii) $r(m_1 + m_2) = rm_1 + rm_2, \quad \forall r \in R, m_1, m_2 \in M$;

(iii) $r_1(r_2m) = (r_1r_2)m, \quad \forall r_1, r_2 \in R, m \in M$;

(iv) $1m = m, \quad \forall m \in M$.

Remark. *In this course, we will only interest ourselves with commutative rings, groups and modules unless stated otherwise. So we will not be referring to left/right modules until the occasion presents itself.*

Example.

(i) Any ring R is a module over itself: $R \times R \to R$.

(ii) Any abelian group is a $\mathbb{Z}$-module.

(iii) When R is a field, a module over R is called a vectorspace.

(iv) $R = \mathbb{R}[x]$, $M = \mathbb{R}^2 = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \middle| \alpha, \beta \in \mathbb{R} \right\}$, then if we fix $A = \begin{pmatrix} 5 & 2 \\ -1 & -3 \end{pmatrix}$, then we can define

$$\left(p(x), \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right) = p(A) \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Definition. A *homomorphism* of R-modules $f : M \to N$ is a map such that

(i) $f(m + m') = f(m) + f(m'), \quad \forall m, m' \in M$;

(ii) $f(rm) = rf(m), \quad \forall r \in R, m \in M$.

An *epimorphism* is a homomorphism which is surjective.
A *monomorphism* is a homomorphism which is injective.
An *isomorphism* is a homomorphism which is bijective.

Proposition 2.2.1. *Let* $\hom_R(M, N)$ *be the set of* R*-homomorphisms from* M *to* N*. Then for* $f, g \in \hom_R(M, N)$*,* $r \in R$*, define*

$$\begin{aligned} f + g : M \to N, &\quad m \mapsto f(m) + g(m); \\ rf : M \to N, &\quad m \mapsto rf(m). \end{aligned}$$

Then, $\hom_R(M, N)$ *is an* R*-module.*

Proof. Checking that $f + g$ is a homomorphism:

$$\begin{aligned} m + m' &\mapsto f(m + m') + g(m + m') \\ &= f(m) + f(m') + g(m) + g(m') \\ &= (f + g)(m) + (f + g)(m'), \end{aligned}$$

$$\begin{aligned} rm &\mapsto (f + g)(rm) \\ &= f(rm) + g(rm) \\ &= rf(m) + rg(m) \\ &= r(f + g)(m). \end{aligned}$$

One can show that rf is a homomorphism by following the method above.

$\hom_R(M, N)$ is an abelian group because N is an abelian group and we are always dealing with elements of N.

Checking module axioms:

$$\begin{aligned} r(f+g)(m) &= r(f(m)+g(m)) \\ &= rf(m)+rg(m) && \because N \text{ is an } R\text{-module} \\ &= (rf+rg)(m) \end{aligned}$$

The distributivity over R and associativity is proved in a similar fashion. Identity is obvious. □

Definition. Let $\{M_i\}_{i\in I}$ be any collection of R-modules.

(i) Consider

$$\prod_{i\in I} M_i = \{(m_i)_{i\in I} | m_i \in M_i \forall i \in I\}.$$

This is an R-module by component-wise addition and multiplication by ring elements. That is

$$(m_i)+(m_i')=(m_i+m_i'), r(m_i)=(rm_i).$$

This is the *direct product* of M_i's.

(ii) Now, consider

$$\bigoplus_{i\in I} M_i = \left\{(m_i)_{i\in I} \in \prod M_i \middle| m_i \neq 0 \text{ for finitely many } i\text{'s, maybe none}\right\}.$$

This the the *direct sum* of M_i's.

Example. If $R=\mathbb{Z}$, $M_i=\mathbb{Z}/3$, then

	$(1,1,1,\ldots)$	$(0,1,2,0,0,1,2,0,\ldots)$	$(1,2,0,0,0,\ldots)$
$\prod M_i$	$\in$	$\in$	$\in$
$\bigoplus M_i$	$\notin$	$\notin$	$\in$

Theorem 2.2.2. *Let $\{M_i\}$ be a collection of R-modules. Let*

$$\alpha_i : M_i \to \bigoplus M_j$$

be given by

$$m_i \mapsto (0,\ldots,0,m_i,0,\ldots).$$

Then, for each R-module N, there is a bijection

$$\left\{\begin{matrix} R\text{-mod homs} \\ f:\bigoplus M_i \to N \end{matrix}\right\} \leftrightarrow \left\{\begin{matrix} \text{Collection of homs} \\ f_i : M_i \to N \end{matrix}\right\}$$
$$f \mapsto (f\circ\alpha_i)_i$$

This says that each homomorphism $f:\bigoplus M_i \to N$ is uniquely defined by the individual component functions.

Proof. Construct the reverse map and check that the composition is the identity. So, given the collection $(f_i : M_i \to N)_{i\in I}$, we need to find a map $g:\bigoplus M_i \to N$. We will define it as

$$g\left((m_i)_{i\in I}\right) = \sum_{i\in I} f_i(m_i).$$

This is a finite sum since (m_i) only have finitely many non-zero components. (This is where the proof will fail without the finiteness assumption.)

Now, let's check that if the composition is the identity, that is,

$$f \to (f \circ \alpha_i)_{i \in I} \to g = \sum_{i \in I} f \circ \alpha_i \implies f = g.$$

Let $(m_j)_{j \in I} \in \bigoplus M_i$, then

$$\begin{aligned} g\big((m_i)_{i \in I}\big) &= \sum_{i \in I} (f \circ \alpha_i)\big((m_i)_{i \in I}\big) \\ &= \sum_{i \in I} (f \circ \alpha_i)(m_i) \\ &= \sum_{i \in I} f(0, \ldots, 0, m_i, 0, \ldots) \\ &= f\left(\sum_{i \in I} (0, \ldots, 0, m_i, 0, \ldots)\right) \\ &= f\big((m_i)_{i \in I}\big). \end{aligned}$$

Hence $f = g$ and the composition is the identity.

Now, for the other direction:

$$(f_i)_{i \in I} \to \sum_{i \in I} f_i \to \left(\left(\sum_{i \in I} f_i\right) \circ \alpha_j\right)_{j \in I}$$

Let $(m_i) \in M_i$, then we want to show that

$$f_i(m_i) = \left(\left(\sum_{i \in I} f_i\right) \circ \alpha_j\right)(m_j).$$

$$\begin{aligned} \left(\left(\sum_{i \in I} f_i\right) \circ \alpha_j\right)(m_j) &= \left(\sum_{i \in I} f_i\right)(0, \ldots, 0, m_j, 0, \ldots) \\ &= \sum_{i \in I} \delta_{ij} f_i(m_j) \\ &= f_i(m_i). \end{aligned}$$

Note that δ_{ij} is the kronecker delta function. Hence, we once again have the identity. ☐

Theorem 2.2.3. *Let $\pi_i : \prod M_j \to M_i$ be given by $(m_j)_{j \in I} \mapsto m_i$. Then, for each R-module N, there is a bijection*

$$\left\{ \begin{matrix} R\text{-mod homs} \\ f : N \to \prod_i M_i \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \text{Collection of homs} \\ f_i : N \to M_i \end{matrix} \right\} \tag{$*$}$$
$$f \mapsto (f_i = \pi_i \circ f)_{i \in I}$$

Proof. The proof is similar to the one above. First, let us identify the reverse map. So given $(f_i : N \to M_i)_{i \in I}$, we define the map $g : N \to \prod M_i$ by

$$g : n \mapsto (f_i(n))_{i \in I}.$$

Since the range is a direct product of modules, we do not have to fret about finiteness. (If we were to replace direct products with direct sums, what would happen?)

Now, we need to proceed as before and check that the compositions result in the identity. Consider this

$$f \to (\pi_i \circ f)_{i \in I} \to (\pi_{i_1} \circ f, \pi_{i_2} \circ f, \dots)_{i_j \in I},$$

where the second term is a collection of maps, and the last term describes the components of a map.

Let $n \in N$, then

$$\begin{aligned} f(n) &= (f(n)_{i_1}, f(n)_{i_2}, \dots)_{i_j \in I} && \text{where } f(n)_{i_j} \in M_{i_j} \\ &= (\pi_{i_1} \circ f(n), \pi_{i_2} \circ f(n), \dots)_{i_j \in I} \\ &= (\pi_{i_1} \circ f, \pi_{i_2} \circ f, \dots)_{i_j \in I}(n), \end{aligned}$$

so we are done.

Now, let's check the other direction:

$$(\pi_i \circ f)_{i \in I} \to (\pi_{i_1} \circ f, \pi_{i_2} \circ f, \dots)_{i_k \in I} \to \Big(\pi_j \circ (\pi_{i_1} \circ f, \pi_{i_2} \circ f, \dots)_{i_k \in I}\Big)_{j \in I}$$

Let $n \in N$, then

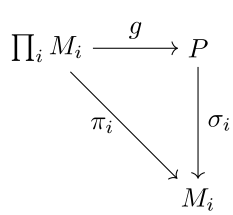
$$\begin{aligned} &\Big(\pi_j \circ (\pi_{i_1} \circ f, \pi_{i_2} \circ f, \dots)_{i_k \in I}\Big)_{j \in I}(n) \\ =&\Big(\pi_j \circ (\pi_{i_1} \circ f(n), \pi_{i_2} \circ f(n), \dots)_{i_k \in I}\Big)_{j \in I} \\ =&\Big(\pi_j \circ \big(f(n)_{i_1}, f(n)_{i_2}, \dots\big)_{i_k \in I}\Big)_{j \in I} \\ =&f(n)_j \\ =&(\pi_i \circ f)(n). \end{aligned}$$

And we are done. Phew! □

Theorem 2.2.4 (Uniqueness of Universal properties for products)**.** *Suppose $\{\sigma_i : P \to M_i\}$ is a collection of R-module homomorphisms such that for all R-modules N, there is a bijection*

$$\left\{\begin{matrix} R\text{-mod homs} \\ f : N \to P \end{matrix}\right\} \leftrightarrow \left\{\begin{matrix} \text{Collection of homs} \\ f_i : N \to M_i \end{matrix}\right\} \qquad (\star)$$
$$f \mapsto (\sigma_i \circ f)_{i \in I}.$$

Then there is a unique isomorphism $g : \prod_i M_i \to P$ such that $\sigma_i \circ g = \pi_i$ and the following diagram commutes.



Proof. By applying $(*)$ of Theorem 2.2.3 to $(P, (\sigma_i))$, we have

$$\begin{Bmatrix} \text{Collection of homs} \\ \sigma_i : P \to M_i \end{Bmatrix} \longmapsto h : P \to \prod M_i \quad (\sigma_i = \pi_i \circ h).$$

Now, by applying $(\star)$ in the statement of the theorem to $(\prod M_i, \pi_i)$, we get

$$\begin{Bmatrix} \text{Collection of homs} \\ \pi_i : \prod M_i \to M_i \end{Bmatrix} \longmapsto g : \prod M_i \to P \quad (\pi_i = \sigma_i \circ g).$$

We now need to show that $h \circ g$ and $g \circ h$ are identity maps.

For $g \circ h$:

$$P \xrightarrow{h} \prod_i M_i \xrightarrow{g} P$$

Then for all i,

$$\sigma_i \circ (g \circ h) = (\sigma_i \circ g) \circ h = \pi_i \circ h = \sigma_i = \sigma_i \circ id.$$

Now apply $(\star)$ to $(P, (\sigma_i))$, then there exists a unique homomorphism $u : P \to P$ such that $\sigma_i \circ u = \sigma_i$. So by the uniqueness of u, $u = Id_P \implies g \circ h = Id_P$. $\square$

Another way of writing universal properties is:

$$\hom_R\left(\bigoplus_i M_i, N\right) \cong \prod_i \hom_R(M_i, N)$$
$$\hom_R\left(N, \prod_i M_i\right) \cong \prod_i \hom_R(N, M_i)$$

where the RHS is the direct product of R-modules.

Tensor Product

In Theorem 2.2.3 we have examined one notion of a product on modules, namely the direct product which is more or less cumbersome to work with. A different product that we can consider is called the *tensor product* and is motivated by the following idea.

Let M, N be two R-modules and consider the free R-module $i : M \times N \to F$ over the direct product $M \times N$. Let T be the submodule generated by all elements of the form

$$\begin{aligned} &i(m + m', n) - i(m, n) - i(m', n), \\ &i(rm, n) - ri(m, n), \\ &i(m, n + n') - i(m, n) - i(m, n'), \\ &i(m, rn) - ri(m, n) \end{aligned} \tag{2.1}$$

where $r \in R$, $m, m' \in M$, and $n, n' \in N$. Let $q : F \to F/T$ be the quotient map, then we claim that $h = q \circ i : M \times N \to F/T$ is what we call the tensor product of M and N which we shall denote with $M \otimes_R N$.

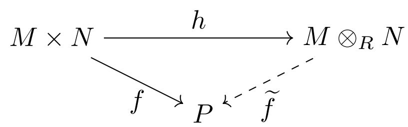
Before we get further into this, let us first define the tensor product in terms of its universal property. For this we require the definition of what is essentially captured in Equation 2.1, namely a *bilinear map* of modules.

Definition. Let M, N, and P be R-modules. A map $f : M \times N \to P$ is called R-bilinear if it is R-linear in each argument, i.e. if

$$\begin{aligned}
&f(m + m', n) = f(m, n) + f(m', n),\\
&f(rm, n) = rf(m, n),\\
&f(m, n + n') = f(m, n) + f(m, n'),\\
&f(m, rn) = rf(m, n)
\end{aligned}$$

for all $r \in R$, $m, m' \in M$, and $n, n' \in N$.

Definition. Let M, N be R-modules. A tensor product of M and N is an R-module denoted by $M \otimes_R N$ together with an R-bilinear map $h : M \times N \to M \otimes_R N$ such that for every R-bilinear map $f : M \times N \to P$ there exists a unique linear map $\widetilde{f} : M \otimes_R N \to P$ which makes the following diagram commute:

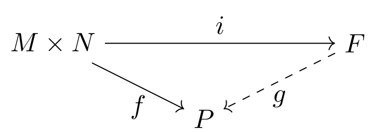


Usually we will drop naming the map h altogether and denote with $m \otimes n$ the image $h(m \times n)$ of an element $m \times n \in M \times N$ in the tensor product $M \otimes_R N$.

