TMA4190 Differential Topology
Lecture Notes
Spring 2018

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These lecture notes are based on the book by Guillemin and Pollack [1] and do not aim to do more than to explain and adjust some of the arguments to the audience.

Please send any comments and suggestions for corrections to gereon.quick@ntnu.no.

Contents

LECTURE 1

Introduction

0.1. Organization. First some general info:

Lectures: Mondays 12.15-14.00 in R92,
and Thursdays 10.15-12.00 in R54.

Exercises: Thursdays 16.15-17.00 in R21, but NOT WEEKLY. We will discuss exercises futher in class.

Important: You will have to solve the exercises yourself. The exercise classes will NOT consist of me giving solutions. If nobody comes up with suggestions, there will be nothing going on. You need to work in order to learn...

General advice: Talk to each other and to me. Ask questions! Interact!!! Solve exercises!!!!!
That's how you learn. Do not sit quiet and just read.

Course webpage (on which I will try to put more information soon):

wiki.math.ntnu.no/tma4190/2018v/start

Office hours: Upon request.

Just send me an email: gereon.quick@ntnu.no

Text books: In the beginning we will follow the book

[GP] V. Guillemin and A. Pollack, Differential Topology.

Another excellent and very short book:

[M] J.W. Milnor, Topology from the Differentiable Viewpoint.

Some other useful books:

[D] B. Dundas, Differential Topology.

[T] L.W. Tu, An Introduction to Manifolds.

There are many other good books out there. Ask me if you need more.

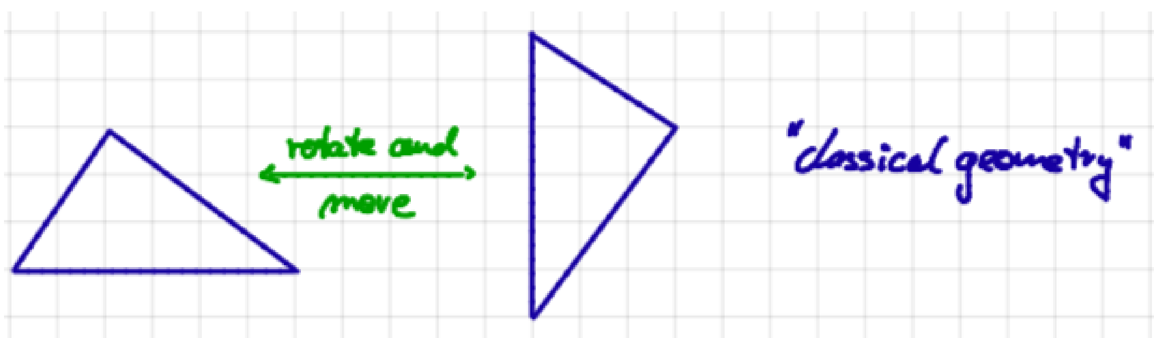
0.2. What is required? We will just assume some knowledge in multivariable calculus, corresponding to **Calculus 1 and 2**. For example, you should know what it means for **a map $\mathbb{R}^n \to \mathbb{R}^m$ to be smooth or differentiable**. We will also assume knowledge on **complex numbers and linear algebra**, corresponding to what you learn in **Calculus 3**. For example, you should know what is a subspace of a vector space, what is the image of a linear map, when is a linear map invertible.
Finally, it would be desirable if you have heard the words:
open, closed, compact in connection with subsets of $\mathbb{R}^n$. Ideally, you also know, for example, what these notions have to do with convergence of sequences. But no worries, I will try to remind you of as much as I can during class. If you want to refresh your knowledge on Topology, you may want to have a look at the books

[J] K. Jänich, Topology.

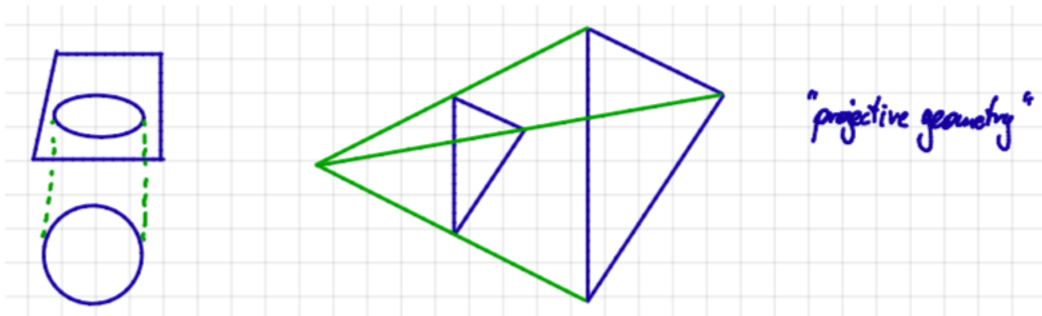
[D] B. Dundas, Appendix in Differential Topology.

As always, ASK ME if you wonder about anything!

0.3. What this class is about? Super roughly speaking, **Topology is some kind of Geometry**. Classical geometers were interested in measruing angles and distances. For example, two things are the "same" (congruent) in **classical geometry** if you can transform one into the other by moving or flipping them over. No stretching allowed. That means angles and lengths of edges stay the same.



A first variation to allow flexibility, is **projective geometry**: Two things are considered the same if they are both **views** of the same object. For example, an ellipse and a circle can be projectively equivalent; for one can look like the other when you look at them from the right prespective.



In **topology**, we take this idea one step further and consider two things the same if we can **continuously transform** one into the other. For example, a triangle is equivalent to a circle is equivalent to a square.

In **differential topology**, the part we will mostly be interested in, we only allow **smooth transformations**. (Then square and circle are different, because a square has edges which are not smooth.)

What Differential Topology is about:

Roughly speaking, differential topology is the study of properties that do not change under diffeomorphisms (specified transformations that are allowed).

We will make sense of all this during the course. This is just a first super rough distinction.

The goal of this class

Learn something about fundamental

- geometric objects, mostly we study smooth manifolds;
- methods and ideas in (differential) topology;
- applications of these objects and methods in different areas of mathematics.

In order to get a first idea, let's look at a fundamental example:

The Circle

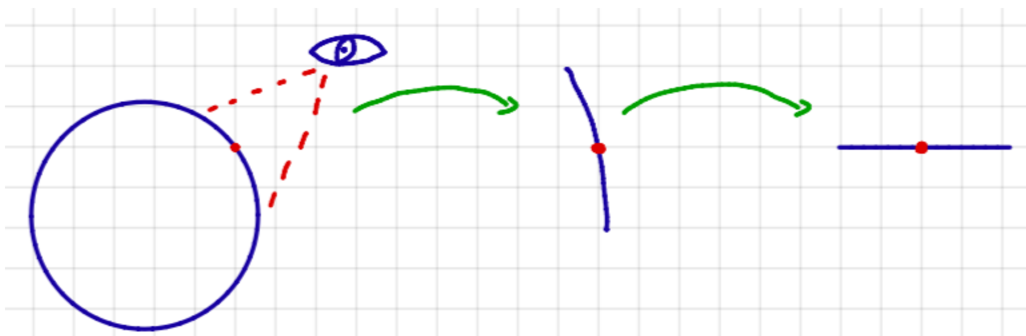
Let us start with the unit circle

$$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2.$$

The circle is something on-dimensional, isn't it? But how do we describe that precisely. Well, it's clear if we **zoom in** at any point, it just looks like a bended line segment. Looking very closely it even looks almost like a striaght line segment.
So, **"locally"** (whatever that means) the circle looks like a segment of $\mathbb{R}^1$.
The unit circle S^1, more generally, the n-dimensional sphere

$$S^n = \{(x_1, \ldots, x_{n+1}) \in R^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}$$

is an example of a smooth manifold.



Let us give a first working definition of what kind of objects we are going to study:

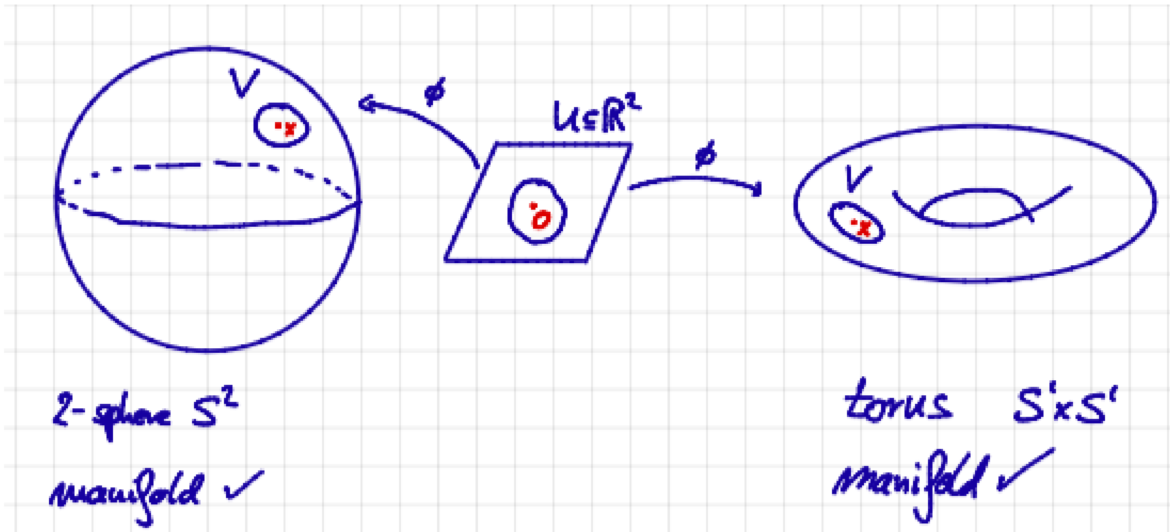
Working definition: What is a manifold

A manifold is a geometric object such that each point has a neighborhood which looks like $\mathbb{R}^n$.
We will make precise what "looks like" means. For smooth manifolds, we need a condition that takes differentiable data into account. The right notion is that of "diffeomorphism".

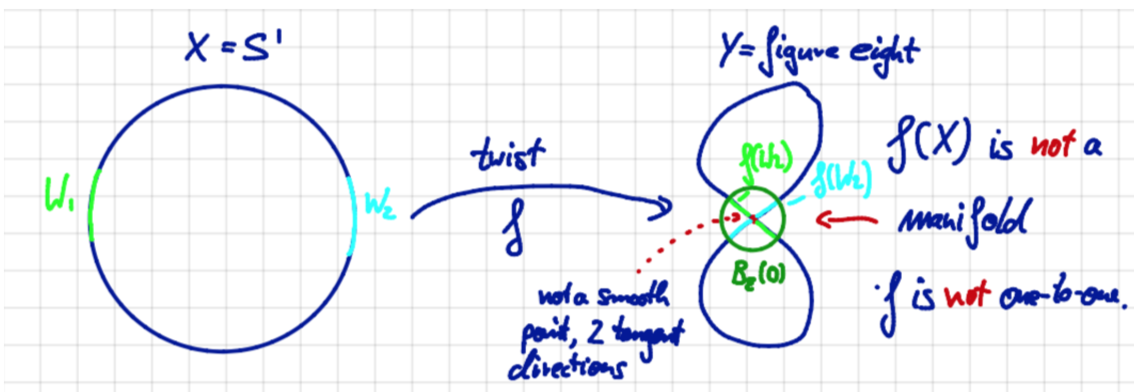
A universe of examples

The previous definition may sound quite strict. Every point looks the same in a small neighborhood. But we will see that there is a huge universe of examples of very different kind. In fact, one of the main goals in topology is to classify all types of manifolds.

Here are two more pictures of examples of smooth manifolds: one of the 2-sphere, the other of the torus:



And here is a NON-Example, the figure eight. The center point does not have any "nice" neighborhood.



0.4. Some nice theorems. Here are some examples of theorems we are going to prove during this class:

Fundamental Theorem of Algebra

Let $P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$ be a polynomial with complex coefficients, i.e. $a_0, \ldots, a_{n-1} \in \mathbb{C}$.
Then $P(X)$ has a zero in $\mathbb{C}$, i.e. there exists at least one complex number $z \in \mathbb{C}$ such that $P(z) = 0$.
That means of course that $P(X)$ has exaclty n zeroes in $\mathbb{C}$ (counted with multiplicities).

This has at first glance nothing to do with topology. But we can do it! (**Fundamental application**)

Brouwer Fixed Point Theorem

Every continuous map $f\colon D^n \to D^n$ has a fixed point, i.e. there is an $x \in D^n$ such that $f(x) = x$. Here D^n is the n-dimensional unit disc

$$D^n = \{(x_1, \dots, x_n) \in R^n : x_1^2 + \cdots + x_n^2 \leq 1\}$$

This may not look so exciting, but HOW can you show that a fixed point always exists? (**Fundamental method**)

Hairy Ball Theorem

Assume you have a ball with hairs attached to it. Then it is impossible to comb the hair continuously and have all the hairs lay flat. Some hair will always be sticking right up.
A more mathematical formulation:
Every smooth vector field on a sphere has a singular point.
An even more general statement:
The n-dimensional sphere S^n admits a smooth field of nonzero tangent vectors if and only if n is odd.

This just sounds like a fun fact. But wind speeds on the surface of the earth is an example of a vector field on a sphere!
(**Fundamental object AND application**)

Something else one can prove using topological methods.

Multiplicative Structures on $\mathbb{R}^n$

Let $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be a bilinear map with two-sided identity element $e \neq 0$ and no zero-divisors. Then $n = 1, 2, 4,$ or 8.
What we are looking for is a "multiplication map". You know the cases $n = 1$ and $n = 2$ very well. It's just $\mathbb{R}$ and $\mathbb{C} \cong \mathbb{R}^2$. These are actually fields.
For $n = 4$, there are the Hamiltonians, or Quaternions, $\mathbb{H} \cong \mathbb{R}^4$ with a multiplication which as almost as good as the one in $\mathbb{C}$ and $\mathbb{R}$, but it is not commutative. (You add elements i, j, k to $\mathbb{R}$ with certain multiplication rules.)

For $n = 8$, there are the Octonions $\mathbb{O} \cong \mathbb{R}^8$. The multiplication is not associative and not commutative.
And that's it!!

This is a really deep result!
The crucial and, at first glance maybe surprising, point to prove this fundamental result is that the statement has something to do with the behavior of tangent spaces on spheres. That's a topological problem. Frank Adams was the first to solve it. The prove goes way beyond the methods of this class, unfortunately. So **stay tuned** on the Topology Chanel and lear more about it in **Advanced Aglebraic Topology...**

LECTURE 2

Topology in $\mathbb{R}^n$ and smooth maps

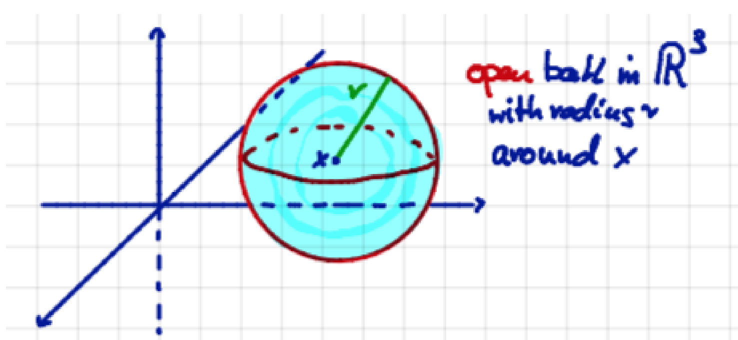
Recall from Calculus 2 that the norm of a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is defined by

$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \in \mathbb{R}.$$

For any n, the space $\mathbb{R}^n$ with this norm is called **n-dimensional Euclidean space**. It is a **topological space** in the following way:

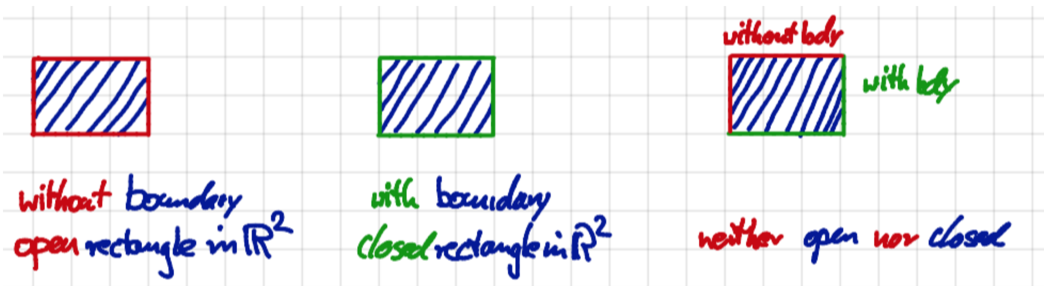
Open sets in $\mathbb{R}^n$

- Let x be a point in $\mathbb{R}^n$ and $r > 0$ a real number. The ball
$$B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$$
with radius ϵ around x is an **open** set in $\mathbb{R}^n$.
- The open balls $B_r(x)$ are the prototypes of open sets in $\mathbb{R}^n$.
- A subset $U \subseteq \mathbb{R}^n$ is called **open** if for every point $x \in U$ there exists a real number $\epsilon > 0$ such that $B_\epsilon(x)$ is contained in U.
- A subset $Z \subseteq \mathbb{R}^n$ is called **closed** if its complement $\mathbb{R}^n \setminus Z$ is open in $\mathbb{R}^n$.



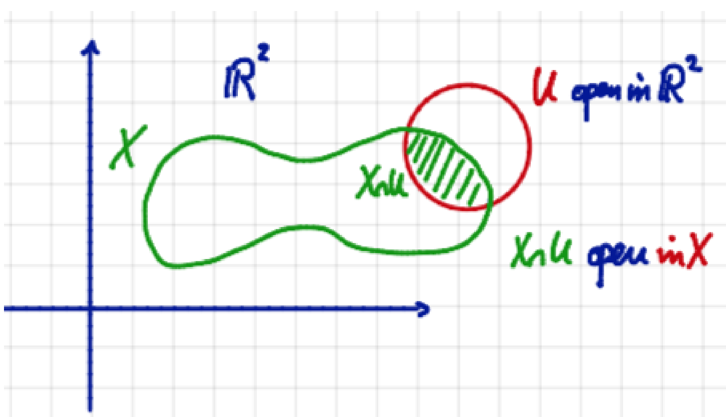
- Familiar examples of open sets in $\mathbb{R}$ are open intervals, e.g. (0,1) etc.
- The cartesian product of n open intervals (an open rectangle) is open in $\mathbb{R}^n$.
- Similarly, closed intervals are examples of closed sets in $\mathbb{R}$.

- The cartesian product of n closed intervals (a closed rectangle) is closed in $\mathbb{R}^n$.
- The empty set $\emptyset$ and $\mathbb{R}^n$ itself are by both open and closed sets.
- Not every subset of $\mathbb{R}^n$ is open or closed. There are a lot of subsets which are neither open nor closed. For example, the interval (0,1] in $\mathbb{R}$; the product of an open and a closed interval in $\mathbb{R}^2$.



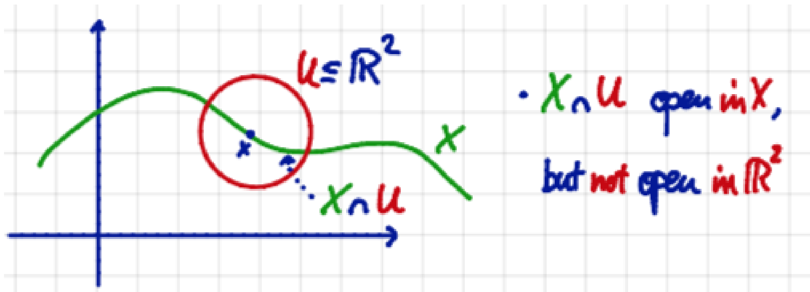
Relative open sets

Let X be a subset in $\mathbb{R}^n$. Then we say that $V \subseteq X$ is **open in** X (or **relatively open**) if there a an open subset $U \in \mathbb{R}^n$ with $V = U \cap X$.



Warning

It is important to note that that the property of being **an open subset** really depends on the bigger space we are looking at. Hence **open** always refers to being **open in** some given space.
For example, a set can be open in a space $X \subset \mathbb{R}^2$, but not be open in $\mathbb{R}^2$, see the picture.



Open sets are nice for a lot of reasons. First of all, they provide us with a way to talk about things that happen **close to** a point.

Open neighborhoods

We say that a subset $V \subseteq X$ containing a point $x \in X$ is a **neighborhood of** x if there is an open subset $U \subseteq V$ with $x \in U$. If V itself is open, we call V an **open neighborhood**.

Second, the collection of all open subsets in a set X, define a **topology** on X. A set together with a topology, is called a **topological space**.

We observe here that the word **"topology"** is used in different ways. On the one hand, it is the name of a **whole area** in mathematics. On the other hand, it is the name for a certain **structure on a set**.
We see that phenomenon happen quite often. For example,

- the term **"algebra"** denotes both a field in mathematics and a certain type of structure on a set;
- the term **"medicine"** denotes the field, but a doctor can also prescribe a specific medicine to cure a desease.

The type of maps that preserve open sets are the continuous maps:

Continuous maps

Let A be a subset in $\mathbb{R}^n$. A map $f\colon A \to \mathbb{R}^m$ is called **continuous at** a if for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon.$$

In our new fancy notation, we can reformulate the last condition as: given any $\epsilon > 0$,

$$\text{there is a } \delta > 0 \text{ such that } \mathbf{x} \in \mathbf{B}_\delta(\mathbf{a}) \cap \mathbf{A} \Rightarrow \mathbf{f}(\mathbf{x}) \in \mathbf{B}_\epsilon(\mathbf{f}(\mathbf{a})).$$

Finally, in terms of limits, we could say: f is continuous at a if

$$\lim_{x \to a} f(x) = f(a).$$

The map f is called **continuous** if it is continuous at every $a \in A$.

A more intrinsic characterization that serves as a definition for arbitrary topological spaces is the following.

Continuous maps: a more general characterization

A map $f\colon A \to \mathbb{R}^m$ is **continuous** if and only if, for every open subset $U \subseteq \mathbb{R}^m$, there is some open subset $V \subseteq \mathbb{R}^n$ with $f^{-1}(U) = V \cap A$ (in other words $f^{-1}(U)$ is open in A).

Proof:

First, assume f is continuous. Let $U \subseteq \mathbb{R}^m$ be an open set in $\mathbb{R}^m$. If $f^{-1}(U)$ is empty, it is open by definition. So let $a \in f^{-1}(U)$ be a point in $f^{-1}(U)$. The fact that U is open means that there is an $\epsilon > 0$ such that $B_\epsilon(f(a)) \subset U$. Given this ϵ, the fact that f is continuous means that

$$\text{there is a } \delta > 0 \text{ such that } x \in B_\delta(a) \cap A \Rightarrow f(x) \in B_\epsilon(f(a)).$$

But

$$f(x) \in B_\epsilon(f(a)) \text{ implies } f(x) \in U \text{ which implies } x \in f^{-1}(U) \cap A.$$

Since x was arbitrary in $B_\delta(a) \cap A$ this means $B_\delta(a) \cap A \subseteq f^{-1}(U)$.

Second, assume that $f^{-1}(U)$ is open in A for every open subset $U \subseteq \mathbb{R}^m$. Given $a \in A$ and $\epsilon > 0$, let $B_\epsilon(f(a)) \subset \mathbb{R}^m$ be the open ball around $f(a)$ with radius ϵ. Since $B_\epsilon(f(a))$ is open in $\mathbb{R}^m$, our assumption tells us that $f^{-1}(B_\epsilon(f(a)))$ is open in A. Since $a \in f^{-1}(B_\epsilon(f(a)))$ this means that

$$\text{there is a } \delta > 0 \text{ such that } B_\delta(a) \subseteq f^{-1}(B_\epsilon(f(a))).$$

But that means

$$x \in B_\delta(a) \Rightarrow f(x) \in B_\epsilon(f(a)).$$

Hence f is continuous at a. Since a was arbitrary, f is continuous. **QED**

Homeomorphisms

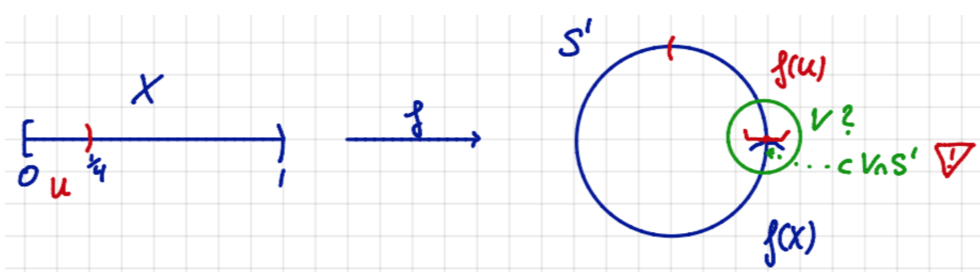
A continuous map $f\colon X \to Y$ is a **homeomorphism** if one-to-one and onto, and its inverse f^{-1} is continuous as well. Homeomorphisms preserve the topology in the sense that $U \subset X$ is open in X if and only if $f(U) \subset Y$ is open in Y.

Examples:

- $\tan\colon (-\pi/2,\pi/2) \to \mathbb{R}$ is a homeomorphism.
- $f\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$ is a homeomorphism.

Example: Bijection which is not a homeomorphism

Let
$$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2$$
be the unit circle considered as a subspace of $\mathbb{R}^2$. Define a map
$$f\colon [0,1) \to S^1,\ t \mapsto (\cos(2\pi t), \sin(2\pi t)).$$
We know that f is bijective and continuous from Calculs and Trigonometry class. But the function **f^{-1} is not continuous**. For example, the image under f of the open subset $U = [0,\frac{1}{4})$ (open **in** $[0,1)$!) is not open in S^1. For the point $y = f(0)$ does not lie in any open subset V of $\mathbb{R}^2$ such that
$$V \cap S^1 = f(U).$$



Spaces

From now on, when we talk about a **space** we mean a set together with a specified topology or collection of open subsets.

Remark:
For topological spaces X and Y, a map $f: X \to Y$ is defined to be **continuous** if for every open set $U \subseteq Y$ the subset $f^{-1}(U)$ is open in X. Just in case you have heard of categories before: **Topological spaces** form a **category** with morphisms given by continuous maps.

Here is another extremely important property a subset in a topological space can have. We are going to use it quite often in fact.

Compact sets in $\mathbb{R}^n$

- A subset Z in a topological space is called **compact** if every open cover $\{U_i\}_i$ of Z has a *finite* subcover. That is, among the $\{U_i\}_i$ it is always possible to pick $U_{i_1}, \ldots, U_{i_n}$ with
$$Z = U_{i_1} \cup \ldots \cup U_{i_n}.$$
- By the Theorem of Heine-Borel, a subset $Z \subset \mathbb{R}^n$ is **compact** if and only if it is **closed and bounded**. Being bounded means, that there is some (possibly huge) $r >> 0$ such that $Z \subset B_r(0)$.

Compactness is an important example of a topological property:

Homeomorphisms preserve

Slogan: **Topology** is the study of properties which are preserved under homeomorphisms. From this point of view, a **topological property** is by definition a property that is preserved under homeomorphisms. For example, if $f: X \to Y$ is a homeomorphism, then $Z \subseteq X$ is compact if and only if $f(Z) \subseteq Y$ is compact.

Finally, open sets are nice because we can say what it means to be differentiable on an open set.

Recall: Smooth maps on open subsets

Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open sets. A map $f: U \to V$ is called **smooth** if it has continuous partial derivatives of **all orders** (i.e. all the partial derivatives $\partial^k f_j / \partial x_{i_1} \ldots \partial x_{i_k}$ exist and are continuous for **all** $k \geq 1$).
Recall also: another way to say that f is **differentiable at** $a \in U$ if there

is a linear map $\lambda\colon R^n \to \mathbb{R}^m$ such that

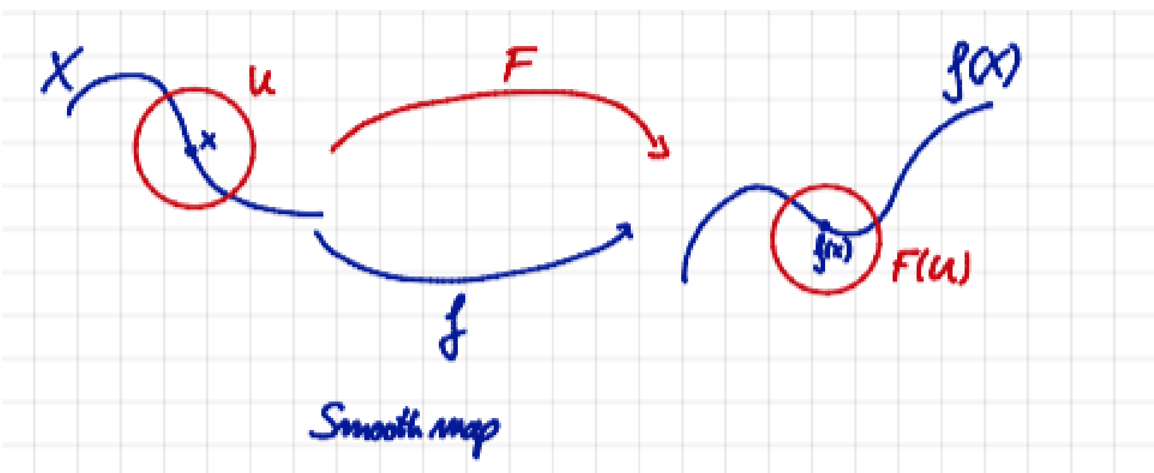
$$\lim_{h\to 0} \frac{|f(a+h) - f(a) - \lambda(h)|}{|h|} = 0.$$

Note that if such a λ exists, it is unique and is often denoted df_a.

Note that a smooth map is in particular also continuous. More generally, we can define smoothness for maps between arbitrary sets subsets of $\mathbb{R}^n$:

Smooth maps

Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be arbitrary subsets. A map $f\colon X \to Y$ is called smooth if for each $x \in X$ there exist an open subset $U \subseteq \mathbb{R}^n$ and a smooth map $F\colon U \to \mathbb{R}^m$ that coincides with f on all of $X \cap U$, i.e. $F_{X\cap U} = f$.