Note that the tensor product induces a functor $M \otimes_R -$ from R-modules to abelian groups. With that being said, we now come back to constructing a tensor product explicitly.

Theorem 2.2.5 (Existence of Tensor Products)**.** *Let M, N be R-modules, then there exists a tensor product $M \otimes_R N$.*

Proof. Recall the map $i : M \times N \to F$ from Equation 2.1. We have that F is a free module and so there exists a unique g such that the diagram



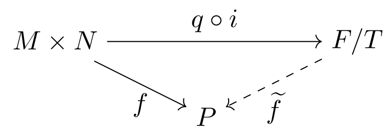
commutes for a bilinear map $f : M \times N \to P$. We want to show that g factors through F/T by $\widetilde{f} : F/T \to P$ such that $g = \widetilde{f} \circ q : F \to P$. To see that, note that f is bilinear and so $g \circ i$ is bilinear, i.e.

$$\begin{aligned}
&(g \circ i)(m + m', n) = (g \circ i)(m, n) + (g \circ i)(m', n),\\
&(g \circ i)(rm, n) = r(g \circ i)(m, n),\\
&(g \circ i)(m, n + n') = (g \circ i)(m, n) + (g \circ i)(m, n'),\\
&(g \circ i)(m, rn) = r(g \circ i)(m, n).
\end{aligned}$$

Since g is linear it follows that

$$\begin{aligned}
&g(i(m + m', n) - i(m, n) - i(m', n)) = 0,\\
&g(i(rm, n) - ri(m, n)) = 0,\\
&g(i(m, n + n') - i(m, n) - i(m, n')) = 0,\\
&g(i(m, rn) - ri(m, n)) = 0.
\end{aligned}$$

Thus g factors through F/T. Let now $\widetilde{f}$ be the map such that $g = \widetilde{f} \circ q$. The composite map $h = q \circ i : M \times N \to F/T$ is bilinear by a similar argument and so we find the following commuting diagram:

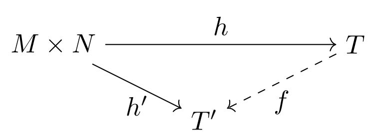


We are left to show that the map $\widetilde{f}$ is unique. This follows from the uniqueness of g and the fact that q is surjective: Assume there exist two $\widetilde{f_1}$ and $\widetilde{f_2}$ such that $\widetilde{f_1} \circ q = g = \widetilde{f_2} \circ q$. For a given $x \in F/T$ let $y \in F$ be such that $q(y) = x$. Then $\widetilde{f_1}(x) = (\widetilde{f_1} \circ q)(y) = g(y) = (\widetilde{f_2} \circ q)(y) = \widetilde{f_2}(x)$ which implies that $\widetilde{f_1} = \widetilde{f_2}$ since q was surjective. $\square$

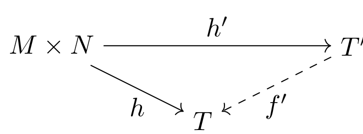
We can actually show a bit more than just existence, namely that tensor products are unique up to isomorphism.

Theorem 2.2.6 (Uniqueness of Tensor Products)**.** *Let M, N be R-modules, and let $h : M \times N \to T$ and $h' : M \times N \to T'$ be two tensor products. Then there exists a unique isomorphism $f : T \to T'$.*

Proof. From the universal property of h we have that h' factors uniquely through T via a linear map, call it $f : T \to T'$ such that the following diagram commutes:



By playing the same game again (with reversed roles of T and T'), we end up with another commuting diagram:



for a different unique linear map $f' : T' \to T$. By combining the two diagrams we find that $f' \circ f = \mathrm{id}_T$ and $f \circ f' = \mathrm{id}_{T'}$ by the universal property of the tensor product. Thus T and T' are isomorphic by f. $\square$

Remark. *The definition and existence of tensor products can be generalised to non-commutative rings. In this case we consider a left R-module M and a right R-module N and form the tensor product $M \otimes_R N$. The construction of the tensor product is now a little more involved as we have to start from a free abelian group and consider a quotient on it rather than directly constructing it from the module F as in Theorem 2.2.5.*

Remark. *The tensor product $M \otimes_R N$ of two R-modules M and N can be generalised to an arbitrary finite number of R-modules $M_1 \otimes_R \cdots \otimes_R M_n$ in the obvious way by extending the definition of the unique property of the tensor product $M \otimes_R N$.*

Example. Let a, b be two non-zero integers. One can show that $\mathbb{Z}/a\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/b\mathbb{Z} \simeq \mathbb{Z}/\gcd(a,b)\mathbb{Z}$. For example, $\mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/5\mathbb{Z} \simeq 0$.

We can generalise this to an arbitrary ring R and ideals I, J of R. Then it holds that $R/I \otimes_R R/J \simeq R/(I+J)$.

Example. The vector space $\mathbb{R}^n$ is an $\mathbb{R}$-module. Thus is makes sense to consider tensor products for it. Take for example $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^3$. Bases for $\mathbb{R}^2$ and $\mathbb{R}^3$ are the sets $\{e_1, e_2\}$ respectively $\{f_1, f_2, f_3\}$ of unit vectors. Using them, we can form a basis $\{g_i\}_{i=1,\ldots,6}$ of $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^3$, namely

$$\begin{aligned} g_1 &= e_1 \otimes f_1, & g_2 &= e_1 \otimes f_2, & g_3 &= e_1 \otimes f_3, \\ g_4 &= e_2 \otimes f_1, & g_5 &= e_2 \otimes f_2, & g_6 &= e_2 \otimes f_3. \end{aligned}$$

We observe, that $\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^3 \simeq \mathbb{R}^6$. In general, one can show that $\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{R}^m \simeq \mathbb{R}^{nm}$.

Adjoint Isomorphism

Let L, M, N be R-modules. We are now interested in the connection between $\hom_R(L \otimes_R M, N)$ and $\hom_R(L, \hom_R(M, N))$ which will become useful in the next section.

Theorem 2.2.7 (Adjoint Isomorphism/Tensor-Hom-Adjunction)**.** *The map*

$$\tau_{L,M,N} : \hom_R(L \otimes_R M, N) \to \hom_R(L, \hom_R(M, N))$$

which takes a homomorphism $\alpha : L \otimes_R M \to N$ to the homomorphism $\alpha' : L \to \hom_R(M, N)$ defined by

$$\alpha'(l)(m) = \alpha(l \otimes m)$$

is an isomorphism of abelian groups.

Proof. We immediately see that $\tau_{L,M,N}$ is a group homomorphism. We are left to show that there exists an inverse group homomorphism. To see this, let $\beta : L \to \hom_R(M, N)$ be a homomorphism. The map $L \times M \to N$ that sends $(l, m) \mapsto \beta(l)(m)$ is bilinear and so by the universal property of the tensor product there exists a group homomorphism $\psi(\beta) : L \otimes_R M \to N$ such that $\psi(\beta)(l \otimes m) = \beta(l)(m)$. It follows that the assignment $\beta \mapsto \psi(\beta)$ defines a group homomorphism. Call it $\sigma_{L,M,N} : \beta \mapsto (l \otimes m \mapsto \beta(l)(m))$. We check that $(\sigma_{L,M,N} \circ \tau_{L,M,N})(\alpha) = \alpha$ and that $(\tau_{L,M,N} \circ \sigma_{L,M,N})(\beta) = \beta$ and so we found the required inverse group homomorphism. $\square$

Theorem 2.2.7 tells us that the functors $- \otimes_R M$ and $\hom_R(M, -)$ are adjoint to each other.

Remark. *The adjoint isomorphism can be generalised for modules over arbitrary rings. In this case one similarly finds that the map*

$$\tau_{L,N} : \hom_S(L \otimes_R M, N) \to \hom_R(L, \hom_S(M, N))$$

is an isomorphism but now L is a left R-module, N is a right S-module and M is a (R, S)-bimodule for rings R and S. In fact there exist even more general forms where all involved modules are taken to be bimodules though which we will not list here and only mention for completeness.

Example. It is easier to visualise the usefulness of these adjoint isomorphism identities by considering vector spaces again. What we then have is that $\hom(U \otimes V, W) \simeq \hom(U, \hom(V, W))$ for finite dimensional vector spaces U, V and W over some common field k. For such vector spaces it holds that $(U \otimes V)^* \simeq U^* \otimes V^*$ and $U^{**} \simeq U$ (where $U^* = \hom(U, k)$, the linear maps from U to k). We write $U^* \otimes V$ as $\hom((U^* \otimes V)^*, k) \simeq \hom(U \otimes V^*, k)$ which using the initial identity we can write as $\hom(U \otimes V^*, k) \simeq \hom(U, \hom(V^*, k))$. But now $\hom(V^*, k)$ is isomorphic to V and thus we find that $U^* \otimes V \simeq \hom(U, V)$, i.e. every linear map $U \to V$ has a representation as an element of $U^* \otimes V$.

Change of Ring

Consider two rings R and S and a ring homomorphism $\varphi : R \to S$. Here we want to answer the question of how to move from a R-module to a S-module via the homomorphism φ. For any S-module M (and thus also S itself), the action of S induces an action of R on M through φ, namely

$$r \cdot m = \varphi(r)m.$$

The idea now is to consider another S-module N and a homomorphism $\psi \in \hom_S(M, N)$. It turns out that ψ can also be regarded as a member of $\hom_R(M, N)$ by the following identity:

$$\psi(r \cdot m) = \psi(\varphi(r)m) = \varphi(r)\psi(m) = r \cdot \psi(m).$$

Thus φ induces a functor φ^* from S-modules to R-modules. Specifically, we call the functor φ^* the *restriction of scalars*. We see why in the following

Theorem 2.2.8 (Extension of Scalars)**.** *There is a natural isomorphism of abelian groups*

$$\hom_S(S \otimes_R N, M) \simeq \hom_R(N, \varphi^*(M)).$$

In particular, the functor $\varphi_! = S \otimes_R -$ is left adjoint to φ^*.

Proof. Observe that $\varphi^*(M) \simeq \hom_S(S, M)$. The theorem then follows from viewing S as a (S, R)-bimodule and the adjoint isomorphism identities. □

Here we call the functor $S \otimes_R -$ the *extension of scalars*. We will see that there exists also a *co-extension of scalars*. Note that we have used φ to define a left action which gave us $\varphi_! = S \otimes_R -$. If we consider a right action instead, we end up with the functor $\varphi_! = - \otimes_R S$. For commutative rings these coincide.

Theorem 2.2.9 (Co-Extension of Scalars)**.** *There is a natural isomorphism of abelian groups*

$$\hom_R(\varphi^*(M), N) \simeq \hom_S(M, \varphi_*(N)).$$

In particular, the functor $\varphi_* = \hom_R(S, -)$ is right adjoint to φ^*.

Proof. Here we view S instead as a (R, S)-bimodule. The theorem then follows from the fact that

$$\tau_{M,S,N} : \hom_R(\varphi^*(M), N) \to \hom_S(M, \hom_R(S, N))$$

is an isomorphism of abelian groups. □

Example. Let S be a commutative ring and R be a subring of S such that $1_S = 1_R$. Then we have $R[x] \otimes_R S = S[x]$. Let I be an ideal of R. Then we have $(R[x]/I) \otimes_R S = S[x]/IS[x]$. In particular we see that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}^2$ for example (by using $\mathbb{C} = \mathbb{R}[x]/(x^2 + 1)$).

Chapter 3

Homology and Cohomology

3.1 Exact Sequences

Remark. *This isn't quite a remark, just letting y'all know that I consistently use xf notation instead of $f(x)$.*

Suppose we have an R-module epimorphism $\varphi : M \to N$. Consider the zero map $0_1 : N \to 0$. By construction, $\ker 0_1 = \operatorname{im} \varphi$. We say that

$$M \xrightarrow{\varphi} N \xrightarrow{0_1} 0$$

is *exact* at N. We now consider the inclusion $\iota : \ker \varphi \to M$, and the zero map $0_2 : 0 \to \ker \varphi$. It is easy to see that

$$0 \xrightarrow{0_2} \ker \varphi \xrightarrow{\iota} M$$

is exact at $\ker \varphi$, because ι is injective. Moreover, the image of ι is precisely the kernel of φ. Therefore

$$0 \longrightarrow \ker \varphi \longrightarrow M \longrightarrow N \longrightarrow 0$$

is exact at each group. Therefore we call it an *exact sequence.*

Definition (Exactness)**.** A sequence

$$\cdots \longrightarrow A_{i+1} \xrightarrow{\varphi_{i+1}} A_i \xrightarrow{\varphi_i} A_{i-1} \longrightarrow \cdots$$

of R-modules and R-homomorphisms is *exact at A_i* if $\ker \varphi_i = \operatorname{im} \varphi_{i+1}$. We say it is *exact* if it is exact at A_i for all i. A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0.$$

Example. In the above example, exactness of $G \to \operatorname{im}\varphi \to 0$ and $0 \to \ker\varphi \to G$ came from the surjectivity and injectivity of the maps in question. The converse also holds:

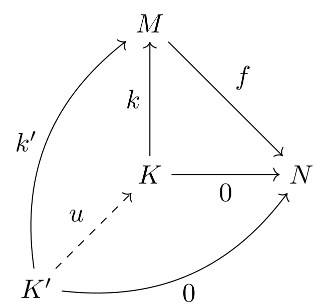
- $M \xrightarrow{\varphi} N \to 0$ is exact if and only if φ is surjective.
- $0 \to M \xrightarrow{\psi} N$ is exact if and only if ψ is injective.

Definition. The *cokernel* of an R-homomorphism $\varphi : M \to N$ is defined by $\operatorname{coker}\varphi := N/\operatorname{im}\varphi$.