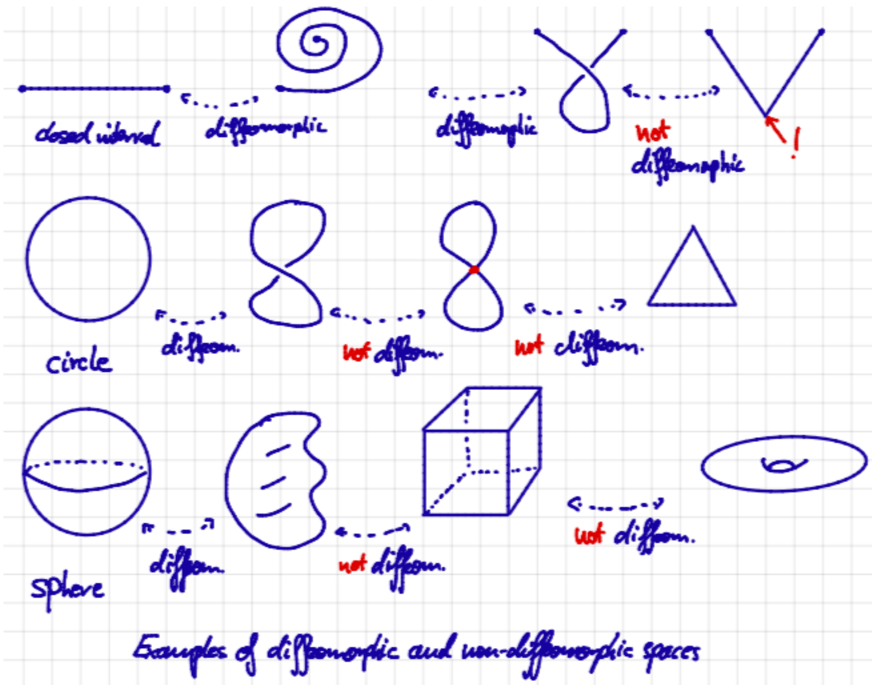
- The identity map of any set X is of course smooth.
- If $f\colon X \to Y$ and $g\colon Y \to Z$ are smooth, then the composition $g \circ f$ is also smooth.
- Note that **smoothness is a local property**, that means we need to check it only in a small neighborhood of any point.

Diffeomorphisms

A smooth map $f\colon X \to Y$ is called a **diffeomorphism** if f one-to-one and onto, and its inverse f^{-1} is smooth as well.

We say that X and Y are **diffeomorphic** if there exists a diffeomorphism $f\colon X \to Y$.

Note that every diffeomorphism is a homeomorphism, but not the other way around. For example, $f\colon \mathbb{R} \to \mathbb{R}$, $x \to x^3$ is a homeomorphism, but not a diffeomorphism. Exercise!



Diffeomorphic spaces are "equivalent"

Differential topology is the study of those properties of spaces which do not change under diffeomorphisms. In other words, from the point of view of differential topology, diffeomorphic spaces are equivalent, and we may (and will) consider them as copies of the same abstract space, which may happen to be differently situated in their surrounding Euclidean spaces.

LECTURE 3

Smooth manifolds

Recall that we defined what it means for subset $X \subseteq \mathbb{R}^n$ to be open. One reason why open sets are useful is that give us a way to talk about things that happen **close to** a point. In order to stress this way of thinking we are going to use the following way of speaking:

Open neighborhoods

We say that a subset $V \subseteq X$ containing a point $x \in X$ is a **neighborhood of x** if there is an open subset $U \subseteq V$ with $x \in U$. If V itself is open, we call V an **open neighborhood**.

Local properties

If we refer to something that happens in the neighborhood of a point $x \in X$, then we are often going to say that it happens **locally** (at x). Moreover, a property of a space or a function that we only need to **test for a neighborhood of each point** is a **local property**. For example, smoothness of a map is a local porperty (for we test it in a neighborhood of each point). In contrast, there are **global** properties which are properties that describe the **whole space**.

Manifolds are now spaces that **locally look like Euclidean spaces** in the following sense.

Smooth manifolds

Let $\mathbb{R}^N$ be some big Euclidean space.

- A subset $X \subseteq \mathbb{R}^N$ is a **k-dimensional smooth manifold** if it is locally diffeomorphic to $\mathbb{R}^k$. The latter means that for every point $x \in X$ there is an open subset $V \subset X$ containing x and an open subset $U \subseteq \mathbb{R}^k$ such that U and V are diffeomorphic.

- Any such diffeomorphism $\phi\colon U \to V$ is called a **(local) parametrization**.
- The inverse diffeomorphism $\phi^{-1}\colon V \to U$ is called a **(local) coordinate system on V**.

The natural number N in the previous definition is not specified. We just assume that there is some $\mathbb{R}^N$ big enough to fit X into it. We are going to discuss what we can say about the minmal N later. It is actually a very interesting question.

Remember that U is a subset of $\mathbb{R}^k$. Hence it makes sense to express a point $u \in U$ by its coordinates $u = (u_1, u_2, \dots, u_k)$. Hence, given a coordinate system $\phi^{-1}\colon V \to U$ on V, we can talk about the coordinates $\phi_1^{-1}(x), \phi_2^{-1}(x), \dots, \phi_k^{-1}(x))$ of a point $x \in V$. Writing $u_i(x) = \phi_i^{-1}(x)$ for $i = 1,\dots,k$, we usually drop mentioning ϕ^{-1} and just talk about the coordinates $(u_1(x), u_2(x), \dots, u_k(x))$ of x. Hence the $u_1, \dots, u_k$ are really **coordinate functions**.)

First examples

- An obvious example of a k-dimensional manifold is an open subset $U \subseteq \mathbb{R}^k$. The identity map $U \to U$ is a parametrization of all of U. For example, any k-dimensional open ball $B_r(x)$ around some point is a manifold of dimension k.
- A 0-dimensional manifold M just consists of a collection of discrete points. Given $x \in M$, the set $\{x\} \subset M$ consisting of x alone is open in M and is diffeomorphic to the one-point set $\mathbb{R}^0$.

A fundamental example that will play an important role during the whole semester is the n-dimensional sphere.

The unit circle

We start with $n = 1$: Let

$$S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2$$

be the unit circle. We are going to show that **S^1 is a 1-dimensional manifold**.

First, suppose that (x,y) lies in the upper semicircle where $y > 0$. Then

$$\phi_1(\mathbf{x}) = (\mathbf{x}, \sqrt{\mathbf{1} - \mathbf{x}^2})$$

maps the open interval $W = (-1,1)$ bijectively onto the upper semicircle. Its inverse is the projection map

$$\phi_1^{-1}(\mathbf{x},\mathbf{y}) = \mathbf{x}$$

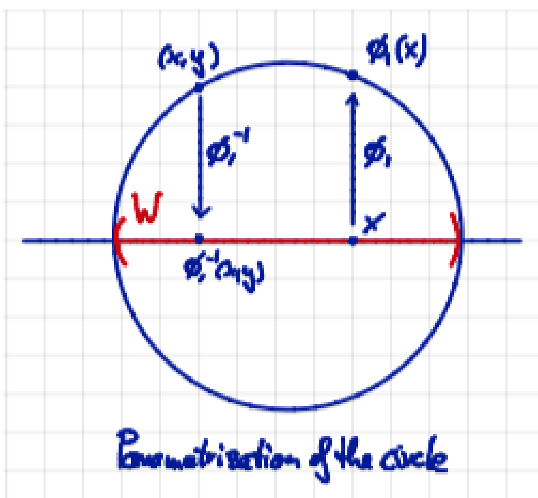
which is defined on the upper semicircle. This ϕ_1^{-1} is smooth, since it extends to a smooth map of all of $\mathbb{R}^2$ to $\mathbb{R}^1$. Therefore, ϕ_1 is a parametrization.
A parametrization of the lower semicircle where $y < 0$ is similarly defined by

$$\phi_2(x) = (x, -\sqrt{1-x^2}) \text{ with inverse } \phi_2^{-1}(x,y) = x.$$

These two maps give local parametrizations of S^1 around any point except the two points $(1,0)$ and $(-1,0)$. To cover these points, we can use the maps

$$\phi_3(y) = (\sqrt{1-y^2}, y) \text{ and } \phi_4(y) = (-\sqrt{1-y^2}, y)$$

which map W to the right and left semicircles, respectively.
This shows that S^1 is a 1-dimensional manifold.



Need at least 2 parametrizations

Note that we have used 4 parametrization maps in the above example. It is an exercise to show that it is possible to cover S^1 with only two parametrizations. (But just one parametrization cannot be enough, because S^1 is compact. For, if such a difeomorphism $\phi\colon S^1 \to U \subset \mathbb{R}^1$ to an open subset existed, it would mean that U is compact contradicting the Theorem of Heine-Borel.)

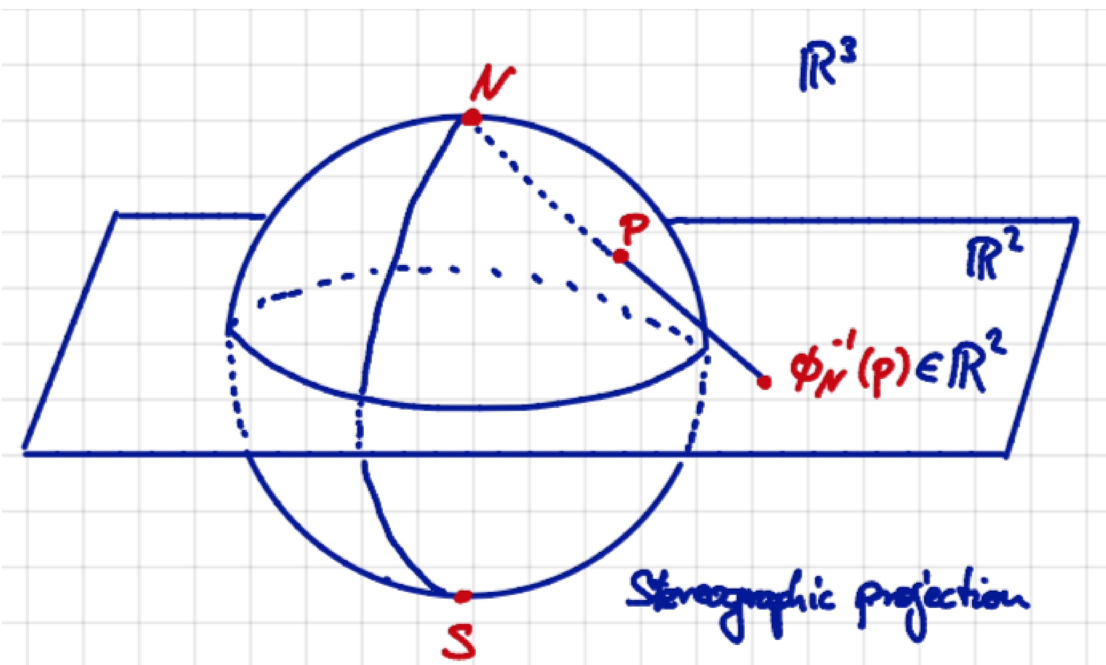
More generally:

n-sphere

The n-sphere
$$S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\} \subset \mathbb{R}^{n+1}$$
is an n-dimensional smooth manifold.

Stereographic projection

The method of stereographic projection yields a cover of the k-sphere with only two parametrizations. It is an exercise to find the formulae for the corresponding diffeomorphisms.



Submanifolds

If Z and X are both manifolds in $\mathbb{R}^N$ and $Z \subset X$, then Z is a **submanifold** of X. In particular, X itself is a submanifold of $\mathbb{R}^N$. Any open subset of X is a submanifold of X.

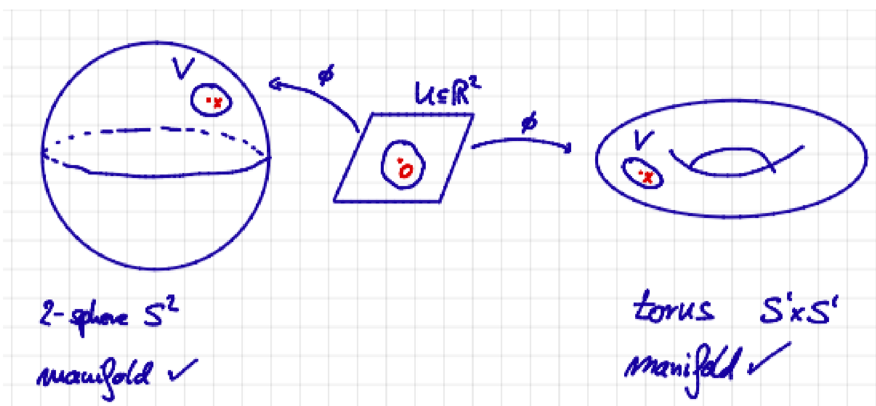
Creating new manifolds out of old ones

Let $X \subseteq \mathbb{R}^N$ and $Y \subseteq \mathbb{R}^M$ be manifolds of dimensions k and l, respectively. Then $X \times Y \subseteq \mathbb{R}^{N+M}$ **is a manifold of dimension** $k+l$. For let $W \subset \mathbb{R}^k$ an open set with $\phi\colon W \to X$ a local parametrization around $x \in X$, and $U \subset \mathbb{R}^k$ an open set with $\psi\colon U \to Y$ a local parametrization around $y \in Y$. Then we can define the map

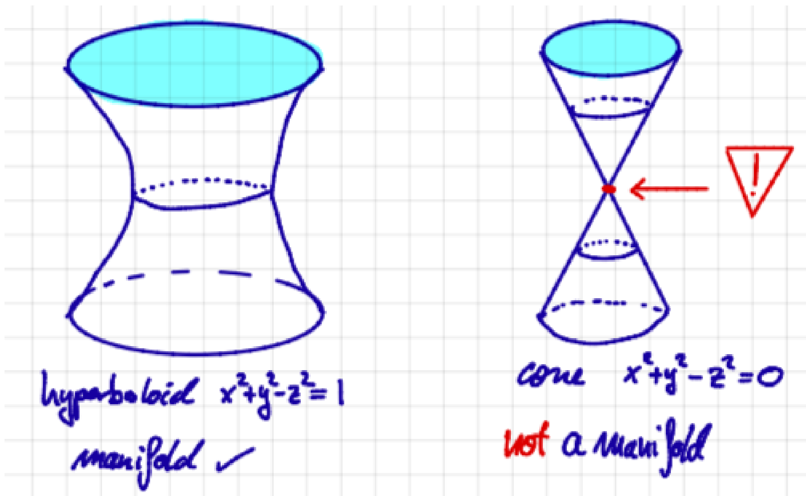
$$\phi \times \psi\colon W \times U \to X \times Y,\ \phi \times \psi(w,u) = (\phi(w), \psi(u)).$$

from the open set $W \times U \subseteq \mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^{k+l}$ to $X \times Y$. This map defines a local parametrization around (x,y). (Check this!)

Here is a picture of two smooth manifolds:



And a picture of a **hyperboloid (a manifold)** and **a cone (not a manifold)**, see the exercises.



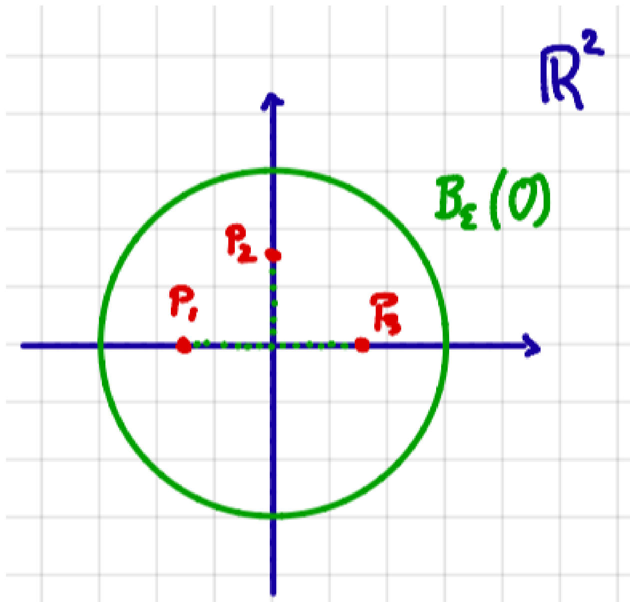
Coordinate axes in $\mathbb{R}^2$

Let us show that the union of the two coordinate axes in $\mathbb{R}^2$ is **not** a manifold.
Let us call the union X. The critical point is of course the origing (0,0), since every other point on X has an open neighborhood which is diffeomorphic to an open intervall in $\mathbb{R}$. But no point in $\mathbb{R}^d$ with $d \geq 2$ has an open neighborhood homeomorphic to an open intervall. Hence X could only be 1-dimensional.
Now let us check the point $O = (0,0)$. **If X was a manifold**, there would be an open subset $V \subseteq X$ around O **diffeomorphic to an open intervall** in $\mathbb{R}$. By definition of open sets in a subset of $\mathbb{R}^2$, there must be an open ball $B_\epsilon(O)$ such that $B_\epsilon(O) \cap X$ contained in V. Let I be the open intervall in $\mathbb{R}$ homeomorphic to $B_\epsilon(O) \cap X$.
The subset $B_\epsilon(O) \cap X$ contains, in particular, the points

$$P_1 = (-\epsilon/2,0),\ P_2 = (0,\epsilon/2), \text{ and } P_3 = (\epsilon/2,0).$$

In $B_\epsilon(O) \cap X$, there are paths
- from P_1 to P_2 **not passing through** P_3
- from P_1 to P_3 **not passing through** P_2
- from P_2 to P_3 **not passing through** P_1.

But there is **no triple of distinct points with this property in the open intervall** $I \subset \mathbb{R}$. Hence I **cannot** be homoeomorphic to $B_\epsilon(O) \cap X$. Hence O does not have a neighborhood homeomorphic to an open intervall in $\mathbb{R}$, and X is not a manifold.



$\mathbb{R}^2$
$B_\varepsilon(0)$
p_2
p_1
p_3

LECTURE 4

Tangent spaces and derivatives

Let us get back to the derivative of a smooth map $f\colon \mathbb{R}^n \to \mathbb{R}^m$. Let x be a point in the domain of f and $h \in \mathbb{R}^n$ be any vector in $\mathbb{R}^n$. Then the **derivative of f** in the direction h can be defined as the limit

$$df_x(h) = \lim_{t\to 0} \frac{f(x+th) - f(x)}{t}.$$

Hence for a fixed x, the derivative is a map

$$df_x\colon \mathbb{R}^n \to \mathbb{R}^m$$

sending a vector $h \in \mathbb{R}^n$ to the vector $df_x(h) \in \mathbb{R}^m$. In Calculus we learned that this map is **linear** (which means $df_x(h+g) = df_x(h) + df_x(g)$ and $df_x(\lambda h) = \lambda df_x(h)$ for all $h, g \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$). Note that **df_x is defined on all of $\mathbb{R}^n$ even if f is not**.

The derivative is a linear approximation

The derivative of f is a map on its own. We think of the parallel translate of df_x to x, i.e. $h \mapsto x + df_x(h)$ as the best **linear approximation** of f at x.

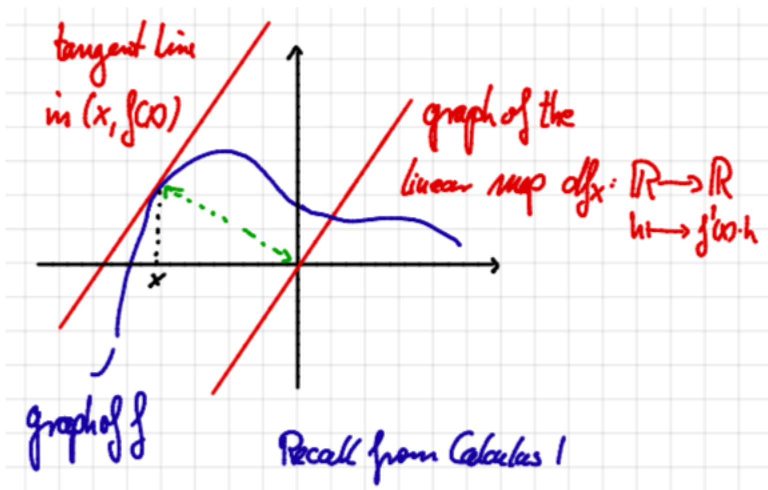
Note that if $f = L\colon U \to \mathbb{R}^m$ is a itself a **linear map**, then

$$df_x = L \text{ for all } x \in U.$$

In particular, the derivative of the inclusion map $U \subseteq \mathbb{R}^n$ at any point $x \in U$ is the identity map on $\mathbb{R}^n$.

df_x and the tangent line

In Calculus 1, we visualized the derivative by saying that $f'(x)$ is the slope of the tangent line at the graph of f at the point $(x,f(x))$. But the derivative $f'(x)$ **really is the linear map** $df_x \colon \mathbb{R} \to \mathbb{R}$ given by multiplying with the real number $f'(x)$. The tangent line at $(x,f(x)$ corresponds to the parallel translate of the linear map df_x, whose graph is the line through the origin with slope $f'(x)$.
We observe that, in order to get a **vector space**, the tangent space to the graph of f at $(x,f(x)$ is the **image of** $\mathbb{R}$ **under** df_x **in** $\mathbb{R}^2$.



We are going to use this picture of parallel translates to define the tangent space of a manifold at a point.

Let $X \subseteq \mathbb{R}^N$ be k-dimensional manifold and $x \in X$ a point. Let $\phi \colon U \to X$ be a **local parametrization around** x (i.e. there is an open subset $V \subseteq X$ containing x and an open subset $U \subseteq \mathbb{R}^k$ together with a diffeomorphism $\phi \colon U \to V$; we then also write $\phi \colon U \to X$ for the composite $U \xrightarrow{\phi} V \hookrightarrow X$).
We assume $\phi(0) = x$.

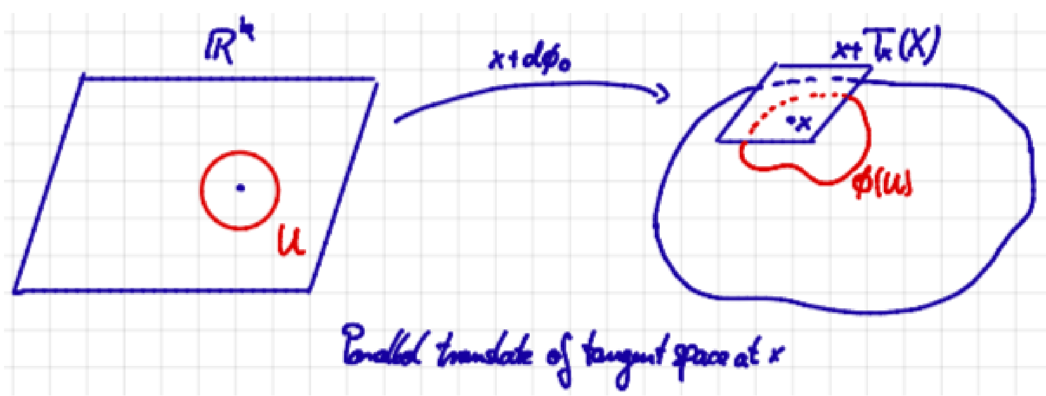
Tangent space

Then the **best linear approximation** to $\phi \colon U \to X$ at 0 is the map

$$u \mapsto \phi(0) + d\phi_0(u) = x + d\phi_0(u).$$

We define the **tangent space** $T_x(X)$ of X at x to be **the image of the linear map** $d\phi_0\colon \mathbb{R}^k \to \mathbb{R}^N$. Note that $T_x(X)$ is a **vector subspace of** $\mathbb{R}^N$.
Its parallel translate $x + T_x(X)$ **is the best linear approximation to** X **through the point** x.

By this definition, a **tangent vector** to $X \subseteq \mathbb{R}^N$ at x is a point $v \in \mathbb{R}^N$ that lies in the vector subspace $T_x(X)$ of $\mathbb{R}^N$. However, we usually picture v geometrically as the arrow running from x to $x+v$ in the translate $x+T_x(X)$.



In order to define $T_x(X)$ we made a **choice** of a parametrization ϕ. We have to check what happens if we choose a different parametrization. Are we getting the same tangent space?

$T_x(X)$ is well-defined

So let $\psi\colon V \to X$ be **another local parametrization** around x with $\psi(0) = x$. By shrinking both U and V we

$$\text{can assume } \phi(\mathbf{U}) = \psi(\mathbf{V})$$

(replace U by $\phi^{-1}(\phi(U) \cap \psi(V)) \subset U$
and V by $\psi^{-1}(\phi(U) \cap \psi(V)) \subset V$). Then the map

$$\theta := \psi^{-1} \circ \phi\colon U \to V$$

is a diffeomorphism (its the composite of two diffeomorphisms). By definition of θ, we have $\phi = \psi \circ \theta$. Differentiating yields

$$d\phi_0 = d\psi_0 \circ d\theta_0$$

(where we have used the chain rule). This implies that the image of $d\phi_0$ is contained in the image of $d\psi_0$:

$$\mathbf{d}\boldsymbol{\phi}_\mathbf{0}(\mathbb{R}^\mathbf{k}) \subseteq \mathbf{d}\boldsymbol{\psi}_\mathbf{0}(\mathbb{R}^\mathbf{k}) \text{ in } \mathbb{R}^\mathbf{N}.$$

By switching the roles of ϕ and ψ in the argument, we also get:

$$\mathbf{d}\boldsymbol{\psi}_\mathbf{0}(\mathbb{R}^\mathbf{k}) \subseteq \mathbf{d}\boldsymbol{\phi}_\mathbf{0}(\mathbb{R}^\mathbf{k}) \text{ in } \mathbb{R}^\mathbf{N}.$$

Hence $d\phi_0(\mathbb{R}^k) = d\psi_0(\mathbb{R}^k)$ in $\mathbb{R}^N$. This shows that whatever local parametrization around x we start with, the vector subspace $T_x(X) \subseteq \mathbb{R}^N$ is always the same. In mathematical terms we say that **$T_x(X)$ is well-defined**.

Dimension of $T_x(X)$

If X is a k-dimensional manifold, then **$T_x(X)$ is a k-dimensional vector space over $\mathbb{R}$**. (For we know from Calculus that the derivative of a diffeomorphism is a linear isomorphism. Hence $d\phi_0$ is an isomorphism of vector spaces $d\phi_0\colon \mathbb{R}^k \xrightarrow{\cong} T_x(X)$.)

Example: Tangent space at the unit circle

Let $p = (a,b) \in S^1$ be a point with $b > 0$. A local parametrization around p with $\phi(0) = p$ is given by

$$\phi\colon (-\epsilon,\epsilon) \to S^1,\ x \mapsto (t + a,\ \sqrt{1 - (x + a)^2}).$$

The derivative at x is the linear map

$$d\phi_x \colon \mathbb{R} \to \mathbb{R}^2,\ d\phi_x = (1, -\frac{x+a}{\sqrt{1-(x+a)^2}}).$$

Hence the image of $\mathbb{R}$ under $d\phi_0$ in $\mathbb{R}^2$ is the line spanned by $(-b,a)$ (writing $b = \sqrt{1-a^2}$).

Example: Tangent space at S^2

Let $p = (a,b,c)$ be point on S^2 which is not the north pole. Then we use the stereographic projection $\phi_N \colon \mathbb{R}^2 \to S^2$ as a local parametrization. (We do not need to translate first to get $\phi_N(0) = p$. That is up to us.)
Recall that

$$\phi_N(x,y) = \frac{1}{1+x^2+y^2}\left(2x,2y,x^2+y^2-1\right).$$

The derivative at (x,y) is the linear map $d\phi_N \colon \mathbb{R}^2 \to \mathbb{R}^3$ defined by the matrix (in the standard basis):

$$d(\phi_N)_{(x,y)} = \frac{2}{(1+x^2+y^2)^2}\begin{pmatrix} 1-x^2+y^2 & -2xy \\ -2xy & 1+x^2-y^2 \\ 2x & 2y \end{pmatrix}.$$

The image of $\mathbb{R}^2$ under the linear map $d(\phi_N)_{(x,y)}$ is the tangent space $T_{\phi_N(x,y)}S^2$. This image is spanned by the two column vectors of the matrix $d(\phi_N)_{(x,y)}$. Let us check that we get the space we would have expected, i.e. a plane which is orthogonal to the vector $\phi_N(x,y)$ (neglecting the first factors):

$$\begin{aligned} &\left(2x,2y,x^2+y^2-1\right)\cdot\begin{pmatrix} 1-x^2+y^2 \\ -2xy \\ 2x \end{pmatrix} \\ &= 2x(1-x^2+y^2) - 2xy^2 + 2x(x^2+y^2-1) \\ &= 2x - 2x^3 + 2xy^2 - 4xy^2 + 2x^3 + 2xy^2 - 2x \\ &= 0. \end{aligned}$$

Similarly,

$$\begin{aligned}
&\left(2x, 2y, x^2+y^2-1\right) \cdot \begin{pmatrix} -2xy \\ 1+x^2-y^2 \\ 2y \end{pmatrix} \\
&= -4x^2y + 2y(1+x^2-y^2) + 2y(x^2+y^2-1) \\
&= -4x^2y + 2y + 2x^2y - 2y^3 + 2x^2y + 2y^3 - 2y \\
&= 0.
\end{aligned}$$

Hence the plane spanned by the column vectors is orthogonal to $\phi_N(x,y)$.

The induced derivative

Now let $f\colon X \to Y$ be a smooth map from a k-dimensional smooth manifold $X \subseteq \mathbb{R}^N$ to an l-dimensional smooth manifold $Y \subseteq \mathbb{R}^M$. We would like to define a map **best linear approximation of f at a point x**. For $y = f(x)$, this should result in a **linear map** of vector spaces

$$T_x(X) \to T_y(Y).$$

Suppose that $\phi\colon U \to X$ is a local parametrization around x with $U \subseteq \mathbb{R}^k$, and $\psi\colon V \to Y$ a local parametrization around y with $V \subseteq \mathbb{R}^l$. We can assume $\phi(0) = x$ and $\psi(0) = y$. Then we define a map $\theta\colon U \to V$ by the following commutative diagram (which means that it does not matter which way we walk around from U to Y):

$$\begin{array}{ccc}
X & \xrightarrow{\quad f \quad} & Y \\
{\scriptstyle \phi}\uparrow & & \uparrow{\scriptstyle \psi} \\
U & \xrightarrow[\theta = \psi^{-1} \circ f \circ \phi]{} & V.
\end{array}$$

Define df_x

Taking derivatives yields a diagram of linear maps and we define df_x to be **the linear map which makes the diagram commutative**:

$$\begin{array}{ccc}
T_x(X) & \xrightarrow{\quad df_x \quad} & T_y(Y) \\
{\scriptstyle d\phi_0}\uparrow & & \uparrow{\scriptstyle d\psi_0} \\
\mathbb{R}^k & \xrightarrow[d\theta_0]{} & \mathbb{R}^l.
\end{array}$$

Since $d\phi_0$ is an isomorphism, we have to **define** df_x as

$$\mathbf{df_x := d\psi_0 \circ d\theta_0 \circ d\phi_0^{-1}}.$$

We call df_x also the **derivative of f at x**.