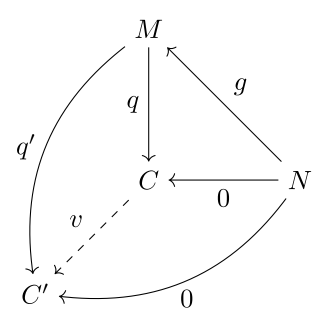
In category theory, like a lot of other things, we define the kernel of a map using a universal property. Suppose M, N are R-modules, and $f : M \to N$ is an R-homomorphism. Then the kernel of f is the map $k : K \to M$ such that:

- $kf = 0$,
- Given any R-homomorphism $k' : K' \to X$ such that $k'f = 0$, there exists a unique R-homomorphism $u : K' \to K$ such that $uk = k'$.

Intuitively, the map $k : K \to M$ takes the role of the inclusion of the classical notion of the kernel into the domain of f. The universal property tells us that K "is the right size". Existence of u tells us that it's large enough, but uniqueness tells us that it's not too large.



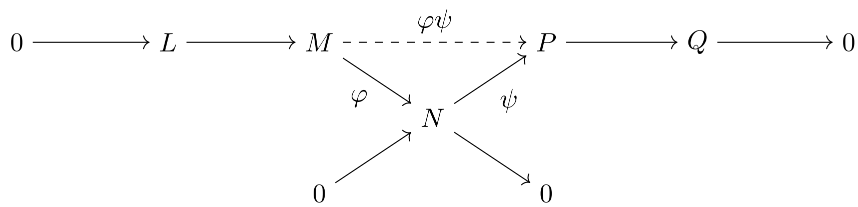
We are now well positioned to define a cokernel. In category theory, dual objects are always defined by reversing arrows. Hence given an R-homomorphism $g : N \to M$, the cokernel of g is the map $q : M \to C$ such that $gq = 0$, and the dashed morphism exists and is unique whenever the diagram below commutes.



Intuitively, q is the quotient map from M to the classical cokernel.

Theorem 3.1.1 (Splicing)**.** *Given two short exact sequences* $0 \to L \to M \to N \to 0$ *and* $0 \to N \to P \to Q \to 0$*, we may form a new (longer!) exact sequence* $0 \to L \to M \to P \to Q \to 0$*. Conversely, every exact sequence* $\cdots \to A_{i-1} \xrightarrow{\varphi_{i-1}} A_i \xrightarrow{\varphi_i} A_{i+1} \to \cdots$ *can be decomposed into short exact sequences* $0 \to \operatorname{im} \varphi_{i+1} \to A_i \to \operatorname{im} \varphi_i \to 0$ *for each* i*.*

Proof. $\Rightarrow$ Consider the following diagram:



We know that ψ is injective, so $\ker \varphi\psi = \ker \varphi$. On the other hand, φ is onto, so $\operatorname{im} \varphi\psi = \operatorname{im} \psi$. Thus the new sequence is exact.

$\Leftarrow$ Consider the exact sequence of R-modules $\cdots \to A_{i-1} \xrightarrow{\varphi_{i-1}} A_i \xrightarrow{\varphi_i} A_{i+1} \to \cdots$. Recall from our first example that, given some n, there is a short exact sequence $0 \to \ker \varphi_n \to A_n \to \operatorname{im} \varphi_n \to 0$. By exactness, $\ker \varphi_n$ is just $\operatorname{im} \varphi_{n+1}$. $\square$

Definition. Let M, N be R-modules. M is a *retract* of N if there exists $r : N \to M$ and $\sigma : M \to N$ such that $\sigma r = \mathrm{id}_M$. We call σ a *section* and r a *retraction.*

Theorem 3.1.2 (Splitting lemma)**.** *Let* L, M, N *be R-modules. Suppose we have a short exact sequence* $0 \to L \xrightarrow{\iota} M \xrightarrow{\pi} N \to 0$*. The following are equivalent, and we call such a short exact sequence* split*:*

(i) *There is a retraction* $r : M \to L$*.*

(ii) *There is a section* $\sigma : N \to M$*.*

(iii) M *is isomorphic to* $L \oplus N$*.*

Proof. It is clear that (iii) implies (i) and (ii).

We now prove (i) implies (iii). Define the map $P : M \to M$ by $P = r\iota$. Let $m \in M$, and write $m = (m - mP) + mP$. $m - mP$ belongs to $\ker r$, since $(m - mP)r = mr - mr\iota r = mr - mr = 0$. Similarly, mP belongs to $\operatorname{im} \iota$. Therefore $M \cong \ker r + \operatorname{im} \iota$.

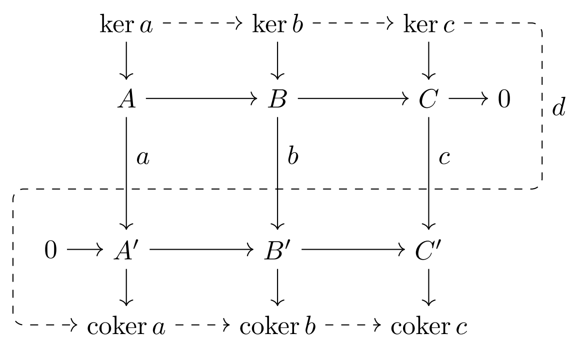
We now have to show that the decomposition is unique, that is, $\ker r \cap \operatorname{im} \iota = 0$. Suppose $l = m\iota$ and $lr = 0$. Then $m = m(\iota r) = (m\iota)r = 0$. It follows that $M \cong \ker r \oplus \operatorname{im} \iota$.

By injectivity, $\operatorname{im} \iota \cong L$. On the other hand, by exactness, $\ker \pi = \operatorname{im} \iota$. Since π is surjective, given any $n \in N$, there exists m such that $m\pi = n$. But recall that $m = l\iota + k$ for some $l \in L$ and $k \in \ker r$, so $n = m\pi = (l\iota + k)\pi = k\pi$. It follows that $\pi|_{\ker r}$ is surjective. To see that it is injective, suppose $k\pi|_{\ker r} = 0$. Then k belongs to $\operatorname{im} \iota$ by exactness, but from earlier we have seen that $\operatorname{im} \iota \cap \ker r = 0$. Thus $\pi_{\ker r}$ is injective, so $\ker r \cong N$.

The proof of (ii) implies (iii) is very similar. $\square$

We now state the snake lemma, because are we really doing homological algebra without our friend snakeboi?

Theorem 3.1.3 (Snake lemma)**.** *Suppose we have a commutative diagram of R-modules and R-homomorphisms as below, denoted by solid morphisms, where the rows are exact.*



Then there exists a morphism d called the connecting homomorphism *so that the sequence denoted by dashed morphisms is exact.*

Proof. It's My Turn, 1980. ☐

Remark. *The snake lemma holds in a more general setting: It is valid in every abelian category.*

Theorem 3.1.4 (Five lemma)**.** *Suppose we have a commutative diagram of R-modules and R-homomorphisms as follows, where the rows are exact and the solid vertical morphisms are isomorphisms. then the dashed morphism is an isomorphism.*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow & & \downarrow & & \vdots & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

Proof. Diagram chase. ☐

Remark. *This also holds in more general settings. We do not in fact need the abelian condition. For example, it holds for groups and group homomorphisms.*

3.2 Exact Functors

Example. As a physicist, a familiar example of an exact sequence is the following diagram.

$$C^\infty(\mathbb{R}^3) \xrightarrow{\text{grad}} \Gamma(T\mathbb{R}^3) \xrightarrow{\text{curl}} \Gamma(T\mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\mathbb{R}^3)$$

First this tells us that given a scalar field ϕ, $\nabla \times \nabla\phi = 0$, and given a vector field A, $\nabla \cdot \nabla \times A = 0$. Moreover, given $A \in \Gamma(T\mathbb{R}^3)$, it tells us that if $\nabla \times A = 0$, then $A = \nabla\phi$ for some $\phi \in C^\infty(\mathbb{R}^3)$, and if $\nabla \cdot A = 0$, then $A = \nabla \times B$ for some $B \in \Gamma(T\mathbb{R}^3)$. This is a special case of the de Rham complex, which may or may not be covered later:

$$C^\infty(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \Omega^2(M) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n(M)$$

To formalise the connection between the two diagrams, we define a map that takes objects from one diagram to objects in the other, as well as taking morphisms from one diagram to morphisms in the other. This is called a functor. A common proof of the exactness of the first diagram is constructive and messy. A cleaner proof uses the fact that the de Rham complex is exact when $M = \mathbb{R}^3$, and the functor between diagrams preserves exactness.

However, it's not generally true that a functor will preserve exactness. To proceed further, we'll formally define some categorical notions along with exactness of functors.

Definition (Category)**.** A *category* C consists of a class of *objects* $\mathrm{ob}(C)$ and a class of *morphisms* $\hom(C)$. Each morphism $f \in \hom(C)$ has a source object a and target object b. $\hom_C(a,b)$ denotes the class of all morphisms from a to b in C. (We drop the subscript C when the category is clear.) There is a binary operation $\hom(a,b) \times \hom(b,c) \to \hom(a,c)$ called *composition*, where $(f,g) \mapsto fg$. Moreover, the following properties must hold:

- (Associativity.) If $f \in \hom(a,b)$, $g \in \hom(b,c)$, and $h \in \hom(c,d)$, then $f(gh) = (fg)h$.
- (Identity.) For each $a \in \mathrm{ob}(C)$, there is a morphism id_a called the *identity*, so that $\mathrm{id}_a f = f$ and $g\,\mathrm{id}_a = g$ for any $f \in \hom(a,b)$ and $g \in \hom(c,a)$, for any $b, c \in \mathrm{ob}(C)$.

Example. The category **Set** consists of sets as its objects and functions as its morphisms. The definition is vague by design: Some consider it to be the category of *all* sets, some consider it to be the category of *small* sets, which are sets contained in a Grothendieck universe. On the other hand, **Ab** is the category consisting of all abelian groups as objects and group homomorphisms as morphisms.

Example. Up until now we have been working in the category of left modules, denoted **R-mod**. We see that our definition of $\hom_R(M,N)$ is in fact $\hom_{\textbf{R-mod}}(M,N)$ equipped with addition of morphisms and left multiplication by elements of R. While $\hom_{\textbf{R-mod}}(M,N)$ is just a set, $\hom_R(M,N)$ is an R-module.

The word "functor" is thrown around a lot, loosely thought of as a map between categories that takes objects to objects and morphisms to morphisms. Here's a precise definition.

Definition (Covariant Functor)**.** Let C, D be categories. A covariant functor (often just called a functor) is a map F that takes each $a \in \mathrm{ob}(C)$ to $aF \in \mathrm{ob}(D)$, and each morphism $f \in \hom_C(a,b)$ to $fF \in \hom_D(aF, bF)$, satisfying

- For all objects $a \in \mathrm{ob}(C)$, $\mathrm{id}_a F = \mathrm{id}_{aF}$.
- For all morphisms $f \in \hom_C(a,b), g \in \hom_C(b,c)$, $(fg)F = (fF)(gF)$.

Definition (Contravariant Functor)**.** A contravariant functor F from C to D is simply a functor from the opposite category C^{op} to D. The opposite category is the category in which the sources and targets of all morphisms is swapped, along with the order of function composition. This means F maps $f \in \hom_C(a,b)$ to $fF \in \hom_D(bF, aF)$. Function composition satisfies $(fg)F = (gF)(fF)$.

Example. We've referred to hom as being a functor, but why is it a functor? Let C be a category, and $a \in \text{ob}(C)$. Then $\hom(a,-)$ defines a functor from C to **Set**. The functor is defined by

- For all $b \in \text{ob}(C)$, $b \mapsto \hom(a,b) \in \text{ob}(\textbf{Set})$.
- For all $f \in \hom(C)$, $(f : c \to d) \mapsto (\hom(a,f) : \hom(a,c) \to \hom(a,d))$, where the map is given by $g \mapsto gf$.

It easily verified that this is a functor. It is also easily shown that $\hom(-,a)$ defines a contravariant functor.

Example. Let $M \in \text{ob}(\textbf{R-mod})$. Then $\hom_R(M,-)$ defines a functor from **R-mod** to itself.

Example. Let $M \in \text{ob}(\textbf{R-mod})$. Then $-\otimes_R M$ defines a functor from **R-mod** to itself. Given $f \in \hom(N,L)$, $f \otimes_R M : N \otimes_R M \to L \otimes_R M$ is defined to be the map $n \otimes_R m \mapsto nf \otimes_R m$.

Definition (Exact Functor)**.** Let C, D be categories of modules, and $F : C \to D$ a functor. We say F is *left exact* if whenever $0 \xrightarrow{f} L \xrightarrow{g} M \xrightarrow{h} N$ is exact in C, $0 \xrightarrow{fF} LF \xrightarrow{gF} MF \xrightarrow{gF} NF$ is exact in D. Similarly, we say F is *right exact* if whenever $L \xrightarrow{g} M \xrightarrow{h} N \xrightarrow{j} 0$ is exact in C, $LF \xrightarrow{gF} MF \xrightarrow{hF} NF \xrightarrow{jF} 0$ is exact in D. A functor is *exact* if it is both left and right exact.

Proposition 3.2.1. *If a functor $F : C \to D$ between categories of modules is left (right) exact, it maps monics to monics (epics to epics).*

Proof. This is clear from the definition. ☐

Example. The functor $\hom(M,-)$ is left exact. The functor $-\otimes_R M$ is right exact. The proofs are bookwork: Simply show that sequences obtained by applying the functors are exact at each module.

3.3 Projective and Injective Modules

3.4 Chain Complexes and Homologies

The notion of exact sequences has been shown to be useful, but many of the sequences we encounter are not exact. How do we measure how "close" we are to exactness, and what information can this measure tell us? If we relax the definition of the exact sequence, we can begin to explore the world of complexes and homologies.

Definition. Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of R-Modules, and let $\{d_n\}_{n\in\mathbb{N}}$ be homomorphisms $d_n : C_n \to C_{n-1}$ such that $d_n \circ d_{n+1} = d_{n+1}d_n = 0$. Then we call $(C_n, d_n)_{n\in\mathbb{N}}$ a **chain complex**, and the d_n the **boundary maps**. We call $\text{Im}d$ the boundaries of C_n and $\ker d$ the cycles of C_n.

Example. Consider the sequence of the modules $C_n = \mathbb{Z}/8\mathbb{Z}$, with the boundary homomorphisms $d_n : x \mapsto 4x$.