Again, we need to check that our definiton of df_x does not depend on the choices of parametrizations. This is left as an exercise. (See below.)

Tangent space of products

Given two smooth manifolds $X \subseteq \mathbb{R}^N$ and $Y \subseteq \mathbb{R}^M$ and points $x \in X$, $y \in Y$, then the tangent space of the product X and Y is the product of the tangent spaces, i.e.

$$T_{(x,y)}(X \times Y) = T_x(X) \times T_y(Y).$$

This follows from the fact that we can choose neighborhoods in $X \times Y$ by taking the product of neighborhoods in X and Y, respectively.
Moreover, it is easy to check that if $: X \to X'$ and $g\colon Y \to Y'$ are smooth maps, then the derivative of the prdoct map is the product of the derivatives, i.e.

$$d(f \times g)_{(x,y)} = df_x \times dg_y$$

for all $(x,y) \in X \times Y$.

Finally, we would like to have that the new derivative satisfies the chain rule. So let $g\colon Y \to Z$ be another smooth map. Let $\eta\colon W \to Z$ be a local parametrization around $z = g(y)$ with an open subset $W \subseteq \mathbb{R}^m$ and $\eta(0) = z$. Then we have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \phi\uparrow & & \uparrow\psi & & \uparrow\eta \\ U & \xrightarrow[\theta=\psi^{-1}\circ f\circ\phi]{} & V & \xrightarrow[\iota=\eta^{-1}\circ g\circ\psi]{} & W \end{array}$$

which gives us the commutative square

$$\begin{array}{ccc} X & \xrightarrow{g\circ f} & Z \\ \phi\uparrow & & \uparrow\eta \\ U & \xrightarrow[\iota\circ\theta]{} & W. \end{array}$$

Thus, by definition,

$$d(g \circ f)_x = d\eta_0 \circ d(\iota \circ \theta)_0 \circ d\phi_0^{-1}.$$

The Chain Rule from Calculus 2 for maps of open sets of Euclidean spaces, then gives

$$d(\iota \circ \theta)_0 = (d\iota_0) \circ (d\theta_0).$$

Thus

$$d(g \circ f)_x = (d\eta_0 \circ d\iota_0 \circ d\psi_0^{-1}) \circ (d\psi_0 \circ d\theta_0 \circ d\phi_0^{-1}) = dg_y \circ df_x.$$

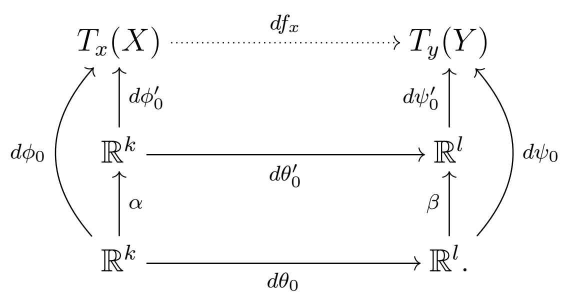
Hence we have in fact the desired rule.

Chain Rule

If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are smooth maps of manifolds, then

$$d(g \circ f)_x = dg_{f(x)} \circ df_x.$$

Let $\phi' \colon U \to X$ and $\psi' \colon V' \to Y$ be another choice of local parametrizations around x and y, respectively. Again by shrinking both U and U', both V and V' accordingly we can assume that $\phi(U) = \phi'(U') \subseteq X$ and $\psi(V) = \psi'(V') \subseteq Y$. Then $d\phi_0$ and $d\phi'_0$ differ by a linear isomorphism of $\mathbb{R}^k$, say α: $d\phi_0 = d\phi'_0 \circ \alpha$. Similarly, there is a linear isomorphism β of $\mathbb{R}^l$ such that $d\psi_0 = d\psi'_0 \circ \beta$. Let $\theta' \colon U \to V$ be defined similarly to θ, i.e. $\theta' = \psi'^{-1} \circ f \circ \phi'$. This gives us the following diagram in which each square commutes



Hence we get the desired identity

$$d\psi'_0 \circ d\theta'_0 \circ {d\phi'_0}^{-1} = d\psi_0 \circ d\theta_0 \circ d\phi_0^{-1} = df_x.$$

For more examples of tangent spaces have a look at the exercises.

LECTURE 5

The Inverse Function Theorem and Immersions

The Inverse Function Theorem

For our quest to understand smooth manifolds, it can be smart to study maps between manifolds (even though it sounds like making things even more difficullt; but if we know something about X and about a map $f\colon X \to Y$ then we might be able to say something interesting about Y). Anyway, there are a lot of interesting problems than can be stated in terms of properties of maps.

We have learned about the derivative of a map as a linear transformation between tangent spaces. We may think of the derivative as the **best linear approximation** at a point.

So let $f\colon X \to Y$ be a smooth map between smooth manifolds. Remember that the derivative at $x \in X$, $df_x\colon T_xX \to T_{f(x)}Y$, is a linear map between vector spaces. Since it is easier to understand linear maps, it would be nice if we could classify maps like f by the behaviour of df_x (with x varying in X).

A natural question:

How much does df_x tell us about the map f?

For the behavior df_x, there are three basic cases:

- $\dim X = \dim Y$ in which case the nicest possbile behavior of f at x is that df_x an isomorphism.
- $\dim X < \dim Y$ in which case the nicest possbile behavior of f at x is that df_x one-to-one.
- $\dim X > \dim Y$ in which case the nicest possbile behavior of f at x is that df_x onto.

We are going to consider these cases separately.

First case: df_x is an isomorphism

We begin with the nicest case when df_x is an isomorphism. This implies in particular: $\dim X = \dim Y$.

Manifolds are characterized by the way they look in a neighborhood around any point (they look like Euclidean space). So let us **think locally**. In the nicest case, f sends a neighborhood of a point x diffeomorphically to a neighborhood of $y = f(x)$. In this case, f is called a **local diffeomorphism at x**.

If f is a diffeomorphism $U \to V$ between neighborhoods U arround $x \in X$ and $y = f(x) \in Y$, respectively, let f^{-1} be its smooth inverse. Then we have $f^{-1} \circ f = \mathrm{Id}_U$ and $f \circ f^{-1} = Id_V$. Then the chain rule implies

$$d(Id_U)_x = d(f^{-1})_y \circ df_x, \text{ and } d(Id_V)_y = df_x \circ d(f^{-1})_y.$$

But we obviously have $d(\mathrm{Id}_X) = \mathrm{Id}_{T_x(X)}$ for any manifold X and any point $x \in X$. Hence df_x is an isomorphism with inverse $d(f^{-1})_{f(x)}$.

Thus a **necessary condition** for f to be a local diffeomorphism at x is that its derivative $df_x \colon T_x(X) \to T_y(Y)$ is an isomorphism.

It is an important result that this is actually a **sufficient condition**.

In order to prove this, we recall the corresponding important result for Euclidean space from Calculus:

The Inverse Function Theorem in Calculus

Suppose that $f \colon \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable in an open set containing a, and $\det df_a \neq 0$, i.e. df_a is an invertible linear map $\mathbb{R}^n \to \mathbb{R}^n$. Then there is an open set $V \subseteq \mathbb{R}^n$ containing a and an open set $W \subseteq \mathbb{R}^n$ containing $f(a)$ such that $f \colon V \to W$ has a continuous inverse $f^{-1} \colon W \to V$ which is differentiable and for all $y \in W$ satisfies

$$d(f^{-1})_y = \left(df_{f^{-1}(y)}\right)^{-1}.$$

Note that this is exactly the formula you are used to from Calculus 1 where we learned

$$(f^{-1})'(y) = (f'(f^{-1}(y)))^{-1}.$$

(You may be used to this formula as $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$. But the fraction here is misleading, since $(f^{-1})'(y)$ is a linear map. The superscript "to the -1"

really means **take the inverse map**! In dimension 1, the inverse map happens to be given by multiplication by the inverse number. But for linear maps or matrices in dimensions > 1, we cannot write the inverse as a fraction.)

The Inverse Function Theorem

Let X and Y be smooth manifolds. Suppose that $f\colon X \to Y$ is a smooth map whose derivative

$$df_x\colon T_x(X) \to T_{f(x)}(Y)$$

at a point $x \in X$ is an isomorphism. Then f is a **local diffeomorphism at x**.

The great thing about the IFT is that it tells us that in order to check that f is a diffeomorphism in a neighborhood of a point x, we just need to check that a **single number**, the determinant of df_x, is **nonzero**.

Idea of Proof: We can assume that X and Y are subsets in $\mathbb{R}^N$ for some large N. Let $\phi\colon U \to X$ be a local parametrization around $x \in X$, and $\psi\colon W \to Y$ a local parametrization around $y = f(x) \in Y$ with $U \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^n$ open and $\phi(0) = x$ and $\psi(0) = y$. (The dimension has to be the same when the tangent spaces are isomorphic.)

We define the map $\theta\colon U \to W$ as in the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \phi \big\uparrow & & \big\uparrow \psi \\ U & \xrightarrow[\theta = \psi^{-1} \circ f \circ \phi]{} & W. \end{array}$$

Then recall that df_x is defined such that the following diagram commutes

$$\begin{array}{ccc} T_x(X) & \xrightarrow{\quad df_x \quad} & T_y(Y) \\ d\phi_0 \big\uparrow & & \big\uparrow d\psi_0 \\ \mathbb{R}^k & \xrightarrow[d\theta_0]{} & \mathbb{R}^l. \end{array}$$

Our assumption is that df_x is an isomorphism which implies that **$d\theta_0$ is an isomorphism**. By the **IFT in Calculus**, this implies that there is

- an open neighborhood $V \subseteq U$ around 0 and
- an open neighborhood $V' \subseteq W$ aroud 0 such that

- $\theta_{|V}: V \to V'$ is a diffeomorphism.

Since ϕ and ψ are diffeomorphisms, $\phi(V) \subseteq X$ and $\psi(V') \subseteq Y$ are open neighborhoods of x and y, respectively. Moreover, $\phi_{|V}$ and $\psi_{|V'}$ are local parametrizations around x and y, respectively, and

$$f_{|\phi(V)}: \phi(V) \to \psi(V')$$

is a diffeomorphism. **QED**

Note that this is a **local statement**, i.e. if df_x is invertible, it only tells us that f is invertible in a **neighborhood of** x. Even if df_x is invertible for every $x \in X$, one cannot conclude that $f: X \to Y$ is globally a diffeomorphism. But such an f is a **local** diffeomorphism for every point $x \in X$. We call such a map a **local diffeomorphism** (without having to refer to a point).

Example 1: A global diffeomorphism

The map

$$(-\pi/2,\pi/2) \to \mathbb{R},\ t \mapsto \tan t$$

is a global diffeomorphism.

Example 2: A local but not global diffeomorphism

A standard example of a local diffeomorphism which is **not** a global diffeomorphism is the map

$$f: \mathbb{R}^1 \to S^1 \subset \mathbb{R}^2,\ t \mapsto (\cos t, \sin t)$$

that we have already met in Lecture 2. Let us check how this example works:

First, f is not a global diffeomorphism because it is not injective. And in Lecutre 2 we have seen that f is not a homeomorphism even when we restrict it to $[0,2\pi) \to S^1$. But anyway, S^1 **is compact** and $\mathbb{R}$ **is not**, so there is no chance of finding a diffeomorphism between them.

Second, the **IFT** tells us that f is indeed a **local diffeomorphism**, since df_t is an isomorphism for every $t \in \mathbb{R}$. For, let $t_0 \in \mathbb{R}$ such that $\cos(t_0) < 0$ (for other points the argument is similar, we just want to be able to choose a parametrization), and consider the local parametrization

$$\psi: (-1,1) \to V,\ y \mapsto (-\sqrt{1-y^2},y)$$

of S^1 around $f(t_0)$ with $V = \{(x,y) \in S^1 : x < 0\}$.

For an $\epsilon > 0$ such that both $\cos(t_0 - \epsilon) < 0$ and $\cos(t_0 + \epsilon) < 0$, we let $\phi\colon U = (t_0 - \epsilon, t_0\epsilon) \to \mathbb{R}$ be the local parametrization around t_0 given by the identity (we don't shift U to be centered around 0). Then the map $\theta\colon U \to W$ (see proof of the IFT) is defined as

$$\theta = \psi^{-1} \circ f \circ \phi, t \mapsto \sin t.$$

Then we get

$$d\theta_t\colon \mathbb{R} \to \mathbb{R},\ z \mapsto (\cos t) \cdot z$$

and

$$d\psi_t\colon \mathbb{R} \to \mathbb{R}^2,\ z \mapsto (-\frac{y}{\sqrt{1-y^2}}, 1) \cdot z.$$

Since ϕ is the identity, we have

$$df_{t_0} = d\psi_{\sin t_0} \circ d\theta_{t_0}$$

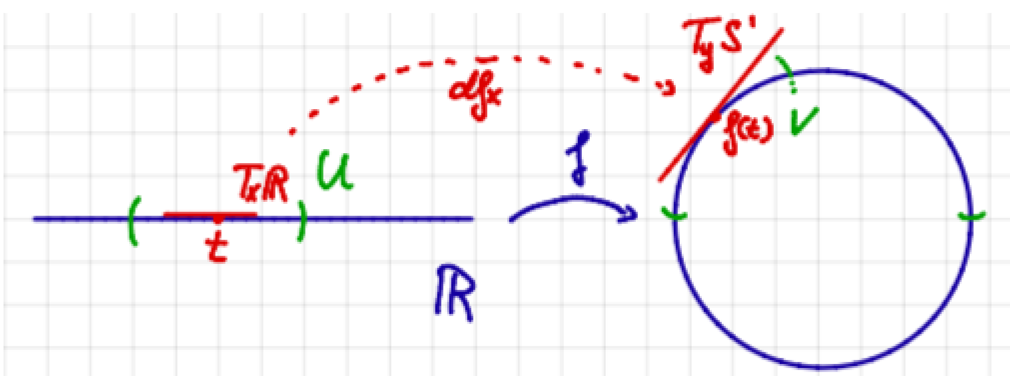
and hence

$$\begin{aligned} df_{t_0}(z) &= (-\frac{\sin t_0}{\cos t_0}, 1)(\cos t_0) \cdot z \\ &= (-\sin t_0, \cos t_0) \cdot z. \end{aligned}$$

Summarizing we have

$$\begin{aligned} df_{t_0}\colon T_{t_0}\mathbb{R} = \mathbb{R} &\to T_{f(t_0)}S^1 = d\psi(\mathbb{R}) = (-\sin(t_0), \cos(t_0)) \cdot \mathbb{R}^2, \\ z &\mapsto (-\sin(t_0), \cos(t_0)) \cdot z \end{aligned}$$

which is an isomorphism (when $\cos(t_0) \neq 0$).
For any other point in $\mathbb{R}$, there is a similar argument.