Definition. A chain complex $(C_\bullet, d_\bullet)$ is **bounded above (below)** if there exists an $n \in \mathbb{N}$ such that $C_i = 0$ for all $i > n$ ($i < n$). If a complex is bounded both above and below, then we say it is **bounded**.

Definition. Given a chain complex $C = (C_\bullet, d_\bullet)$, define its **homology** groups as

$$H_k(C) = \frac{\ker d_k}{\text{im} d_{k+1}}$$

If a complex has only trivial Homology groups, then we say that the chain complex is **exact**.

Example. In the setting of the previous example, the homology groups are all isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Definition. Let $C = (C_\bullet, d_\bullet)$ and $D = (D_\bullet, \delta_\bullet)$ be chain complexes. A sequence of homomorphisms $f_n : C_n \to D_n$ is a **chain map** if each f_n commutes with the boundary operators:

$$f_{n-1} \circ d_n = \delta_{n-1} \circ f_n$$

i.e., for which the following diagram commutes:

$$\begin{array}{ccccccccc}
\longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \longrightarrow \\
 & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & \\
\longrightarrow & D_{n+1} & \xrightarrow{\delta_{n+1}} & D_n & \xrightarrow{\delta_n} & D_{n-1} & \longrightarrow
\end{array}$$

These are important because of the following:

Lemma 3.4.1. *Chain maps map cycles to cycles and boundaries to boundaries.*

Proof. Let $c \in \ker d$. Then by the commutativity of f, $cf\delta = cdf = 0f = 0$, so indeed cycles are preserved. Now take $c = bd \in \mathrm{Im}d$. Then again by commutativity $cf = bdf = bf\delta \in \mathrm{Im}\delta$, so boundaries are mapped to boundaries too. ☐

Now consider a commutative diagram of the form:

$$\begin{array}{ccccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & D_{n+1} & \xrightarrow{d_{n+1}} & D_n & \xrightarrow{d_n} & D_{n-1} & \longrightarrow \\
 & \downarrow{\scriptstyle j_{n+1}} & & \downarrow{\scriptstyle j_n} & & \downarrow{\scriptstyle j_{n-1}} & \\
\longrightarrow & E_{n+1} & \xrightarrow{d_{n+1}} & E_n & \xrightarrow{d_{n+1}} & E_{n-1} & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

If the columns are exact, then we call this a **short exact sequence of chain complexes**. It is possible to relate the homology groups together to form a long exact sequence of homology groups:

$$\longrightarrow H_n(C) \xrightarrow{i_*} H_n(D) \xrightarrow{j_*} H_n(E) \xrightarrow{\delta} H_{n-1}(C) \xrightarrow{i_*} H_{n-1}(D) \xrightarrow{j_*} H_{n-1}(E) \longrightarrow$$

To see this, note that the commutativity of the diagram means that i and j are chain maps, as defined above. They induce homomorphisms $i_* : H_n(C) \to H_n(D)$ and $j_* : H_n(D) \to H_n(E)$ given by, $[c] \mapsto [ci]$ and $[d] \mapsto [jd]$. All that we need is the boundary map $\delta : H_n(E) \to H_{n-1}(C)$.

Take $[e] \in H_n(E)$. Since j is onto, $e = aj$ for some $a \in D$. Now consider $adj = ajd = ed = 0$, so $ad \in \ker j = \mathrm{Im} i$. But then there exists a $c \in C$ so that $ci = ad$. Moreover, c is unique because i is injective. Let δ map $[e] \mapsto [c]$.

Lemma 3.4.2. *$\delta : H_n(E) \to H_{n-1}(C)$ is a well defined homomorphism.*

Proof. First, see that $cdi = cid = add = 0$, so by injectivity of i, $c \in \ker d$.

Secondly, we need that $[c]$ as defined is an invariant of the choice of a. Suppose both a and a' have $aj = a'j = e$. Then $a - a' \in \ker j = \operatorname{Im} i$, so $a - a' = c'i$ for some $c' \in C$. Thus $a' = a + c'i$, and so $a'd = ad + c'id = ci + c'di = (c + c'd)i$. But $[c] = [c + c'd]$, so indeed this does not depend on the choice of a.

Thirdly, see that the map does not depend on the choice of representative from the cohomology class of e. Take $e, e' \in [e]$, with preimages (under j) a and a' respectively. Then we must have $a' = a + a_0 d$ so $ad = a'd$, and thus we are done.

Lastly, δ is indeed a homomorphism. Let $[e_1] \mapsto [c_1]$ and $[e_2] \mapsto [c_2]$ via the choices a_1 and a_2 as above. See that $(a_1 + a_2)j = a_1 j + a_2 j = e_1 + e_2$ and that $(c_1 + c_2)i = c_1 i + c_2 i = a_1 d + a_2 d = (a_1 + a_2)d$. Then

$$([e_1] + [e_2])\delta = ([e_1 + e_2])\delta = [c_1 + c_2] = [c_1] + [c_2]$$

□

Theorem 3.4.3. *The long exact sequence of homology groups (given below) is exact.*

$$\longrightarrow H_n(C) \xrightarrow{i_*} H_n(D) \xrightarrow{j_*} H_n(E) \xrightarrow{\delta} H_{n-1}(C) \xrightarrow{i_*} H_{n-1}(D) \xrightarrow{j_*} H_{n-1}(E) \longrightarrow$$

Proof. We must show that the kernel for each homomorphism is indeed the image of the prior one. We do this for each three below.

It is clear that $\operatorname{Im} i_* \subseteq \ker j_*$ because $ij = 0$ and so $j_* i_* = 0$. So let $[a] \in \ker j_*$. Then $aj = ed$ for some $e \in E$. By the sujectivity of j, there exists some $b \in D_{n+1}$ with $bj = e$. Then

$$(a - bd)j = aj - adj = aj - ajd = aj - ed = aj - aj = 0$$

so $a - bd \in \ker j = \operatorname{Im} i$. Thus $a - bd = ci$ for some $c \in C$, and so

$$cdi = cid = (a - bd)d = ad = 0$$

But then by the injectivity of i, $cd = 0$. Thus $[c]i_* = [a - bd] = [a]$, so $[a] \in \operatorname{Im} i_*$.

We now show $\operatorname{Im} j_* = \ker \delta$. First, see that if $[e] = [b]j_* \in \operatorname{Im} j_*$ then we must have $b \in \ker d$, so $bd = 0i$ (by injectivity of i), and so $[e]\delta = [0]$, so $\operatorname{Im} j_* \subseteq \ker \delta$. Now let $[e] \in \ker \delta$, and take $b \in D$ such that $bj = e$. Then $[c]\delta = [a] = 0$ so $a \in \ker d$, and thus $a = a'd$ for some $a' \in C$. Now

$$(b - a'i)d = bd - a'id = bd - a'di = bd - ai = bd - bd = 0$$

so $(b - a'i) \in \ker d$. Then $(b - a'i)j = bj - a'ij = bj = e$ and so $[b - a'i]j_* = [e] \in \operatorname{Im} j_*$.

Lastly, see that $\operatorname{Im}\delta \subseteq \ker i_*$ as $[a]\delta i_* = [c]i_* = [bd] = 0$. So take $[a] \in \ker i_*$, so $ai = bd$ for some $b \in D$. Then $bdj = bdj = aij = 0$, so $bj \in \ker d$. Then δ maps $[bj] \mapsto [a] \in \operatorname{Im}\delta$, finishing the proof. □

This could also be done by two applications of the snake lemma, and we leave the details of this to the reader.

3.5 Cochains and Cohomology

Definition. Let $\{C^n\}_{n\in\mathbb{N}}$ be a sequence of R-Modules, and let $\{d^n\}_{n\in\mathbb{N}}$ be homomorphisms $d^n : C^n \to C^{n+1}$ such that $d^n \circ d^{n-1} = 0$. Then we call $(C^n, d^n)_{n\in\mathbb{N}}$ a **cochain complex**.

Definition. Given a co-chain complex $C = (C^\bullet, d^\bullet)$, define its **cohomology** groups as

$$H^k(C) = \frac{\ker d^k}{\mathrm{im} d^{k-1}}$$

Example. Consider the de Rham complex on a closed smooth n-manifold $(\Omega^\bullet(M), \mathrm{d})$. Where

$$dd^k\omega(V_1, \ldots, V_{k+1}) = \sum_{i=1}^{k+1}(-1)^i\, V_i\, \omega(V_1, \ldots, V_{i-1}, V_{i+1}, \ldots V_{k+1})$$
$$+ \sum_{i=1}^{k+1}\sum_{j=1}^{i}(-1)^{i+j}\omega([V_j, V_i], V_1, \ldots, V_{j-1}, V_{j+1}, \ldots, V_{i-1}, V_{i+1}, \ldots V_{k+1})$$

Then in local co-ordinates (for global co-ordinates, the proof is much longer and more involved), we see that

$$\mathrm{d}^{k+1} \circ \mathrm{d}^k\omega = -(\partial_i\partial_j\omega_{i_1\cdots i_k})(\mathrm{d}u^i \wedge \mathrm{d}u^j \wedge \mathrm{d}u^{i_1} \wedge \cdots \wedge \mathrm{d}u^{i_k}) = -\mathrm{d}^{k+1} \circ \mathrm{d}^k\omega$$

Thus $\mathrm{d}^2 = 0$, and so we have a cochain complex. The cohomology groups of this complex contain important topological information about the structure of the manifold, as we will see when we consider singular homologies.

How do we construct cochains from chains? One way is to reindex; are there any nontrivial ways? Let (C_n, d_n) be a chain complex and let G be an abelian group (considered as a $\mathbb{Z}$-Module). Consider the modules $C_n^* = \mathrm{Hom}(C_n, G)$. Then define the **coboundary** map $\delta^n : C_{n-1}^* \to C_n^*$ by $\phi \mapsto d^*\phi = (c \mapsto cd\phi)$. It is then clear that

$$\phi\delta^n\delta^{n+1} = d_{n+1}d_n\phi = 0\phi = 0$$

So (C_n^*, δ^n) is indeed a cochain complex, closely related to (C_n, d_n). The question now is how are the homology groups related to the cohomology groups? It may be tempting to think that $H^n(C, G) \simeq \mathrm{Hom}(H_n(C), G)$, but unfortunately this is not usually the case. We start see how much of this we can recover with the following proposition.

Definition. If $[\phi] \in H^n(C, G)$, then $\phi|_{\ker d} : \ker d \to G$ induces a homomorphism $\phi_0 : H_n(C) \to G$. Define the map $h : H^n(C_n, G) \to \mathrm{Hom}(H_n(C_n), G)$ given by taking $[\phi] \mapsto \phi_0$.

Proposition 3.5.1. *h is a homomorphism.*

Theorem 3.5.2. *h is surjective.*

3.6 Recap of Chain Complexes and Maps

We firstly recall some important definitions and results from earlier. Reference: *A Course in the Theory of Groups* by Derek J.S. Robinson.

Let $\mathbf{C} = (C_n, \partial_n)_{n\in\mathbb{Z}}$ and $\overline{\mathbf{C}} = (\overline{C}_n, \overline{\partial}_n)_{n\in\mathbb{Z}}$ be chain complexes of R-modules. Sometimes we will just call $\mathbf{C}$, $\overline{\mathbf{C}}$ *complexes*. Recall that a chain map $\gamma\colon \mathbf{C} \to \overline{\mathbf{C}}$ is a sequence $(\gamma_n\colon C_n \to \overline{C}_n)_{n\in\mathbb{Z}}$ of R-homomorphisms such that $\partial_n\gamma_{n-1} = \gamma_n\overline{\partial}_n$ for $n \in \mathbb{Z}$. We sometimes call a chain map a *morphism* of complexes.

Since $\partial_{n+1}\partial_n = 0$ one has $\mathrm{im}\,\partial_{n+1} \subseteq \ker\partial_n$. Note the image and kernel of an R-homomorphisms are R-submodules, hence it makes sense to define the homology group $H_n(\mathbf{C})$ of a complex by $H_n(\mathbf{C}) \coloneqq$

$\ker \partial_n / \operatorname{im} \partial_{n+1}$. In fact these may be equipped with an R-module structure, but they are typically just called homology groups.

There is a reason that ∂, δ, or d are commonly used letters for the homomorphisms in a chain complex. In 3.12, we will see that there is a natural chain complex (the singular complex of a topological space) in which one may view ∂ as a "boundary" operator. The identity $\partial_{n+1}\partial_n = 0$ then says that the boundary of a boundary is empty.

In view of this, one sometimes refers to the elements of $\operatorname{im} \partial_{n+1}$ as the boundaries, and the elements of $\ker \partial_n$ as the cycles (cycles have no boundary). Lemma 3.4.1 then states that if $\gamma\colon \mathbf{C} \to \overline{\mathbf{C}}$ is a morphism of complexes, then γ_n maps $\ker \partial_n$ into $\ker \overline{\partial_n}$, and $\operatorname{im} \partial_{n+1}$ into $\operatorname{im} \overline{\partial}_{n+1}$.

It follows from this that $\gamma\colon \mathbf{C} \to \overline{\mathbf{C}}$ induces group (indeed R-module) homomorphisms $\gamma_{n,*}\colon H_n(\mathbf{C}) \to H_n(\overline{\mathbf{C}})$. Why? Elements of $H_n(\mathbf{C})$ are of the form $a + \operatorname{im} \partial_{n+1}$ where $a \in \ker \partial_n$. For $a \in \ker \partial_n$, the mapping $a + \operatorname{im} \partial_{n+1} \mapsto a\gamma_n + \operatorname{im} \overline{\partial}_{n+1}$ is

- well defined, because γ_n maps $\operatorname{im} \partial_{n+1}$ into $\operatorname{im} \overline{\partial}_{n+1}$,
- maps $H_n(\mathbf{C}) = \ker \partial_n / \operatorname{im} \partial_{n+1}$ into $H_n(\overline{\mathbf{C}}) = \ker \overline{\partial}_n / \operatorname{im} \overline{\partial}_{n+1}$, because γ_n maps $\ker \partial_n$ into $\ker \overline{\partial}_n$,
- is an R-module homomorphism from $H_n(\mathbf{C}) \to H_n(\overline{\mathbf{C}})$.

We invite the reader to verify the above claims. Once this is done, one has the following lemma.

Lemma 3.6.1. *A morphism $\gamma\colon \mathbf{C} \to \overline{\mathbf{C}}$ of complexes induces homomorphisms $\gamma_{n,*}\colon H_n(\mathbf{C}) \to H_n(\overline{\mathbf{C}})$.*

3.7 Homotopy of Chain Complexes

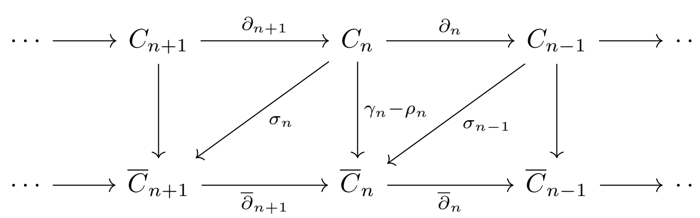
Homotopy is a way of comparing maps. Recall that in topology, if X and Y are topological spaces, and $f, g\colon X \to Y$ are continuous maps, then we say that f and g are *homotopic* if there exists a continuous function $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. We call H a *homotopy.* This induces an equivalence relation on the space of continuous maps from $X \to Y$. A continuous map is *null-homotopic* if it is homotopic to a constant function.

Our goal is to define a meaningful notion of homotopy on chain maps between two complexes $\mathbf{C}$ and $\overline{\mathbf{C}}$. It is difficult (for me) to fully motivate this in advance so here we will firstly just state the definition.

Definition (Homotopy)**.** Let $\gamma, \rho\colon \mathbf{C} \to \overline{\mathbf{C}}$ be morphisms of complexes (i.e. chain maps). We say that γ, ρ are *homotopic* if there exist R-homomorphisms $\sigma_n\colon C_n \to \overline{C}_{n+1}$ such that

$$\gamma_n - \rho_n = \sigma_n \overline{\partial}_{n+1} + \partial_n \sigma_{n-1}$$

for all $n \in \mathbb{Z}$. One may visualise this as "partial commutativity" in the two middle triangles of the below diagram (NB: this diagram does NOT commute).



Why is this a good definition for homotopy? One answer is that to each topological space one can associate a chain complex (known as the singular complex). A continuous map $f\colon X \to Y$ of topological spaces induces a chain map between the corresponding singular complexes. If two continuous maps $X \to Y$ are homotopic in the topological sense, then one can show that the induced chain maps are homotopic in the algebraic sense.

A second answer is that there is an appropriate generalisation of homotopy which agrees with the above algebraic definition. Instead of working in the category of topological spaces, one works in the category of R-module chain complexes (the morphisms are naturally chain maps). One must generalise the notion of an "interval object" to replace $[0,1]$ with an appropriate object in this new category.

Theorem 3.7.1. *Let $\gamma\colon \mathbf{C} \to \overline{\mathbf{C}}$ and $\rho\colon \overline{\mathbf{C}} \to \mathbf{C}$ be morphisms of complexes. If $\gamma\rho$ and $\rho\gamma$ are homotopic to identity morphisms, then the induced homomorphisms $\gamma_{n,*}\colon H_n(\mathbf{C}) \to H_n(\overline{\mathbf{C}})$ are isomorphisms.*

Proof. Since $\gamma\rho$ is homotopic to $\mathrm{id}_{\mathbf{C}}$, there exist R-module homomorphisms $\sigma_n\colon C_n \to C_{n+1}$ such that $\gamma_n\rho_n - \mathrm{id}_{C_n} = \sigma_n\partial_{n+1} + \partial_n\sigma_{n-1}$, for $n \in \mathbb{Z}$. If $a \in \ker\partial_n$, then $a\gamma_n\rho_n - a = a\sigma_n\partial_{n+1} \in \operatorname{im}\partial_{n+1}$. Therefore one has $a\gamma_n\rho_n + \operatorname{im}\partial_{n+1} = a + \operatorname{im}\partial_{n+1}$, i.e. $(\gamma\rho)_{n,*}\colon H_n(\mathbf{C}) \to H_n(\mathbf{C})$ is the identity map. Similarly $(\rho\gamma)_{n,*}\colon H_n(\overline{\mathbf{C}}) \to H_n(\overline{\mathbf{C}})$ is the identity map. Hence $\gamma_{n,*}\colon H_n(\mathbf{C}) \to H_n(\overline{\mathbf{C}})$ is an isomorphism. □

3.8 Resolutions

Definition. A chain complex $\mathbf{C}$ is *positive* if $C_n = 0$ for all $n < 0$. We often write a positive complex $\mathbf{C}$ as $\cdots \to C_2 \to C_1 \to C_0 \to 0$.

Definition (Resolution)**.** Let M be an R-module. An *R-resolution* of M is a positive chain complex $\mathbf{C}$ and an epimorphism $\varepsilon\colon C_0 \twoheadrightarrow M$ such that

$$\cdots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

is exact. We may abbreviate the resolution by $\mathbf{C} \overset{\varepsilon}{\twoheadrightarrow} M$. We call the resolution free (projective) if the complex $\mathbf{C}$ is free (projective), which means that each of the R-modules $C_0, C_1, \ldots$ are free (projective).

Theorem 3.8.1. *Every R-module has a free R-resolution.*

Proof. Let M be an R-module. There exists a free R-module C_0 and an epimorphism $\varepsilon\colon C_0 \twoheadrightarrow M$. For example, let X be a set which generates M as an R-module, let C_0 be free on X, then the identity map $X \to X$ extends to the required epimorphism. We then have a short exact sequence $0 \hookrightarrow \ker\varepsilon \hookrightarrow C_0 \overset{\varepsilon}{\twoheadrightarrow} M \twoheadrightarrow 0$.

Replacing M with $\ker\varepsilon$, we obtain a short exact sequence $0 \to \ker\varepsilon_1 \to C_1 \xrightarrow{\varepsilon_1} \ker\varepsilon \to 0$, where C_1 is a free R-module. Similarly we obtain short exact sequences $0 \to \ker\varepsilon_n \to C_n \xrightarrow{\varepsilon_n} \ker\varepsilon_{n-1} \to 0$ for $n = 1, 2, \ldots$, with each C_n free. Concatenating these using the splicing lemma 3.1.1 we obtain an exact sequence

$$\cdots \to C_2 \to C_1 \to C_0 \xrightarrow{\varepsilon} M \to 0,$$

as required. □

In view of this theorem and the above proof, we may imagine R-resolutions to be a generalisation of R-module presentations. Indeed, let M be an R-module, and let X be a set which generates M as an R-module. Let F be a free R-module on the set X, and let K be the kernel of the natural map $F \twoheadrightarrow M$. Then we say $\langle X \mid K \rangle$ is a presentation of M. (More generally, if Y is any set which generates K, then $\langle X \mid Y \rangle$ is a presentation of M.)

In group theory there is an important theorem (Nielsen-Schreier) which asserts that every subgroup of a free group is free. However as we have already seen in 3.3, submodules of free modules are not necessarily free. Thus in our identification $F/K \cong M$, while F is free, K may not be, and so we may wish to construct a presentation of K to better understand it. This yields an epimorphism $F_1 \twoheadrightarrow K$ with some kernel K_1 and F_1 free. Continuing this process yields the free R-resolution of M described above.

Theorem 3.8.2. *Let $\mathbf{P} \overset{\varepsilon}{\twoheadrightarrow} M$ and $\overline{\mathbf{P}} \overset{\overline{\varepsilon}}{\twoheadrightarrow} \overline{M}$ be two projective R-resolutions. If $\alpha \colon M \to \overline{M}$ is an R-homomorphism, there is a morphism $\pi \colon \mathbf{P} \to \overline{\mathbf{P}}$ such that $\pi_0 \overline{\varepsilon} = \varepsilon\alpha$. Moreover π is unique up to homotopy.*

$$\begin{array}{ccc} \mathbf{P} & \overset{\varepsilon}{\twoheadrightarrow} & M \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\alpha} \\ \overline{\mathbf{P}} & \overset{\overline{\varepsilon}}{\twoheadrightarrow} & \overline{M} \end{array}$$

Proof. Since P_0 is projective and $\overline{\varepsilon}$ is an epimorphism, we may lift $\varepsilon\alpha \colon P_0 \to M$ along $\overline{\varepsilon} \colon \overline{P}_0 \to M$ to obtain an R-homomorphism $\pi_0 \colon P_0 \to \overline{P}_0$ such that $\pi_0 \overline{\varepsilon} = \varepsilon\alpha$. Now let $n \geq 0$ and suppose homomorphisms $\pi_i \colon P_i \to \overline{P}_i$ have been constructed for $1 \leq i \leq n$, such that $\pi_i \overline{\partial}_i = \partial_i \pi_{i-1}$. For convenience, call $\varepsilon = \partial_0$, $\overline{\varepsilon} = \overline{\partial}_0$, $\alpha = \pi_{-1}$, so that this holds for $i = 0$ as well.

Note that $\overline{\partial}_{n+1}$ is a an epimorphism when viewed as mapping from P_{n+1} onto $\operatorname{im} \overline{\partial}_{n+1} = \ker \overline{\partial}_n$. We would like to show that $\operatorname{im}(\partial_{n+1}\pi_n) \subseteq \ker \overline{\partial}_n$, because then since P_{n+1} is projective, it would follow that map $\partial_{n+1}\pi_n$ will factor through $\overline{\partial}_{n+1}$ via some map $\pi_{n+1} \colon P_{n+1} \to \overline{P}_{n+1}$.

One has $\partial_{n+1}\pi_n\overline{\partial}_n = \partial_{n+1}\partial_n\pi_{n-1} = 0$, since $\partial_{n+1}\partial_n = 0$. Hence $\operatorname{im} \partial_{n+1}\pi_n \subseteq \ker \overline{\partial}_n = \operatorname{im} \overline{\partial}_{n+1}$. Since P_{n+1} is projective, the map $\partial_{n+1}\pi_n \colon P_{n+1} \to \operatorname{im} \overline{\partial}_{n+1}$ factors via the epimorphism $\overline{\partial}_{n+1} \colon \overline{P}_{n+1} \twoheadrightarrow \operatorname{im} \overline{\partial}_{n+1}$. In other words, there exists a map $\pi_{n+1} \colon P_{n+1} \to \overline{P}_{n+1}$ such that $\pi_{n+1}\overline{\partial}_{n+1} = \partial_{n+1}\pi_n$.

Inductively this produces a morphism $\pi \colon \mathbf{P} \to \overline{\mathbf{P}}$ as required. Suppose that $\pi' \colon \mathbf{P} \to \overline{\mathbf{P}}$ is another such morphism. Recall that $\overline{\partial}_1$ is an epimorphism from $\overline{P}_1$ onto $\operatorname{im} \overline{\partial}_1 = \ker \overline{\varepsilon}$. We would like to show that $\operatorname{im}(\pi_0 - \pi_0') \subseteq \ker \overline{\varepsilon}$, so that by the projectivity of P_0, the map $\pi_0 - \pi_0'$ lifts along $\overline{\partial}_1$ to a map $\sigma_0 \colon P_0 \to \overline{P}_1$ such that $\sigma_0 \overline{\partial}_1 = \pi_0 - \pi_0'$. Notice that $(\pi_0 - \pi_0')\overline{\varepsilon} = \pi_0\overline{\varepsilon} - \pi_0'\overline{\varepsilon} = \varepsilon\alpha - \varepsilon\alpha = 0$, hence $\operatorname{im}(\pi_0 - \pi_0') \subseteq \ker \overline{\varepsilon}$ as required.

Let $n \geq 0$ and suppose that σ_i has been defined for $1 \leq i \leq n$, such that $\pi_i - \pi_i' = \sigma_i \overline{\partial}_{i+1} + \partial_i \sigma_{i-1}$. We want to construct $\sigma_{n+1} \colon P_{n+1} \to \overline{P}_{n+2}$ such that $\pi_{n+1} - \pi_{n+1}' = \sigma_{n+1}\overline{\partial}_{n+2} + \partial_{n+1}\sigma_n$. If we can show that $\operatorname{im}(\pi_{n+1} - \pi_{n+1}' - \partial_{n+1}\sigma_n) \subseteq \operatorname{im} \overline{\partial}_{n+2} = \ker \overline{\partial}_{n+1}$, then by the projectivity of P_{n+1} such a map σ_{n+1} indeed exists. Observe we have the following

$$(\pi_{n+1} - \pi_{n+1}' - \partial_{n+1}\sigma_n)\overline{\partial}_{n+1} = (\pi_{n+1} - \pi_{n+1}')\overline{\partial}_{n+1} - \partial_{n+1}\sigma_n\overline{\partial}_{n+1} = \partial_{n+1}(\pi_n - \pi_n' - \sigma_n\overline{\partial}_{n+1}) = \partial_{n+1}\partial_n\sigma_{n-1} = 0.$$

Hence $\operatorname{im}(\pi_{n+1} - \pi_{n+1}' - \partial_{n+1}\sigma_n) \subseteq \ker \overline{\partial}_{n+1}$, as required. Inductively we construct the maps $\sigma_n \colon P_n \to \overline{P}_{n+1}$ as required, and thus π and π are homotopic. This completes the proof. $\square$

3.9 Double complexes, and the Ext and Tor functors

We have seen over the course the last few sections/weeks the increasing complexity of complexes (pun intended). We first saw short exact sequences (SES), and moved on to long exact sequences (LES) before generalising those into chain complexes. We then studied maps between chain complexes and moved on to cover SESs of chain complexes. Now, we will follow the trajectory from before and define chain complexes

of chain complexes!
These are easy to define, but difficult to draw!

Definition. A *double complex* of R-modules C^{**} is a family $\{C^{i,j}\}_{i,j\in\mathbb{Z}}$ of modules together with morphisms

$$h^{i,j} : C^{i,j} \to C^{i,j+1} \quad \text{and} \quad v^{i,j} : C^{i,j} \to C^{i+1,j}$$

such that we still have cochain complexes in both directions: $h^{i,j} \circ h^{i,j-1} = 0$ and $v^{i,j} \circ v^{i-1,j} = 0$; and we have anti-commutativity in all squares: $v^{i,j+1} \circ h^{i,j} = -h^{i+1,j} \circ v^{i,j}$.