We close this first case, with an observation and some new terminology (way of speaking).

In some situations it would be nice if we could assume that the linear isomorphism df_x was the identity. This is usually not the case of course. But our freedom of choosing local parametrizations allows us to do the following. Assume that df_x is an isomorphism as in the IFT. Then, after possibly shrinkning U, we can assume $U = V$ and find a diffeomorphism $\gamma\colon U \to U$ such that $d\theta_0$ composed with $d\alpha_0$ becomes the identity.

df_x looks like the identity

If df_x is an isomorphism, we can choose local parametrizations $\phi\colon U \to X$ and $\psi\colon U \to Y$ around x and $f(x)$, respectively, with the same open domain $U \subset \mathbb{R}^n$, such that the diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle\phi}\uparrow & & \uparrow{\scriptstyle\psi} \\ U & \xrightarrow[\mathrm{Id}_U]{} & U. \end{array}$$

For example, in Example 2 above, we would replace

- $(-1,1)$ with $U = (t_0 - \epsilon, t_0 + \epsilon)$ and
- ψ with

$$\psi \circ \theta\colon t \mapsto (-\sqrt{1-\sin^2 t}, \sin t) = (\cos t, \sin t)$$

(remember $\cos t < 0$ for $t \in (t_0 - \epsilon, t_0 + \epsilon)$ by our choice of t_0 and ϵ).

In general, we are going to explain how to choose suitable parametrizations in the next section.

We would like to reformulate the IFT by saying that f is equivalent to the identity. To make this precise, we introduce the following terminology:

Equivalence of maps

We say that two maps $f\colon X \to Y$ and $g\colon X' \to Y'$ are **equivalent** if there exist diffeomorphisms α and β completing a commutative square

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\beta} \\ X' & \xrightarrow[g]{} & Y'. \end{array}$$

(One might also want to say that f and g are the same up to diffeomorphism.)

Hence the IFT says that if df_x is an isomorphism, then f is **locally equivalent at x** to the identity. Since a linear map is equivalent to the identity if and only if it is an isomorphism, we get:

IFT revisited

f is locally equivalent to the identity precisely when df_x is.

Immersions

We move on to the next case:

Second case: df_x is injective

Let us now assume $\dim X < \dim Y$. Then the best possible behavior of df_x is that

$$df_x \colon T_x(X) \to T_{f(x)}(Y)$$

is an **injective linear map**.

Let us introduce some terminology for this case.

Immersions

If df_x is injective, we say that f is an **immersion at x**. If f is an immersion at every point, we say that f is an **immersion**.

The **canonical immersion** is the standard inclusion for $n \leq m$:

$$\mathbb{R}^n \hookrightarrow \mathbb{R}^m,\ (a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n, 0, \ldots, 0).$$

Following our previous observation (i.e. up to diffeomorphism), the canonical immersion is **locally** the only immersion:

Local Immersion Theorem

Suppose that $f \colon X \to Y$ is an **immersion at x**, and $y = f(x)$. Then there exist local coordinates around x and y such that

$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0).$$

In other words, f is **locally equivalent to the canonical immersion**.

How to read the Local Immersion Theorem

We should read the statement in the LIT as follows: We can choose local parametrizations $\phi\colon U \to X$ around x and $\psi\colon V \to Y$ around y such that in the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle\phi}\uparrow & & \uparrow{\scriptstyle\psi} \\ U & \xrightarrow[\theta=\psi^{-1}\circ f\circ\phi]{} & V \end{array}$$

the map θ is the **canonical immersion** restricted to U.

Proof of the Local Immersion Theorem:
We start by choosing any local parametrization $\phi\colon U \to X$ with $\phi(0) = x$ and $\psi\colon V \to Y$ with $\psi(0) = y$:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle\phi}\uparrow & & \uparrow{\scriptstyle\psi} \\ U & \xrightarrow[\theta=\psi^{-1}\circ f\circ\phi]{} & V \end{array}$$

Now the plan is to manipulate ϕ and ψ such that g becomes the canonical immersion.

By the assumption, we know $d\theta_0\colon \mathbb{R}^n \to \mathbb{R}^m$ is injective. Now recall that we can represent a linear map between the vector spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ by an $m \times n$-matrix. In order to do that we have to choose a basis for the vector spaces.
(For $\mathbb{R}^n$ we usually use the standard basis. That's why we often don't think about bases when we look at a linear map $\mathbb{R}^n \to \mathbb{R}^m$.)
By choosing a suitable basis for $\mathbb{R}^m$, we can assume that $d\theta_0$ is the matrix

$$M \left(\begin{array}{c} I_n \\ \hline 0 \end{array}\right)$$

which consists of the $n \times n$-identity matrix sitting in the first n rows, and the zero $(m-n) \times n$-matrix occupying the remaining rows.

Choosing a basis

Recall that choosing a suitable basis works as follows:
Let $e_1^n, \ldots, e_n^n \in \mathbb{R}^n$ be the standard basis, and let $b_1 = d\theta_0(e_1^n), \ldots, b_n =$

$d\theta_0(e_n^n) \in \mathbb{R}^m$ be the images under $d\theta_0$. In terms of the standard basis of $\mathbb{R}^m$, the matrix for $d\theta_0$ is given by the $m \times n$-matrix A with b_i as ith column vector.
Since $d\theta_0$ is injective, the vectors $b_1, \ldots, b_n$ are linearly independent. We would to extend these vectors to a suitable basis of $\mathbb{R}^m$. Let $\mathrm{span}(b_1, \ldots, b_n)$ be the image of dg_0 in $\mathbb{R}^m$, and let $\mathrm{span}(b_1, \ldots, b_n)^\perp$ be its orthogonal complement in $\mathbb{R}^m$. Let $c_{n+1}, \ldots, c_m$ be a basis for $\mathrm{span}(b_1, \ldots, b_n)^\perp$. (You learned in Matte 3 how to find such a basis: $\mathrm{span}(b_1, \ldots, b_n)^\perp$ is the null space or kernel of the matrix A above.) We define a new basis for $\mathbb{R}^m$ as $b_1, \ldots, b_n, c_{n+1}, \ldots, c_m \in \mathbb{R}^m$.
In terms of this basis, the matrix of $d\theta_0$ is exactly $M\left(\frac{I_n}{0}\right)$. (Recall also that, in order to switch from the standard basis of $\mathbb{R}^m$ to that new basis, we apply the $m \times m$-matrix B whose first n columns are $b_1, \ldots, b_n$ and remaining $m - n$ columns are $c_{n+1}, \ldots, c_m$. Again, since $d\theta_0$ is injective, B is an invertible matrix which sends the standard basis $e_1^m, \ldots, e_m^m \in \mathbb{R}^m$ to the basis $b_1, \ldots, b_n, c_{n+1}, \ldots, c_m \in \mathbb{R}^m$.)

Back to the proof: We define a new map

$$\Theta \colon U \times \mathbb{R}^{m-n} \to \mathbb{R}^m, \text{ by } \Theta(x,z) = \theta(x) + (0,z).$$

It is related to θ by the picture

$$\begin{array}{ccc} U & \xrightarrow[\text{immersion}]{\text{canonical}} & U \times \mathbb{R}^{m-n} \\ & {\scriptstyle\theta}\searrow \quad \swarrow{\scriptstyle\Theta} & \\ & \mathbb{R}^m. & \end{array}$$

Since θ is a local diffeomorphism at 0, we can choose U and V small enough such that θ sends open sets to open sets. Moreover, the matrix representing $d\Theta_0$ (in our chosen basis) is just the $m \times m$-identity matrix I_m (it's $M\left(\frac{I_n}{0}\right)$ with the zero replaced with the remainind standard basis vectors $e_{n+1}^m, \ldots, e_m^m$). By the Inverse Function Theorem, this implies that Θ is a local diffeomorphism of $\mathbb{R}^m$ of itself at 0. Since ψ and Θ are local diffeomorphisms at 0, so is the composition $\psi \circ \Theta$. Hence we can use $\psi \circ \Theta$ as a local parametrization around y. Finally, after possibly shrinking U and V we get the desired commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\phi}\uparrow & & \uparrow{\scriptstyle\psi\circ\Theta} \\ U & \xrightarrow[\text{immersion}]{\text{canonical}} & V \end{array}$$

which proves the theorem. **QED**

Still an immersion in a neighborhood

We observe from the proof of the theorem that if $f\colon X \to Y$ is an immersion at x, then it is also an **immersion for all points in a neighborhood of** x. For, local parametrization $\phi\colon U \to X$ of the proof also parametrizes any point in the image of ϕ which is an open subset around x (open because ϕ is a diffeomorphism onto its image).

Local nature

To be an immersion is a **local condition**. For example, if $\dim X = \dim Y$, then being an immersion means being a local diffeomorohism. Hence in order to say more about f we need to add some (more global) topological properites to the local differential data.
For example, for a local diffeomorphism to be a global one, it has to be one-to-one and onto.

LECTURE 6

Immersions and Embeddings

Last time we studied immersions. Recall:

Local nature

To be an immersion is a **local condition**. For example, if $\dim X = \dim Y$, then being an immersion means being a local diffeomorohism. Hence in order to say more about f we need to **add some (more global) topological properties to the local differential data**.
For example, for a **local** diffeomorphism to be a **global** one, it has to be one-to-one and onto.

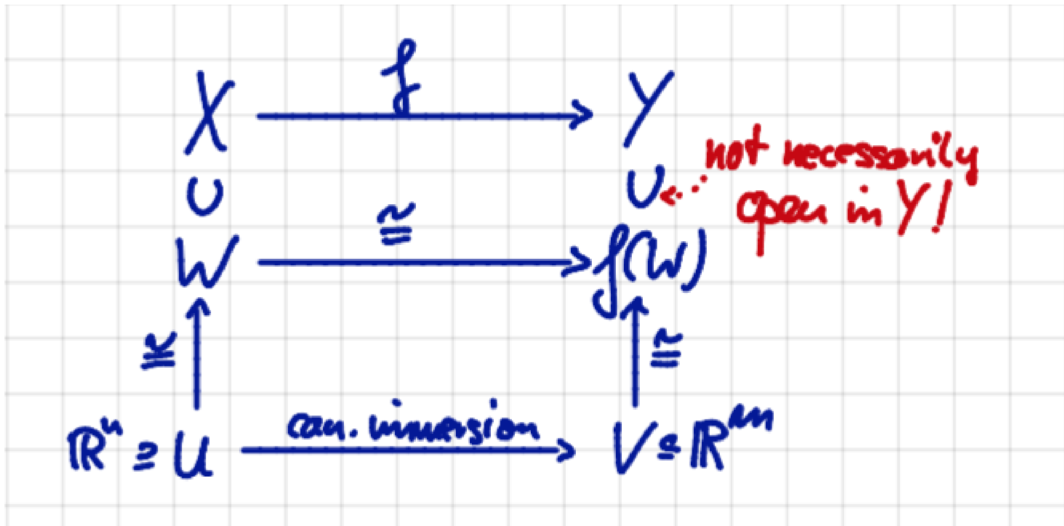
Let us look at the image of an immersion. The nicest possible case is the image of the canonical immersion $\mathbb{R}^n \hookrightarrow \mathbb{R}^m$. The Local Immersion Theorem tells us that **locally** any immersion looks like **the canonical one**. But we are now going to see:

Be aware!

The image of an immersion is **not always a submanifold**.

Let us try to understand **what can go wrong**:

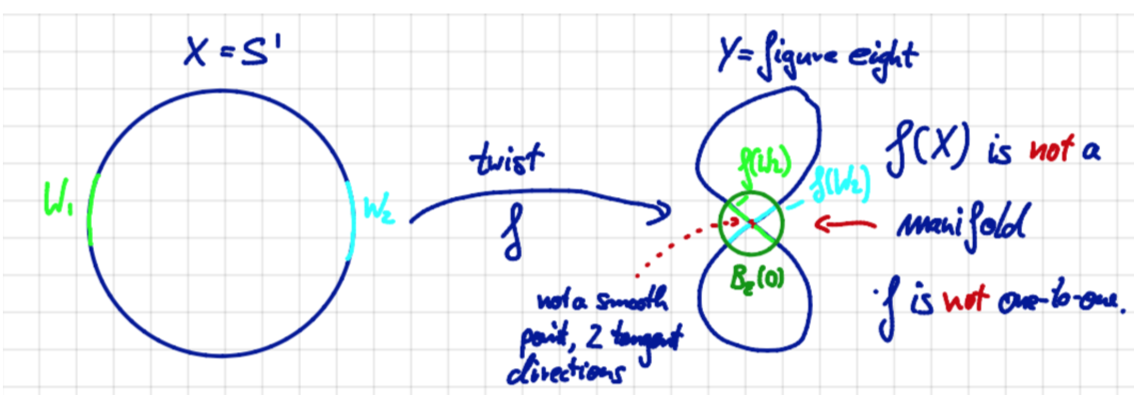
Let $f\colon X \to Y$ be an immersion. Then we know from the Local Immersion Theorem that f maps any sufficiently **small neighborhood** W of any point x in X **diffeomorphically onto its image** $f(W) \subset Y$. (By the LIT, W is diffeomorohic to a $U \subset \mathbb{R}^n$ which sits canonically in $V \subset \mathbb{R}^m$ which is diffeomorphic to $f(W)$, see the picture.)



Not open in Y?

Hence every point in $f(X)$ lies in a subset which is diffeomorphic to an open subset in $\mathbb{R}^n$. Isn't that the definition of $f(X)$ being a submanifold? **No**. The problem is that $f(W)$ does **not need to be open in** Y. Hence we cannot garantuee that points in $f(X)$ are in **parametrizable open neighborhoods**. UGH!

Before we try to find a global condition to fix this issue, let us look at **some examples of immersions whose image is not a submanifold**.



In the example above, f is not one-to-one and $f(X)$ has a point that is not smooth.

But even when f is one-to-one, this can happen, as the next example demonstrates. The image $f(X)$ is the same as above and not a manifold.