A double complex would look something like:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow {\scriptstyle v^{i-2,j-1}} & & \downarrow {\scriptstyle v^{i-2,j}} & & \downarrow {\scriptstyle v^{i-2,j+1}} & & \\
\cdots & \xrightarrow{h^{i-1,j-2}} & C^{i-1,j-1} & \xrightarrow{h^{i-1,j-1}} & C^{i-1,j} & \xrightarrow{h^{i-1,j}} & C^{i-1,j+1} & \xrightarrow{h^{i-1,j+1}} & \cdots \\
 & & \downarrow {\scriptstyle v^{i-1,j-1}} & & \downarrow {\scriptstyle v^{i-1,j}} & & \downarrow {\scriptstyle v^{i-1,j+1}} & & \\
\cdots & \xrightarrow{h^{i,j-2}} & C^{i,j-1} & \xrightarrow{h^{i,j-1}} & C^{i,j} & \xrightarrow{h^{i,j}} & C^{i,j+1} & \xrightarrow{h^{i,j+1}} & \cdots \\
 & & \downarrow {\scriptstyle v^{i,j-1}} & & \downarrow {\scriptstyle v^{i,j}} & & \downarrow {\scriptstyle v^{i,j+1}} & & \\
\cdots & \xrightarrow{h^{i+1,j-2}} & C^{i+1,j-1} & \xrightarrow{h^{i+1,j-1}} & C^{i+1,j} & \xrightarrow{h^{i+1,j}} & C^{i+1,j+1} & \xrightarrow{h^{i+1,j+1}} & \cdots \\
 & & \downarrow {\scriptstyle v^{i+1,j-1}} & & \downarrow {\scriptstyle v^{i+1,j}} & & \downarrow {\scriptstyle v^{i+1,j+1}} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

We say that a double complex C^{**} is *bounded* if for each n, there are only finitely many non-zero modules $C^{i,j}$ such that $n = i + j$.

We say that a double complex C^{**} is *positive* if $C^{i,j} = 0$ for all $i, j < 0$.

Remark. *The use of the anti-commutative property seems odd here, but this definition would become apparent after the definition of the total complex to come.*

We can turn a positive double complex C^{**} into a single one by defining

$$\mathrm{Tot}(C)^n = \bigoplus_{i+j=n} C^{i,j}\,.$$

Now consider the maps

$$\begin{aligned} D^n : \mathrm{Tot}(C)^n &\to \mathrm{Tot}(C)^{n+1} \\ C^{i,j} \ni x &\mapsto h^{i,j}(x) + v^{i,j}(x)\,. \end{aligned}$$

Then let's consider what happens to the map $D^{n+1} \circ D^n$. Let $x \in C^{i,j} \subset \mathrm{Tot}(C)^n$, then

$$\begin{aligned} D^{n+1} \circ D^n(x) &= D^{n+1}(\underset{\substack{\cap \\ C^{i,j+1}}}{h^{i,j}(x)} + \underset{\substack{\cap \\ C^{i+1,j}}}{v^{i,j}(x)}) \\ &= h^{i,j+1}(h^{i,j}(x)) + v^{i,j+1}(h^{i,j}(x)) + h^{i+1,j}(v^{i,j}(x)) + v^{i+1,j}(v^{i,j}(x)) \\ &= 0\,. \end{aligned}$$

Hence $\{\mathrm{Tot}(C)^n\}_{n\in\mathbb{N}}$ with the maps defined above forms a cochain complex.

For positive double complexes, we have the following picture:

$$\begin{array}{ccccccc} C^{0,0} & \xrightarrow{h^{0,0}} & C^{0,1} & \xrightarrow{h^{0,1}} & C^{0,2} & \xrightarrow{h^{0,2}} & \cdots \\ \downarrow{\scriptstyle v^{0,0}} & & \downarrow{\scriptstyle v^{0,1}} & & \downarrow{\scriptstyle v^{0,2}} & & \\ C^{1,0} & \xrightarrow{h^{1,0}} & C^{1,1} & \xrightarrow{h^{1,1}} & C^{1,2} & \xrightarrow{h^{1,2}} & \cdots \\ \downarrow{\scriptstyle v^{1,0}} & & \downarrow{\scriptstyle v^{1,1}} & & \downarrow{\scriptstyle v^{1,2}} & & \\ C^{2,0} & \xrightarrow{h^{2,0}} & C^{2,1} & \xrightarrow{h^{2,1}} & C^{2,2} & \xrightarrow{h^{2,2}} & \cdots \\ \downarrow{\scriptstyle v^{2,0}} & & \downarrow{\scriptstyle v^{2,1}} & & \downarrow{\scriptstyle v^{2,2}} & & \\ \vdots & & \vdots & & \vdots & & \end{array}$$

The diagram has omitted the $0 \to *$ maps for brevity. Note that we can extend/augment this diagram by including the kernels of the maps $h^{*,0}$ and $v^{0,*}$ on the left and the top respectively. Then setting

$$H^0_v(C^{0,*}) = \ker v^{0,*} \quad \text{and} \quad H^0_h(C^{*,0}) = \ker h^{*,0}$$

we have

$$\begin{array}{ccccccc} & & H^0_v(C^{0,0}) & \xrightarrow{d^0_h} & H^0_v(C^{0,1}) & \xrightarrow{d^1_h} & \cdots \\ & & \downarrow & & \downarrow & & \\ H^0_h(C^{0,0}) & \longrightarrow & C^{0,0} & \xrightarrow{h^{0,0}} & C^{0,1} & \xrightarrow{h^{0,1}} & \cdots \\ \downarrow{\scriptstyle d^0_v} & & \downarrow{\scriptstyle v^{0,0}} & & \downarrow{\scriptstyle v^{0,1}} & & \\ H^0_h(C^{1,0}) & \longrightarrow & C^{1,0} & \xrightarrow{h^{1,0}} & C^{1,1} & \xrightarrow{h^{1,1}} & \cdots \\ \downarrow{\scriptstyle d^1_v} & & \downarrow{\scriptstyle v^{1,0}} & & \downarrow{\scriptstyle v^{1,1}} & & \\ \vdots & & \vdots & & \vdots & & \end{array}$$

Now note that by inheriting the operations on C^{**}, we have that

$$H_v^0(C^{**}) = H_v^0(C^{0,*}) \quad \text{and} \quad H_h^0(C^{**}) = H_h^0(C^{*,0})$$

are cochain complexes.

Lemma 3.9.1 (Double Complex Lemma)**.** *If C^{**} is a positive double complex with exact rows and columns, then there are canonical isomorphisms for all $n \geq 0$:*

$$H^n(\mathrm{Tot}(C)^*) \cong H_v^n(H_h^0(C^{**})) \cong H_v^h(H_v^0(C^{**}))\,.$$

Proof. The notes have already constructed an isomorphism between $H_v^n(H_h^0(C^{**})) \cong H^n(\mathrm{Tot}(C)^*)$. We will construct the other isomorphism.

Consider the map

$$\begin{aligned} H_h^n(H_v^0(C^{**})) &\to H^n(\mathrm{Tot}(C)^*) \\ [a^{0,n}] &\mapsto [(a^{0,n}, 0, \dots, 0)]\,. \end{aligned}$$

We follow the proof strategy in the notes and aim to show that an arbitrary element in $H^n(\mathrm{Tot}(C)^*)$ can be written in the above form, then the inverse map is obvious.

So let $x = [(a^{0,n}, a^{1,n-1}, \dots, a^{k,n-k}, 0, \dots, 0)]$ and let $\pi : \mathrm{Tot}(C)^{n+1} \to C^{k+1,n-k}$. Then notice that $D(x) = 0$ since D is now restricted to the cohomology groups, and so we have

$$0 = \pi \circ D(x) = h^{k,n+k}(0) + v^{k,n+k}(a^{k,n-k})\,.$$

Then using the exactness of the $(n-k)$-th column, there exists $e \in C_{k-1,n-k}$ such that $v^{k-1,n-k+1}(e) = a^{k,n-k}$. Hence

$$\begin{aligned} &[(a^{0,n}, a^{1,n-1}, \dots, a^{k,n-k}, 0, \dots, 0)] - [(a^{0,n}, a^{1,n-1}, \dots, a^{k-1,n-k+1} - h^{k-1,n-k+1}(e), 0, \dots, 0)] \\ =&[(0, \dots, 0, h^{k-1,n-k+1}(e), a^{k,n-k}, 0, \dots, 0)] \\ =&[(0, \dots, 0, h^{k-1,n-k+1}(e), v^{k-1,n-k+1}(e), 0, \dots, 0)] \\ =&D[(0, \dots, 0, e, 0, \dots, 0)]\,. \end{aligned}$$

□

An example of a double complex is by constructing them via resolutions:
Let A, B be R-modules and consider the projective $P_* \xrightarrow{\epsilon} A$ and injective $B \xrightarrow{\eta} I^*$ resolutions of A and B respectively. Then consider the positive double exact complex obtained by using the left-exactness of the covariant functor $\hom(M, -)$ and its dual, we have

$$
\begin{array}{ccccccc}
 & & \hom_R(A, I^0) & \xrightarrow{d_h^0} & \hom_R(A, I^1) & \xrightarrow{d_h^1} & \cdots \\
 & & \Big\downarrow{\scriptstyle \epsilon^*} & & \Big\downarrow{\scriptstyle \epsilon^*} & & \\
\hom_R(P_0, B) & \xrightarrow{\eta_*} & \hom_R(P_0, I^0) & \xrightarrow{h^{0,0}} & \hom_R(P_0, I^1) & \xrightarrow{h^{0,1}} & \cdots \\
\Big\downarrow{\scriptstyle d_v^0} & & \Big\downarrow{\scriptstyle v^{0,0}} & & \Big\downarrow{\scriptstyle v^{0,1}} & & \\
\hom_R(P_1, B) & \xrightarrow{\eta_*} & \hom_R(P_1, I^0) & \xrightarrow{h^{1,0}} & \hom_R(P_1, I^1) & \xrightarrow{h^{1,1}} & \cdots \\
\Big\downarrow{\scriptstyle d_v^1} & & \Big\downarrow{\scriptstyle v^{1,0}} & & \Big\downarrow{\scriptstyle v^{1,1}} & & \\
\vdots & & \vdots & & \vdots & &
\end{array}
$$

So using the Double Complex Lemma, we have

$$H^n(\hom(A, I^*)) \cong H^n(\hom(P_*, B)) \,.$$

Definition. The cohomology group we obtained above is defined to be the *Ext groups*, and are denoted $\mathrm{Ext}^n(A, B)$.

Analogously, we can take two projective resolutions $P_* \xrightarrow{\epsilon} A$ and $Q_* \xrightarrow{\delta} B$ and define the *Tor groups*, $\mathrm{Tor}^n(A, B)$ as the homology groups

$$\mathrm{Tor}^n(A, B) \cong H^n(A \otimes Q_*)) \cong H^n(P_* \otimes B) \,.$$

Now, the natural thing to do here is to take two injective resolutions and look to build more double complexes. It is clear that we will not be able to use the hom-functor twice as this would cause the complexes to point the wrong way. One can see that this is true for using a mix of the two functors as in the construction of Ext and Tor. This leaves us with two applications of the tensor functor. But one will see that this construction does not work because of the lack of left-exactness of the tensor functor.

The last result in the notes show that the Tor-functor measures the obstruction of the left-exactness of the tensor-functor. However, I will not pull on that thread and will instead motivate Ext and Tor from a different direction: derived functors.

Derived functors

Definition. Let F be a right-exact (resp. left-exact) covariant map and M be any module. We take a projective resolution P_* (resp. injective resolution I^*) of M and define $H_i(F(P_*))$ (resp. $H^i(F(I^*))$) to be the *i-th left derived functor* (resp. *right derived functor*) of M, written as $F_i(M)$ (resp. $F^i(M)$).

Remark. *If F is contravariant, swap the words* left *and* right *in the definitions above.*

Lemma 3.9.2. *The $F_i(M)$ depends only on M and F (and not on the projective resolution). Furthermore, $F_0(M) = F(M)$.*

Proof. Suppose we have two projective resolutions on M given by $P_* \to M \to 0$ and $Q_* \to M \to 0$. Then using results from the Section 3.8, we can extend $1 : M \to M$ to $\phi : P_* \to Q_*$, and also $\psi : Q_* \to P_*$. And applying Theorem 3.8.2 twice, we have that $\psi \circ \phi$ and $\phi \circ \psi$ are homotopic to the identity, hence Theorem 3.7.1 gives us the isomorphisms between the homology groups.

The fact that F is right-exact shows that the last four terms

$$F(P_1) \to F(P_0) \to F(M) \to 0$$

is exact, hence $F(M) = F_0(M)$. ☐

Lemma 3.9.3. *If $\alpha : M \to N$ is R-linear and F is as above, then we get $F_i(\alpha) : F_i(M) \to F_i(N)$.*

Proof. Use Proposition 1.10.6 and Lemma 1.10.12 of Moerdijk's notes to get the result. ☐

From these results, it is clear that if $M \xrightarrow{\alpha} N \xrightarrow{\beta} P$ is a sequence, then $F_i(\beta\alpha) = F_i(\beta)F_i(\alpha)$. So we have a functor between R-modules that takes M to $F_i(M)$.

Theorem 3.9.4. *If there is a SES $0 \to L \to M \to N \to 0$ of modules, then we get a long exact sequence:*

$$\cdots \to F_2(N) \to F_1(L) \to F_1(M) \to F_1(N) \to F_0(L) \to F_0(M) \to F_0(N) \to 0\,.$$

Definition. Let M be an R-module, then $\hom_R(M, -)$ is a left-exact covariant functor. The n-th right-derived functor for this is written as $\mathrm{Ext}^n(M, -)$. Explicitly, for a module N, take any injective resolution $0 \to N \to I^*$, and applying the functor, we get the complex

$$0 \to \hom(M, N) \to \hom(M, I^0) \to \hom(M, I^1) \to \dots$$

and $\mathrm{Ext}^n(M, N)$ is the n-th cohomology group of this complex.

Similarly, $M \otimes_R -$ is a right-exact covariant functor, and its n-th left-derived functor is written as $\mathrm{Tor}^n(M, -)$. Explicitly, for a module N, take any projective resolution $P_* \to N \to 0$, and we have a complex

$$\cdots \to M \otimes_R P_1 \to M \otimes_R P_0 \to M \otimes_R N \to 0$$

and $\mathrm{Tor}^n(M, N)$ is the n-th homology group of this sequence.

Computing with Ext and Tor

In this section, we will try to compute $M \otimes N$, $\hom(M, N)$, $\mathrm{Tor}(M, N)$ and $\mathrm{Ext}(M, N)$ for $\mathbb{Z}$-modules $M, N = \mathbb{Z}, \mathbb{Q}, \mathbb{Z}/n\mathbb{Z}$.

Before doing so, we will state some results that I may or may not prove:

Proposition 3.9.5. *The following are equivalent for a module M:*

(i) *M is projective.*

(ii) $\mathrm{Ext}^n(M, N) = 0$ *for all $i > 0$ and modules N.*

(iii) $\mathrm{Ext}^1(M, N) = 0$ *for all modules N.*

Proposition 3.9.6. *The following are equivalent for a module N:*

(i) *N is injective.*

(ii) $\text{Ext}^n(M,N) = 0$ *for all* $i > 0$ *and modules* N*.*

(iii) $\text{Ext}^1(M,N) = 0$ *for all modules* N*.*

Proposition 3.9.7. *The following are equivalent for a left-module* N*:*

(i) N *is flat.*

(ii) $\text{Tor}^n(M,N) = 0$ *for all* $i > 0$ *and modules* M*.*

(iii) $\text{Tor}^1(M,N) = 0$ *for all modules* M*.*

Tensor and Tor

Firstly, note that $0 \otimes N = 0$ since $0 \otimes n = 0 \cdot 0 \otimes n = 0 \otimes 0 \cdot n = 0 \otimes 0 = 0$.
Next, check that we have

(1) $\mathbb{Z} \otimes N = N$ is given by $a \otimes n \mapsto an$,

(2) $\mathbb{Q} \otimes \mathbb{Q} = \mathbb{Q}$ is given by $a \otimes b \mapsto ab$.

Proposition 3.9.8. *(3)* $\mathbb{Z}/n\mathbb{Z} \otimes N = N/nN$*,*

(4) $\text{Tor}(\mathbb{Z}/n\mathbb{Z}, N) = N[n]$*.*

Proof. Consider the SES

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

and apply the right-exact covariant functor $- \otimes N$ to get

$$\cdots \to \text{Tor}(\mathbb{Z}/n\mathbb{Z}, N) \to \mathbb{Z} \otimes N \xrightarrow{n} \mathbb{Z} \otimes N \to \mathbb{Z}/n\mathbb{Z} \otimes N \to 0\,.$$

Using what we got from (1) and noting that $\text{Tor}(\mathbb{Z}, N) = 0$ since $\mathbb{Z}$ is projective and hence flat, we can simplify it to get

$$0 \to \text{Tor}(\mathbb{Z}/n\mathbb{Z}, N) \to N \xrightarrow{n} N \to \mathbb{Z}/n\mathbb{Z} \otimes N \to 0\,.$$

Then

$$\begin{aligned} \text{Tor}(\mathbb{Z}/n\mathbb{Z}, N) &= \ker(\mathbb{Z} \xrightarrow{n} \mathbb{Z}) = N[n] \\ \mathbb{Z}/n\mathbb{Z} \otimes N &= \text{coker}(\mathbb{Z} \xrightarrow{n} \mathbb{Z}) = N/nN\,. \end{aligned}$$

□

Now notice that we can use this result to get

(5) $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Q} = 0$.

(6) $\text{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}) = 0$.

Finally, we have from the first section of this chapter that

(7) $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/d\mathbb{Z}$, where $d = \gcd(m,n)$.

(8) Then using (4), we also have that $\text{Tor}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$, where $d = \gcd(m,n)$.

We have (9) $\text{Tor}(M, \mathbb{Q}) = 0$ since $\mathbb{Q}$ is flat.

Hom and Ext

We have from basis results that

(1) $\hom(\mathbb{Z}, N) = N$ is given by $\phi \mapsto \phi(1) = n$,

(2) $\operatorname{Ext}(\mathbb{Z}, N) = 0$ since $\mathbb{Z}$ is projective.

Next, we have

Proposition 3.9.9. *(3)* $\hom(\mathbb{Z}/n\mathbb{Z}, N) = N[n]$,

(4) $\operatorname{Ext}(\mathbb{Z}/n\mathbb{Z}, N) = N/nN$.

The proof is almost identical to the proof above and we will not repeat it.

Using this result, we get:

(5) $\hom(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$.

(6) $\operatorname{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$.

(7) $\hom(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$, where $d = \gcd(m, n)$.

(8) $\operatorname{Ext}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$, where $d = \gcd(m, n)$.

(9) $\hom(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}) = 0$.

(10) $\operatorname{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}) = 0$.

Next, we have $\hom(\mathbb{Q}, \mathbb{Q}) = \mathbb{Q}$ via $\phi \mapsto \phi(1)$ and $\operatorname{Ext}(M, \mathbb{Q}) = 0$ since $\mathbb{Q}$ is injective.

Lastly, we come to $\operatorname{Ext}(\mathbb{Q}, \mathbb{Z})$... this is where we stop!

3.10 The Künneth Formula

We are now reaching a stage where computing actual objects becomes a reality. In this section we will give more tools that are handy for calculating (co)homology groups involving tensor products.

Informally, let C be a chain complex of free modules over a ring R and let M be an R-module. What we will derive are the following short exact sequences:

$$0 \to H_n(C) \otimes M \to H_n(C \otimes M) \to \operatorname{Tor}_1(H_{n-1}(C), M) \to 0 \tag{3.1}$$

and more generally, for R a principal ideal domain and a second chain complex C' again of free R-modules

$$0 \to \bigoplus_{p+q=n} H_p(C) \otimes H_q(C') \to H_n(C \otimes C') \to \bigoplus_{p+q=n-1} \operatorname{Tor}_1(H_p(C), H_q(C')) \to 0. \tag{3.2}$$

We have not yet defined what it means to take a tensor product of chain complexes, but we see that taking $C'_0 = M$ and $C'_n = 0$ for $n \neq 0$ in Equation 3.2 immediately yields Equation 3.1. So instead of deriving the first equation directly, we will prove a theorem about the more general second one first.

Tensor Product of Complexes Consider the double complex $C_{p,q} = C_p \otimes C'_q$ of chain complexes C and C' of left and right R-modules with the boundary maps

$$d \otimes \mathrm{id} : C_{p,q} \to C_{p-1,q}, \qquad (-1)^p \,\mathrm{id} \otimes d : C_{p,q} \to C_{p,q-1}.$$

The boundary map of its total complex is given by

$$D(c \otimes c') = dc \otimes c' + (-1)^p c \otimes dc' \tag{3.3}$$

for all $c \in C_p$ and $c' \in C'_q$.

Definition. The total complex of the double complex $C_{p,q} = C_p \otimes C'_q$ is called the *tensor product* of C and C' and is denoted by $C \otimes_R C'$.

Recall, that for the tensor product of modules, a certain associativity holds; let A be a right R-module, B an (R, S)-bimodule, and C a left R-module. Then there is an isomorphism

$$(A \otimes_R B) \otimes_S C \to A \otimes_R (B \otimes_S C),$$
$$(a \otimes b) \otimes c \mapsto a \otimes (b \otimes c).$$

Further, let A and B be R-bimodules, then there is an isomorphism

$$A \otimes B \to B \otimes A,$$
$$a \otimes b \mapsto b \otimes a.$$

This generalises neatly to complexes as follows.

Corollary 3.10.1. *Let C' and C'' be chain complexes of right R-modules respectively left S-modules and let C be a chain complex of (R, S)-bimodules. Then there is an isomorphism of chain complexes*

$$(C' \otimes_R C) \otimes_S C'' \to C' \otimes_R (C \otimes_S C'').$$

Further, let C and C' be chain complexes of R-bimodules, then there is an isomorphism

$$C \otimes C' \to C' \otimes C,$$
$$c \otimes c' \mapsto (-1)^{pq} c' \otimes c,$$

for all $c \in C_p$ and $c' \in C'_q$.

Let us now come back to the tensor product of chain complexes. Let C and C' be two chain complexes of left respectively right R-modules. From Equation 3.3 we see that the tensor product of two cycles is again a cycle and that the tensor product of a cycle and a boundary is a boundary. Thus we can make the following

Definition. Let C and C' be two chain complexes of left respectively right R-modules. The homomorphism

$$\bigoplus_{p+q=n} H_p(C) \otimes H_q(C') \to H_n(C \otimes C') : [c] \otimes [c'] \mapsto [c \otimes c']$$

is called the *homology product.*

Before we proceed, we require another definition.

Definition. Let $(C_i)_{i \in I}$ be a family of chain complexes of R-modules. The *direct sum* $\bigoplus_{i \in I} C_i$ is given by

$$\left(\bigoplus_{i \in I} C_i \right)_n = \bigoplus_{i \in I} (C_i)_n, \qquad d(c_i) = (dc_i).$$

The direct sum $\bigoplus_{i\in I} C_i$ is a chain complex of R-modules. There exists a canonical isomorphism

$$H_n(\bigoplus_{i\in I} C_i) \simeq \bigoplus_{i\in I} H_n(C_i).$$

If all of the C_i are chain complexes of left R-modules and we have an additional C' a chain complex of left R-modules, then there exists a canonical isomorphism

$$(\bigoplus_{i\in I} C_i) \otimes C' \simeq \bigoplus_{i\in I}(C_i \otimes C').$$

The Künneth Formula We are now ready to state and proof the Künneth formula given by Equation 3.2. Intuitively, the torsion term $\bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(C), H_q(C'))$ gives us a measure of how much the homology product fails to be an isomorphism.

Remark. *If a ring R is a principal ideal domain, then every submodule of a free R-module is free as well.*

Theorem 3.10.2 (Künneth Formula)**.** *Let R be a principal ideal domain and let C and C' be chain complexes of right and left R-modules respectively. If the R-modules C_i are all free then, for each n, there exists a natural short exact* Künneth sequence

$$0 \to \bigoplus_{p+q=n} H_p(C) \otimes H_q(C') \to H_n(C \otimes C') \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(C), H_q(C')) \to 0.$$

Proof. This is a proof in two parts.

(i) We first proof the statement in a special case. Assume that the chain complex C has trivial boundary morphisms (i.e. $d_n \equiv 0$) such that $H_p(C) = C_p$ is free for all p. Then the torsion term vanishes and it suffices to show that the homology product is an isomorphism.

Now $D(c \otimes c') = (-1)^p c \otimes dc'$ for all $c \in C_p$ and $c' \in C'_q$. We also have a canonical isomorphism

$$C \otimes C' \simeq \bigoplus_{p\in\mathbb{Z}} C_p \otimes C'[p]$$

where the $C'[p]$ is given by $C'[p]_n := C'_{n-p}$. By our assumption, C_p is free and so it can be written as a disjoint sum of of a family of R-modules $(R_i)_{i\in I}$, and all of the R_i are isomorphic to R. Thus, using $R_i \otimes C'_q \simeq C'_q$ and $H_n(C'[p]) = H_{n-p}(C')$, we find

$$H_n(C_p \otimes C'[p]) \simeq \bigoplus_i H_{n-p}(C') \simeq C_p \otimes H_{n-p}(C') = H_p(C) \otimes H_{n-p}(C').$$

Now applying the homology functor $H_n(-)$ to both sides of Equation 1 and summing over p gives us the required isomorphism

$$H_n(C \otimes C') \simeq \bigoplus_{p+q=n} H_p(C) \otimes H_q(C').$$

(ii) Now we show the general case. Recall that $H_n(C) = Z_n/B_n$ where we denote with $Z_n \subset C_n$ the kernel of d_n and with $B_n \subset C_n$ the image of d_{n+1}. Notice that they form chain complexes Z and B with trivial boundary morphisms and that we have a short exact sequence

$$0 \to Z_p \to C_p \xrightarrow{d} B_{p-1} \to 0$$

for each p. Since B_{p-1} is free, we have $\mathrm{Tor}_1(B_{p-1}, C'_q) = 0$ in the associated Tor-sequence. Therefore,

$$0 \to Z_p \otimes C'_q \to C_p \otimes C'_q \to B_{p-1} \otimes C'_q \to 0$$

is exact for all pairs (p, q). By taking the chain complex tensor product (i.e. summing over $p+q=n$) we find

$$0 \to (Z \otimes C')_n \to (C \otimes C')_n \to (B \otimes C')_{n-1} \to 0$$

By case (i) we have

$$Z \otimes C' \simeq \bigoplus_{p \in \mathbb{Z}} Z_p \otimes C'[p],$$
$$B \otimes C' \simeq \bigoplus_{p \in \mathbb{Z}} B_p \otimes C'[p]$$

and thus we obtain the exact sequence

$$0 \to \bigoplus_{p \in \mathbb{Z}} Z_p \otimes C'[p] \to C \otimes C' \to \bigoplus_{p \in \mathbb{Z}} B_{p-1} \otimes C'[p] \to 0.$$

Taking homologies, we find the long exact sequence

$$\cdots \to \bigoplus_{p+q=n} B_p \otimes H_q(C') \xrightarrow{\delta_n} \bigoplus_{p+q=n} Z_p \otimes H_q(C') \to H_n(C \otimes C') \tag{3.4}$$
$$\to \bigoplus_{p+q=n-1} B_p \otimes H_q(C') \xrightarrow{\delta_{n-1}} \bigoplus_{p+q=n-1} Z_p \otimes H_q(C') \to \cdots \tag{3.5}$$

The exactness in $H_n(C \otimes C')$ is equivalent to the exactness of the following short exact sequence

$$0 \to \operatorname{coker} \delta_n \to H_n(C \otimes C') \to \ker \delta_{n-1} \to 0.$$

We will determine $\operatorname{coker} \delta_n$ and $\ker \delta_{n-1}$. Start with the exact sequence $0 \to B_p \to Z_p \to H_p(C) \to 0$ to find the exact sequence

$$B_p \otimes H_q(C') \to Z_p \otimes H_q(C') \to H_p(C) \otimes H_q(C') \to 0.$$

Again, summing over $p+q=n$ we obtain the exact sequence

$$\bigoplus_{p+q=n} B_p \otimes H_q(C') \to \bigoplus_{p+q=n} Z_p \otimes H_q(C') \to \bigoplus_{p+q=n} H_p(C) \otimes H_q(C') \to 0$$

and hence by matching with Equation 3.4 we find that $\operatorname{coker} \delta_n = \bigoplus_{p+q=n} H_p(C) \otimes H_q(C')$.