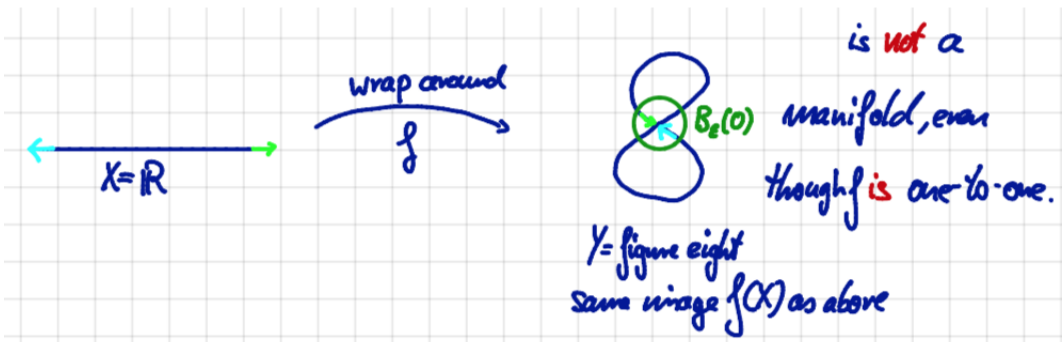


Figure eight immersion

In this example, the map f can be defined as

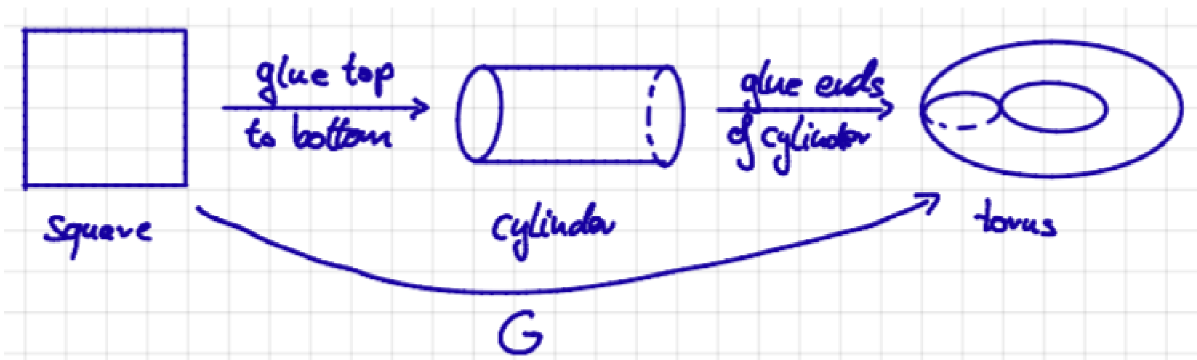
$$f\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (\sin(4\arctan t)), \sin(2\arctan t)).$$

(The image of f is called a lemniscate, the locus of points (x,y) satisfying $x^2 = 4y^2(1-y^2)$.)
We can check that f is **smooth**, **one-to-one** and **an immersion** (df_t is never zero and hence as a linear map between one-dimensional vector spaces an isomorphism).
But $f(X)$ is not a submanifold and f is not a diffeomorphism onto its image, because $f(X)$ is compact while X is not (an open interval in $\mathbb{R}$).

Torus by gluing:
Let $g\colon \mathbb{R} \to S^1$ be the local diffeomorphism $t \mapsto (\cos(2\pi t), \sin(2\pi t))$. We define

$$G\colon \mathbb{R}^2 \to S^1 \times S^1 =: T^2,\ G(x,y) = (g(x),g(y))$$

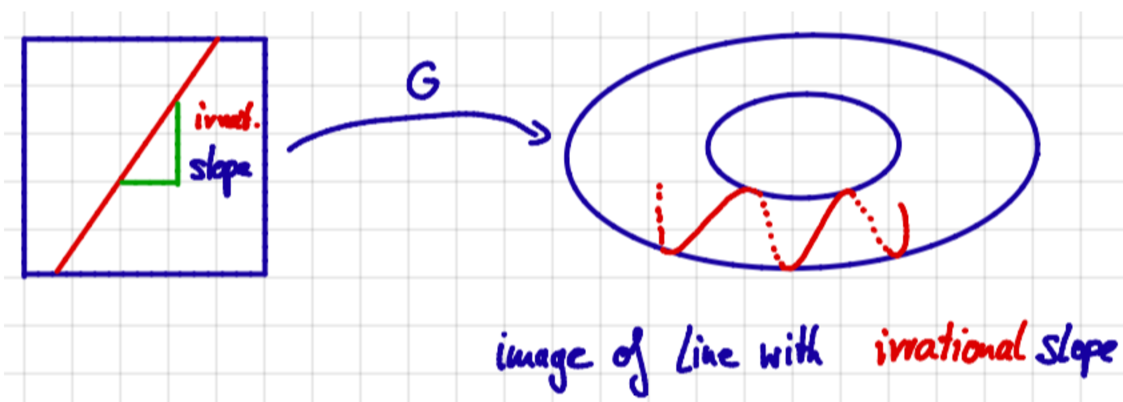
The map G is a local diffeomorphism from the plane onto the torus T^2. (G "glues" opposite sides of the square together, see the picture.)



We define the map γ by

$$\gamma\colon \mathbb{R} \to T^2,\ \gamma(t) = (g(t), g(\alpha \cdot t))$$

where α is an **irrational** number.



Note that another way to describe $\gamma = \gamma_\alpha$ would be to define it by

$$\gamma_\alpha\colon \mathbb{R} \to S^1 \times S^1,\ t \mapsto (e^{2\pi i t}, e^{2\pi i \alpha t})$$

where we consider S^1 as a subset of $\mathbb{C} \cong \mathbb{R}^2$. Then we require that the quotient α is irrational.

Image of a line with irrational slope

The map γ is an **immersion** because $d\gamma_t$ **is nonzero for every** t (and as before a nonzero linear map from a one-dimensional vector space to another is automatically injective; its image is a line in that other vector space).

And γ **is injective**, since $\gamma(t_1) = \gamma(t_2)$ implies

$$\begin{aligned} & g(t_1) = g(t_2) \text{ and } g(\alpha t_1) = g(\alpha t_2) \\ \Rightarrow & \cos(2\pi t_1) = \cos(2\pi t_2) \text{ and } \cos(2\pi\alpha t_1) = \cos(2\pi\alpha t_2) \\ \Rightarrow & t_1 - t_2 \in \mathbb{Z} \text{ and } \alpha(t_1 - t_2) \in \mathbb{Z} \end{aligned}$$

which is impossible, since α **is irrational**, unless $t_1 = t_2$.
Actually, one can show that the image of γ is a dense subset in T^2. But γ is **not a diffeomorphism onto its image**, since it is not even a homeomorphism:
For, look at the set $\gamma(\mathbb{Z}) = \{\gamma(n) : n \in \mathbb{Z}\}$. By Dirichlet's approximation theorem, for every $\epsilon > 0$, there are integers n and m such that

$$|\alpha n - m| < \epsilon.$$

Since the line segment between two points $(\cos t_1, \sin t_1)$ and $(\cos t_2, \sin t_2)$ on the unit circle is shorter than the circular arc of length $|t_1 - t_2|$ we have

$$\begin{aligned} & |(\cos(2\pi\alpha n), \sin(2\pi\alpha n)) - (1,0)| \\ = & |(\cos(2\pi\alpha n), \sin(2\pi\alpha n)) - (\cos(2\pi m), \sin(2\pi m))| \\ \leq & 2\pi|\alpha n - m| \\ \leq & 2\pi\epsilon. \end{aligned}$$

Therefore,

$$\begin{aligned} & |\gamma(n) - \gamma(0)| \\ = & |((g(n),g(\alpha n)) - (g(0),g(0))| \\ = & |\,((1,0),(\cos(2\pi\alpha n), \sin(2\pi\alpha n))) - ((1,0),(1,0))\,| \\ = & |(\cos(2\pi\alpha n), \sin(2\pi\alpha n)) - (\cos(2\pi m), \sin(2\pi m))| \\ \leq & 2\pi|\alpha n - m| \\ \leq & 2\pi\epsilon. \end{aligned}$$

Thus, there is a sequence of integers such that $\gamma(n)$ converges to $\gamma(0)$, i.e. $\gamma(0)$ is a limit point in $\gamma(\mathbb{Z})$. But $\mathbb{Z}$ does not have any limit points in $\mathbb{R}$.
But note that the image of a convergent sequence under a continuous map is again a convergent sequence. Hence if γ^{-1} was continuous, then $0 = \gamma^{-1}(\gamma(0))$ had to be a limit point as well. Hence γ is **not a homeomorphism onto its image**.

Aside: LIT for the above example

Let $t_0 = 0$ for simplicity. We apply the LIT to the map

$$\gamma\colon \mathbb{R} \to S^1 \times S^1$$

above. First, we parametrize $\mathbb{R}$ by the identity and pick some $U = (-1,1)$. Then we parametrize $S^1 \times S^1$ around $\gamma(0) = (1,0,1,0)$ by

$$\begin{aligned}\psi\colon V = (-1,1) \times (-1,1) &\to S^1 \times S^1,\\ (x,y) &\mapsto (\sqrt{1-x^2},x,\sqrt{1-y^2},y).\end{aligned}$$

The corresponding map $\theta\colon U \to V$ is then

$$t \mapsto (\sin(2\pi t), \sin(2\pi\alpha t)).$$

Now we would like to modify the local parametrization ψ around $\gamma(0)$ such that θ becomes

$$U \to U \times \mathbb{R},\ t \mapsto (t,0).$$

For that we define a new map

$$\Theta\colon U \times \mathbb{R} \to \mathbb{R}^2,\ (t,s) \mapsto \theta(t) + (0,s).$$

Then we compose ψ with Θ to get a new local parametrization around $\gamma(0)$:

$$\begin{aligned}\psi \circ \Theta\colon (t,s) \mapsto &\Big(\sqrt{1-\sin^2(2\pi t)}, \sin(2\pi t),\\ &\sqrt{1-(\sin(2\pi\alpha t)+s)^2}, \sin(2\pi\alpha t)+s\Big)\\ =&\Big(\cos(2\pi t), \sin(2\pi t),\\ &\sqrt{1-(\sin(2\pi\alpha t)+s)^2}, \sin(2\pi\alpha t)+s\Big).\end{aligned}$$

Finally, in order to make everything work, we have to make U and V small enough such that $\sin(2\pi t)$ and $\sin(2\pi\alpha t) + s$ stay in $(-1,1)$ for all $t \in U$ and $\theta(t) + (0,s) \in V$.

The pathologies of the last two examples arise because the map sends **points near infinity** in $\mathbb{R}$ into **small regions of the image**. So if we want to tame our immersions we have to try to avoid such a behavior. It will turn out that this is the only problem.

The topological analog of **points near infinity** in a topological space X is the exterior or complement of a compact set.

Proper maps

A map $f\colon X \to Y$ between topological spaces is said to be **proper** if the **preimage** of any compact subset is a compact subset.
(Recall: For a general continuous map, the image of any compact set is compact. Check that you understand why!)

Let $f\colon X \to Y$ be a proper map and let $Z \subset Y$ be a compact subset of Y. Then $f^{-1}(Z) \subset X$ is a compact subset of X, since f is proper. The complement $X \setminus f^{-1}(Z)$ of $f^{-1}(Z)$ in X is the largest subset of X which is not mapped to Z under f. Since f is proper, every point $x \in X \setminus f^{-1}(Z)$ is contained in the complement of a compact set and $f(x) \notin Z$. Thus f sends x to the complement of a compact subset in Y. Therefore, morally speaking, a proper map sends the complement of a compact set to the complement of a compact set. In other words:

Proper maps respect infinity

Proper maps send **points near infinity** to **points near infinity**.

Let us give proper immersions a name:

Embeddings

An immersion that is **one-to-one and proper** is called an **embedding**.

Properness turns out to be a sufficient global topological constraint for a local immersion. For proper maps we have the following extension of the Local Immersion Theorem.

Embedding theorem

An embedding $f\colon X \to Y$ maps X diffeomorphically onto a **submanifold** of Y.

Proof of the theorem:
By the assumption of f being a one-to-one immersion, we know that f is a **local diffeomorphism** from X to $f(X)$. Moreover, $f\colon X \to f(X)$ is **bijective** (injective by assumption and obviously surjective onto its image), and the inverse

f^{-1} exists as a map of sets. But locally f^{-1} **is smooth**, since f is a local diffeomorphism.

Hence in order to prove that $f(X)$ is a manifold, it remains to show that **the image of any open subset W of X is an open subset of $f(X)$**. For then f maps local parametrizations diffeomorphically to local parametrizations. Hence we need to show the general statement: **A bijective proper map is a homeomorphism**.

If $f(W)$ was not an open subset, then there would be a point $y \in f(W)$ and an open neighborhood of y which is not contained in $f(W)$. In different words, there would be a point $y \in f(W)$ such that in any small neighborhood of y there would be points y_i which are not in $f(W)$. We can rephrase this by saying:

If $f(W)$ is not an open subset, then there **exists a sequence of points $y_i \in f(X)$ that do not belong to $f(W)$, but converge to a point y in $f(W)$**.

The set $S := \{y, y_i\}_i$ is compact (a countable union of compact sets). Since **f is proper**, the preimage $f^{-1}(S)$ of S in X must be compact, too.

Since **f is injective**, there is exactly one preimage x of y in X and exaclty one preimage x_i for each y_i. Since $y \in f(W)$, x must belong to W.

Since **$f^{-1}(S) = \{x, x_i\}_i$ is compact**, after possibly restricting to a subsequence, we may assume that **the sequence of the x_i converges to a point $z \in X$**, we write $x_i \to z$. That implies $f(x_i) \to f(z)$ (since f is continuous). But since $f(x_i) \to f(x)$, the **injectivity of f implies $x = z$**.

Now **W is open**, which implies that, for large i, $x_i \in W$. But this implies $y_i = f(x_i) \in W$ and **contradicts $y_i \notin f(W)$**. Hence $f(W)$ is open in Y, and $f(X)$ is indeed a manifold. QED

A corollary for compact domains

If X is compact, then any continuous map $f\colon X \to Y$ is proper (closed subsets of compact sets are compact).
Hence, for compact X, every one-to-one immersion $f\colon X \to Y$ is an embedding and f maps X diffeomorphically onto a submanifold of Y.

LECTURE 7

Submersions

Let $f\colon X \to Y$ be a smooth map between smooth manifolds. Remember that the derivative at $x \in X$, $df_x\colon T_xX \to T_{f(x)}Y$, is a linear map between vector spaces, and we are trying to answer the question:

A natural question:

How much does df_x tell us about f?

We move to the third case:

Third case: df_x is surjective

Assume $\dim X > \dim Y$. The best possible behavior of df_x is then that

$$df_x\colon T_x(X) \to T_{f(x)}(Y)$$

is a **surjective linear map**.

Again, there is a name for this case:

Submersions

If df_x is surjective, we say that f is a **submersion at x**. If f is a submersion at every point, we say that f is an **submersion**.

The **canonical submersion** for $n \geq m$ is the standard projection

$$\mathbb{R}^n \to \mathbb{R}^m,\ (a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_m)$$

onto the first m coordinates (i.e. omitting the remaining $n - m$ coordinates).

Up to diffeomorphism the canonical submersion is **locally** the only submersion:

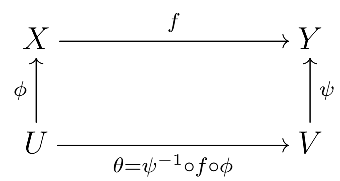
Local Submersion Theorem

Suppose that $f\colon X \to Y$ is a submersion at x, and $y = f(x)$. Then there exist local coordinates around x and y such that

$$f(x_1, \dots, x_n) = (x_1, \dots, x_m).$$

In other words, f is **locally equivalent to the canonical submersion**.

Proof of the Local Submersion Theorem:
As for the immersion case, we start by choosing any local parametrization $\phi\colon U \to X$ with $\phi(0) = x$ and $\psi\colon V \to Y$ with $\psi(0) = y$:



Now we are going to manipulate ϕ and ψ such that θ becomes the canonical submersion.

By the assumption, we know $d\theta_0\colon \mathbb{R}^n \to \mathbb{R}^m$ is surjective. Hence, after choosing a suitable basis for $\mathbb{R}^m$, we can assume that $d\theta_0$ is the matrix

$$M(I_m|0)$$

which consists of the $m \times m$-identity matrix sitting in the first n columns, and the zero $n \times (n - m)$-matrix occupying the remaining columns.