On the other hand, start again with the exact sequence $0 \to B_p \to Z_p \to H_p(C) \to 0$ and consider the associated Tor-sequence. By our assumption, Z_p is free and so $\operatorname{Tor}_1(Z_p, H_q(C')) = 0$. Hence we obtain the exact sequence

$$0 \to \operatorname{Tor}_1(H_p(C), H_q(C')) \to B_p \otimes H_q(C') \to Z_p \otimes H_q(C') \to H_p(C) \otimes H_q(C') \to 0.$$

Similarly, by summing over $p+q=n-1$ and matching with Equation 3.4 we find that $\ker \delta_{n-1} = \bigoplus_{p+q=n-1} \operatorname{Tor}_1(H_p(C), H_q(C'))$.

□

The Universal Coefficient Theorem We are now ready to state and proof our second result about Equation 3.1. But before we do so, we quickly define a notation often found in literature.

Definition. Let C be a chain complex of left R-modules. If M is any right R-module, the homology of C with coefficients in M is given by

$$H_*(C; M) = H_*(C \otimes M).$$

Theorem 3.10.3 (Universal Coefficient Theorem)**.** *Let C be a chain complex of free left R-modules and let M be a right R-modules. Then there is a natural short exact sequence*

$$0 \to H_n(C) \otimes M \to H_n(C; M) \to \mathrm{Tor}_1(H_{n-1}(C), M) \to 0.$$

Proof. This now follows immediately from Theorem 3.10.2. □

Remark. *Theorem 3.10.2 and Theorem 3.10.3 have some interesting implications. If K is a field and C and C' are chain complexes of K-modules, then the homology product is an isomorphism. It also holds that $H_n(C \otimes M) \simeq H_n(C) \otimes M$ for some K-module M.*

Remark. *The Künneth-sequence splits:*

$$H_n(C \otimes C') \simeq \left(\bigoplus_{p+q=n} H_p(C) \otimes H_q(C') \right) \oplus \left(\bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(C), H_q(C')) \right),$$

but this splitting is not natural.

Example. Anticipating later results, let us state the homology groups for the real projective plane $\mathbb{R}\mathrm{P}^2$:

$$\begin{aligned} &H_0(\mathbb{R}\mathrm{P}^2, \mathbb{Z}) \simeq \mathbb{Z}, \\ &H_1(\mathbb{R}\mathrm{P}^2, \mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}, \\ &H_i(\mathbb{R}\mathrm{P}^2, \mathbb{Z}) = 0, \; i \geq 2. \end{aligned}$$

All Tor groups are zero but

$$\mathrm{Tor}_1(H_1(\mathbb{R}\mathrm{P}^2, \mathbb{Z}), H_1(\mathbb{R}\mathrm{P}^2, \mathbb{Z})) \simeq \mathbb{Z}/2\mathbb{Z}.$$

Using this and the Künneth formula, we can compute the homology groups of $\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2$:

$$\begin{aligned} &H_0(\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2, \mathbb{Z}) \simeq H_0(\mathbb{R}\mathrm{P}^2, \mathbb{Z}) \otimes H_0(\mathbb{R}\mathrm{P}^2, \mathbb{Z}) \simeq \mathbb{Z}, \\ &H_1(\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2, \mathbb{Z}) \simeq H_0(\mathbb{R}\mathrm{P}^2, \mathbb{Z}) \otimes H_1(\mathbb{R}\mathrm{P}^2, \mathbb{Z}) \oplus H_1(\mathbb{R}\mathrm{P}^2, \mathbb{Z}) \otimes H_0(\mathbb{R}\mathrm{P}^2, \mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \\ &H_2(\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2, \mathbb{Z}) \simeq H_1(\mathbb{R}\mathrm{P}^2, \mathbb{Z}) \otimes H_1(\mathbb{R}\mathrm{P}^2, \mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}, \\ &H_3(\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2, \mathbb{Z}) \simeq \mathrm{Tor}_1(H_1(\mathbb{R}\mathrm{P}^2, \mathbb{Z}), H_1(\mathbb{R}\mathrm{P}^2, \mathbb{Z})) \simeq \mathbb{Z}/2\mathbb{Z}, \\ &H_i(\mathbb{R}\mathrm{P}^2 \times \mathbb{R}\mathrm{P}^2, \mathbb{Z}) = 0, \; i \geq 4. \end{aligned}$$

3.11 Group cohomology

3.12 Simplicial cohomology

Chapter 4

Abelian Categories and Derived Functors

4.1 Categories and functors

4.2 Abelian categories

Recall the definition of a category:

Definition (Category)**.** A *category* C consists of a class of *objects* $\mathrm{ob}(C)$ and a class of *morphisms* $\hom(C)$. Each morphism $f \in \hom(C)$ has a source object a and target object b. $\hom_C(a, b)$ denotes the class of all morphisms from a to b in C. (We drop the subscript C when the category is clear.) There is a binary operation $\hom(a, b) \times \hom(b, c) \to \hom(a, c)$ called *composition*, where $(f, g) \mapsto fg$. Moreover, the following properties must hold:

- (Associativity.) If $f \in \hom(a, b)$, $g \in \hom(b, c)$, and $h \in \hom(c, d)$, then $f(gh) = (fg)h$.
- (Identity.) For each $a \in \mathrm{ob}(C)$, there is a morphism id_a called the *identity*, so that $\mathrm{id}_a f = f$ and $g\,\mathrm{id}_a = g$ for any $f \in \hom(a, b)$ and $g \in \hom(c, a)$, for any $b, c \in \mathrm{ob}(C)$.

Example. The category **Set** consists of sets as its objects and functions as its morphisms. The definition is vague by design: Some consider it to be the category of *all* sets, some consider it to be the category of *small* sets, which are sets contained in a Grothendieck universe. On the other hand, **Ab** is the category consisting of all abelian groups as objects and group homomorphisms as morphisms.

Example. Up until now we have been working in the category of left modules, denoted **R-mod**. We see that our definition of $\hom_R(M, N)$ is in fact $\hom_{\textbf{R-mod}}(M, N)$ equipped with addition of morphisms and left multiplication by elements of R. While $\hom_{\textbf{R-mod}}(M, N)$ is just a set, $\hom_R(M, N)$ is an R-module.

There are obvious connections between **Ab** and **R-mod**: every abelian group can be made into a $\mathbb{Z}$-module and every R-module is, at its core, an abelian group. Both of these are prototypical examples of an **abelian category**, a special type of categories where the morphisms are enriched with additional structure.

Example. Consider **Ab** the category of Abelian groups with group homomorphisms. Take two abelian groups A and B, and consider $f, g \in \hom(A, B)$. We define a new map $f + g : A \to B$ by $(f + g)(a) = f(a) + g(a)$, as if $a, b \in A$, then $(f + g)(a + b) = f(a + b) + g(a + b) = f(a) + f(b) + g(a) + f(g)$. If A were not abelian, then this new map is not necessarily a Homomorphism, but here we luckily get: $(f + g)(a + b) = (f + g)(a) + (f + g)(b)$. We also get identity via the trivial Homomorphism, associativity inverses and commutativity via the associativity of the group. Therefore $\hom(A, B)$ can be "enriched" with the structure of an Abelian group.

This structure is exactly what we want to capture with of abelian categories. It will turn out that abelian categories are the most general (in a sense) place where we can do Homological Algebra. We begin by introducing new category theory definitions.

Definition. A **preadditive category** if every $\hom(A, B)$ has the structure of an abelian group (as outlined above), where the composition of morphisms is bilinear: if $\phi \in \hom(A, B)$, $f, g \in \hom(B, C)$ and $\psi \in \hom(C, D)$, then both:

$$(f + g) \circ \phi = f \circ \phi + g \circ \phi; \text{ and}$$
$$\psi \circ (f + g) = \phi \circ f + \psi \circ g.$$

Now we need additional structure:

Definition. An **initial object** in a category C is an object I such that for every object A, there exists a unique morphism $\phi \in \hom(I, A)$. Similarly, a **terminal object** is is an object T such that for every object A, there exists a unique morphism $\phi \in \hom(A, T)$. A **zero object** is any object that is both an initial and a zero object.

Example. In **Gp** and **Ab** the trivial group is a zero object. **Set** has $\varnothing$ as an initial object, and singletons $\{x\}$ as terminal objects, so there is no zero object in **Set**

Definition. A **zero morphism** for objects A, B and zero object Z is a morphism $0_{A,B,Z} \in \hom(A, B)$ such that $0_{A,B,Z} = \phi \circ \psi$ where $\psi \in \hom(A, Z)$ and $\phi \in \hom(Z, B)$. Since ψ and ϕ are unique, the zero morphism through Z is unique too.

Example. The trivial homomorphism is a zero morphism in **Ab** through the zero object $\{0\}$.

Proposition 4.2.1. *A zero morphism between A and B is independent of the choice of zero object.*

Proof. Take two zero objects Z, Z' in C. Define ϕ, ψ as above and ϕ', ψ' as the unique morphisms corresponding instead to Z'. Also take φ as the unique morphism in $\hom(Z, Z')$. Then by the uniqueness of these homomorphisms, we have that $\psi' = \varphi \circ \psi$ and $\phi = \phi' \circ \varphi$. Thus by the associativity:

$$0_{A,B,Z} = \phi \circ \psi = (\phi' \circ \varphi) \circ \psi = \phi' \circ (\varphi \circ \psi) = \phi' \circ \psi' = 0_{A,B,Z'}$$

□

Proposition 4.2.2. *Composition with a zero morphism is another zero morphism.*

Proof. Exercise. □

Definition. If A, B are objects, a **product** of A and B, denoted $A \times B$, is an object paired with morphism $\pi_1 \in \hom(A \times B, A)$ and $\phi_2 \in \hom(A \times B, B)$ such that if C is another object,$f_1 \in \hom(C, A)$ and $f_2 \in \hom(C, B)$ then there exists a unique $f \in \hom(C, A \times B)$ such that $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$.

A **coproduct** of A and B, denoted $A \coprod B$ (or sometimes in abelian categories, $A \oplus B$) is the dual of a product: i.e., the same definition as above with the arrows reversed.

Example. In **Gp** , the direct product along with the component wise projections forms the product. Similarly

Definition. An **additive category** is a preadditive category that also:

(i) has at least one zero object; and

(ii) every finite set of objects $\{X_i\}$ has both a product and coproduct.

Definition. a **kernel** of $f \in \hom(A, B)$ is a pair of an object K and a morphsim $k \in \hom(K, A)$ such that $f \circ k$ is a zero morphism. A **cokernel** of $f \in \hom(A, B)$ is a pair of an object C and a morphism $c \in \hom(B, C)$ such that $c \circ f$ is the zero morphism.

Example. Every morphism $\phi \in \hom(A, B)$ in **Gp** has a kernel, namely $\ker \phi \subseteq A$. The morphism is then just inclusion $\iota : \ker \phi \to A$. We also know that $\text{im}\phi \leq B$. Let N be the smallest normal subgroup of B that contains $\text{im}\phi$. Then the cokernel of f is B/N with morphism $x \mapsto xN$. In **Ab** this is just $B/\text{im}\phi$ as every subgroup is normal.

Definition. A **pre-abelian** category is an additive category where every morphism has both a kernel and cokernel.

Definition. A monomorphism is **normal** if it is the kernel of some morphism. An epimorphism is **normal** if it is the cokernel of some morphism.

Definition. An **abelian category** is a preabelian category where every monomorphism and epimorphism are normal.

Example. **Ab**, **R-mod**. and **Vect** are all abelian categories.

We now wish to characterise abelian categories.

Definition. An **additive functor** $F : A \to B$ is a functor for which the induced maps $\hom_A(a, a') \to \hom_B(Fa, Fa')$ are homomorphisms of abelian groups. F is called a **full embedding** if F is injective on objects and the induced homomorphisms (on the hom groups) are bijections for every pair of objects.

Theorem 4.2.3. *Let $\mathcal{A}$ be a small abelian category. Then there exist a ring R and a full exact embedding $F : \mathcal{A} \to$ **R-mod**.*

We now establish homological algebra in our abelian categories.

Definition. Define **cocomplexes** as in **R-mod** but where each C_i is an object rather than a module specifically, and boundary maps $d_n \in \hom(C_n, C_{n+1})$. Define $\iota_n : \ker d_n \to C_n$ as the natural embedding. By the universal property there exists a unique a_{n-1} such that $\iota_n \circ a_{n-1} = d_{n-1}$. Define **cohomologies** as $H^n(C) = \text{coker}(a_{n-1})$.

Example. See that this definition agrees with the prior definition in **R-mod**.

Proposition 4.2.4. *If $\mathcal{C}$ is an abelian category, then define $\mathbf{C}(\mathcal{C})$ as the category of complexes in $\mathcal{C}$. $\mathbf{C}(\mathcal{C})$ is an abelian category.*

Now see that any chain map $f : X \to Y$ in $\mathbf{C}(\mathcal{C})$ induces a map $H^n(f) : H^n(X) \to H^n(Y)$. (e.g., in **R-mod**, the map is given by $x + \text{Im } d^{n-1} \mapsto f(x) + \text{Im } \delta^{n-1}$). Thus H^n is a functor from $\mathbf{C}(\mathcal{C})$ to $\mathcal{C}$.

4.3 Derived functors

4.4 Sheaf cohomology

4.5 Derived categories

Chapter 5

Spectral Sequences

5.1 Motivation

5.2 Serre spectral sequence

5.3 Grothendieck spectral sequence