Choosing a basis

This time we need to choose a suitable basis for $\mathbb{R}^n$. Let $e_1^m, \dots, e_m^m \in \mathbb{R}^m$ be the standard basis. Since $d\theta_0$ is surjective, the induced linear map

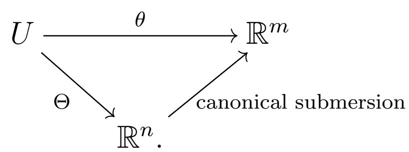
$$d\bar{\theta}_0\colon \mathbb{R}^n/\mathrm{Ker}\,(d\theta_0) \to \mathbb{R}^m$$

from the quotient vector space $\mathbb{R}^n$ modulo the kernel of $d\theta_0$ to $\mathbb{R}^m$ is an isomorphism. Hence we can choose unique vectors $b_1, \dots, b_m \in \mathbb{R}^n/\mathrm{Ker}\,(d\theta_0)$ with $d\bar{\theta}_0(b_i) = e_i^m$ for $i = 1, \dots, m$, and these $b_1, \dots, b_m$ form a basis of $\mathbb{R}^n/\mathrm{Ker}\,(d\theta_0)$. Now we choose a basis vector $b_{m+1}, \dots, b_n$ of $\mathrm{Ker}\,(d\theta_0)$. This gives us a basis $b_1, \dots, b_n$ of $\mathbb{R}^n$ such that $d\theta_0(b_i) = e_i^m$ for $i = 1, \dots, m$ and $d\theta_0(b_i) = 0$ for $i = m+1, \dots, n$. Hence in this basis for $\mathbb{R}^n$ and the standard basis for $\mathbb{R}^m$ the matrix for $d\theta_0$ is exactly $M(I_m|0)$ (remember: the columns are the images of the basis vectors).

Back to the proof: We define a new map

$$\Theta\colon U \to \mathbb{R}^n, \text{ by } \Theta(a) = (\theta(a), a_{m+1}, \ldots, a_n)$$

for a point $a = (a_1, \ldots, a_n)$. It is related to θ by the commutative diagram

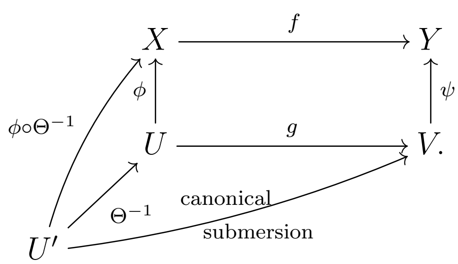


The derivative $d\Theta_0$ at 0 is given by the identity matrix I_n. Hence Θ is a **local diffeomorphism at** 0. Thus we can find a small neighborhood U' around 0 in $\mathbb{R}^n$ such that Θ^{-1} **exists** as a diffeomorphism from U' onto some small neighborhood around 0 in U.

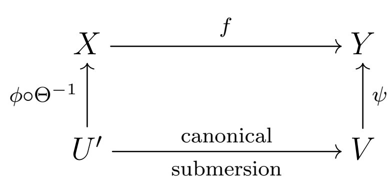
By construction,

$$\theta = \text{canonincal submersion} \circ \Theta, \text{ i.e. } \theta \circ \boldsymbol{\Theta^{-1}} = \text{canonincal submersion}.$$

This gives us the commutative diagram



Hence it suffices to replace U with U' and ϕ with $\phi \circ \Theta^{-1}$ to get the desired commutative diagram



which proves the theorem. **QED**

We observe from the proof of the theorem that if $f\colon X \to Y$ is a submersion at x, then it is also a **submersion for all points in a neighborhood of** x. For, local parametrization $\phi\colon U \to X$ of the proof also parametrizes any point in the image of ϕ which is an open subset around x (open because ϕ is a diffeomorphism onto its image).

Given a map $f\colon X \to Y$ and a point $y \in Y$, we would like to study the **fiber of f over y**, i.e. the set

$$f^{-1}(y) = \{x \in X : f(x) = y\} \subseteq Y.$$

Be aware

In general, there is no reason for that set $f^{-1}(y)$ has any nice geometric structure.

But life is much nicer in the world of submersions. So suppose that $f\colon X \to Y$ is a **submersion at a point $x \in X$** with $f(x) = y$ or in other words $x \in f^{-1}(y)$. By the Local Submersion Theorem, we can choose local coordinates around x and y such that, expressed in these local coordinates, $y = (0, \dots, 0)$ and f becomes the canonical submersion. Let $V \subset X$ be the chosen local neighborhood around x on which the local coordinates are defined. We write $u_1, \dots, u_n$ for the local coordinate functions. Expressed in these local coorindates f becomes

$$f(u_1, \dots, u_n) = (u_1, \dots, u_m).$$

Moreover, still in these coordinates, the fiber over y is the set of points

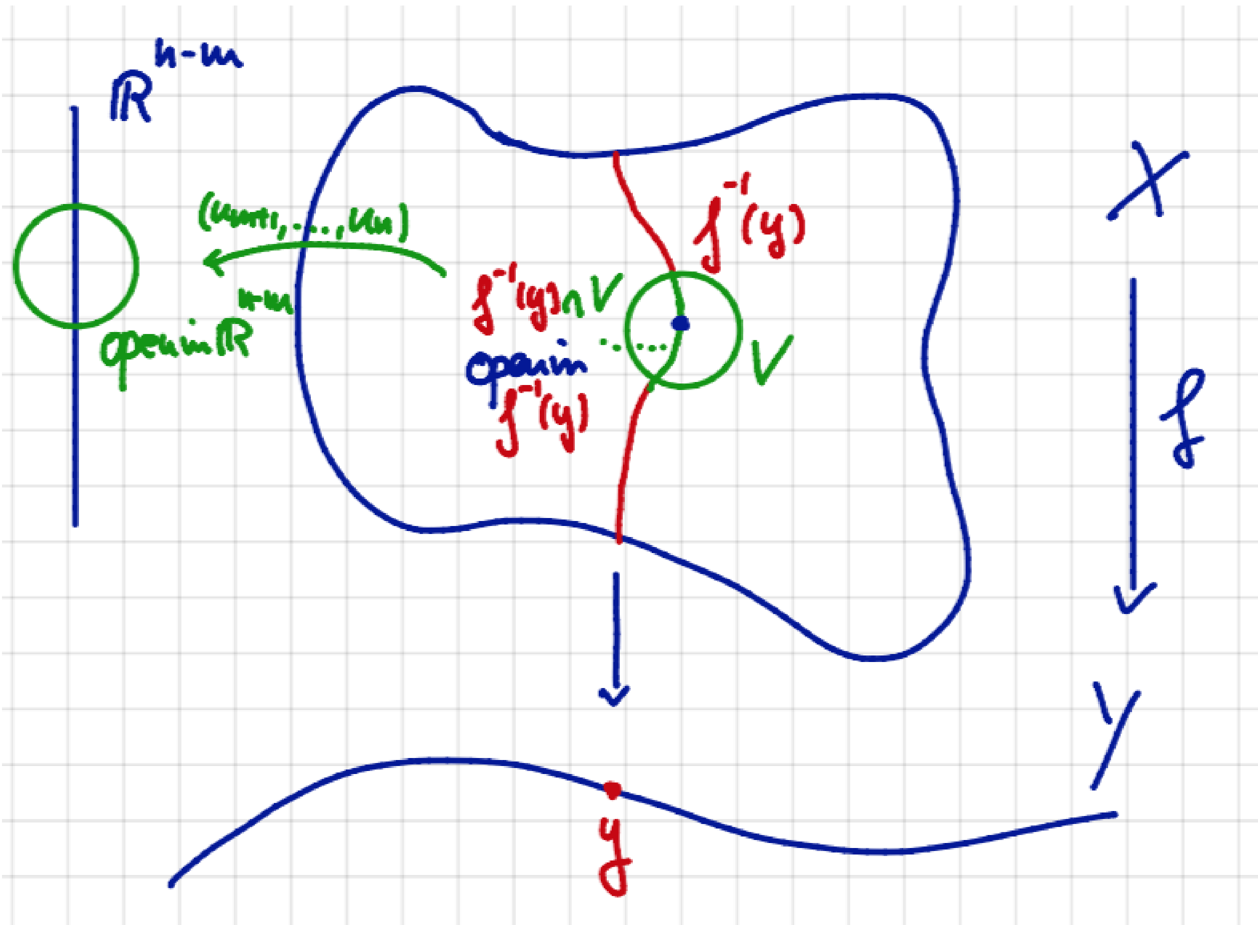
$$f^{-1}(y) \cap V = \{p \in V : u_1(p) = \cdots = u_m(p) = 0\}.$$

Hence we can use the remaining functions $u_{m+1}, \dots, u_n$ to define a local coordinate system on $f^{-1}(y) \cap V$ which is an **open subset in $f^{-1}(y)$**. With these local coordinates, $f^{-1}(y)$ looks like Euclidean space $\mathbb{R}^{m-n}$ in a neighborhood of x.

We would like this to be the case for every point in the fiber $f^{-1}(y)$. This is not always the case. So let us give the desired case a name:

Regular values

For a smooth map of manifolds $f\colon X \to Y$, a point $y \in Y$ is called a **regular value for f** if $df_x\colon T_x(X) \to T_y(Y)$ at every point $x \in X$ such that $f(x) = y$.



Then the above argument shows the following important result:

Preimage Theorem

If y is a regular value for $f\colon X \to Y$, then the fiber $f^{-1}(y)$ over y is a **submanifold of X**, with $\dim f^{-1}(y) = \dim X - \dim Y$.

As a first application, we can show once again that spheres are smooth manifolds.

Example: Spheres at preimages

Let $f\colon \mathbb{R}^{k+1} \to \mathbb{R}$ be the map

$$x = (x_1, \ldots, x_{k+1}) \mapsto |x|^2 = x_1^2 + \cdots + x_{k+1}^2.$$

The derivative dfa at the point $a = (a_1, \ldots, a_{k+1})$ has the matrix $(2a_1 \ldots 2a_{k+1})$. Thus $df_a\colon \mathbb{R}^{k+1} \to \mathbb{R}$ is surjective unless $f(a) = 0$, so every nonzero real number is a regular value of f. In particular, we get again that the sphere $S^k = f^{-1}(1)$ is a k-dimensional manifold.

Since regular values are so nice, we also want to have a name for other values:

Critical values

For a smooth map of manifolds $f\colon X \to Y$, a point $y \in Y$ which is not a regular value, is called a **critical value for f**.
Note that critical values got their name from the fact that $f^{-1}(y)$ can be very complicated if y is critical.

Note that all values y which are not in the image of f are also regular values for f. For, if $f^{-1}(y)$ is the **empty set**, then there is no condition to be satisfied.

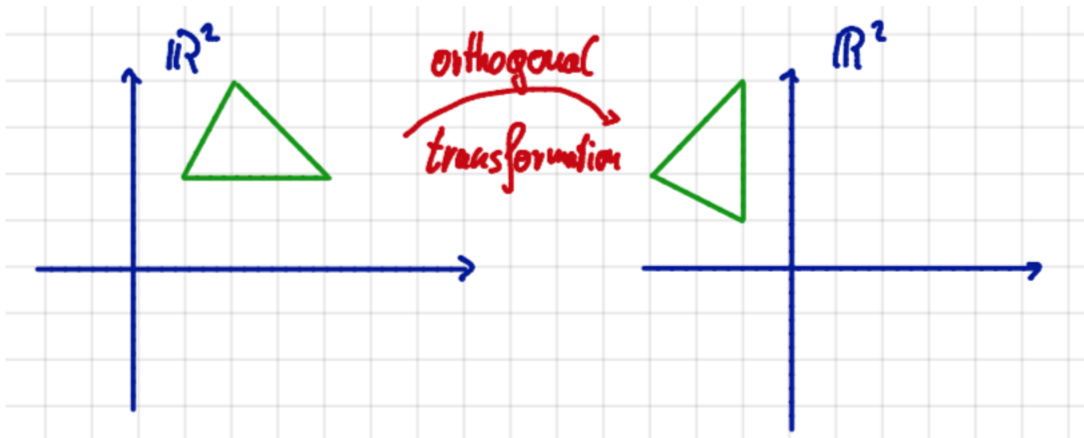
Summary for regular values

Suppose $f\colon X \to Y$ is a smooth map of manifolds. Then **y being a regular value for f** has the following meaning:

- when $\dim X > \dim Y$, then f is a submersion at each point $x \in f^{-1}(y)$;
- when $\dim X = \dim Y$, then f is a local diffeomorphism at each point $x \in f^{-1}(y)$;
- when $\dim X < \dim Y$, then y is not in the image of f; for, all values in the image are critical (df_x cannot be surjective when $\dim T_x(X) < \dim T_{f(x)}(Y)$).

Matrix subgroups are manifolds

A very important application of the Preimage Theorem, is that we can use it to show that various matrix groups sare smooth manifolds. Let $M(n)$ denote the space of real $n \times n$-matrices. It is isomorphic as a vector space to R^{n^2} (we can write every $n \times n$-matrix as a column vector of length n^2). Let $O(n)$ be the subgroup of matrices A in $M(n)$ which satisfy $AA^t = I$ where A^t denotes the transpose of A and I is the $n \times n$-identity matrix. Note that $O(n)$ is the subgroup of matrices which preserve the scalar product of vectors. In particular, matrices in $O(n)$ preserve distances in $\mathbb{R}^n$.



Our goal is to show that $O(n)$ **is a smooth manifold of dimension** $n(n-1)/2$.

First, we note that AA^t is a symmetric matrix. For

$$(AA^t)^t = (A^t)^t A^t = AA^t.$$

The subspace $S(n)$ of symmetric matrices in $M(n)$ is a smooth submanifold of $M(n)$ of dimension $\mathbb{R}^k$ with $k = n(n+1)/2$ (everything below the diagonal is determined by what happens above the diagonal such that there are $n(n+1)/2$ free entries). We define the map

$$f \colon M(n) \to S(n),\ A \mapsto AA^t.$$

This map is smooth, since multiplication of matrices is smooth and taking transposes is obviously smooth as well.

Now we observe $O(n) = f^{-1}(I)$. Hence, in order to show that $O(n)$ is a smooth manifold, we just need to show that I **is a regular value for** f. So let us compute the derivative of f at a matrix A:

$$\begin{aligned} df_A(B) &= \lim_{s\to 0} \frac{f(A+sB)-f(A)}{s} \\ &= \lim_{s\to 0} \frac{(A+sB)(A+sB)^t - AA^t}{s} \\ &= \lim_{s\to 0} \frac{(A+sB)(A^t+sB^t) - AA^t}{s} \\ &= \lim_{s\to 0} \frac{AA^t + sBA^t + sAB^t + s^2BB^t - AA^t}{s} \\ &= \lim_{s\to 0} \frac{sBA^t + sAB^t + s^2BB^t}{s} \\ &= \lim_{s\to 0} BA^t + AB^t + sBB^t \\ &= AB^t + BA^t. \end{aligned}$$

In order to check that I is a regular value, we need to show that

$$df_A \colon T_A(M(n)) \to T_{f(A)}(S(n))$$

is surjective for all $A \in O(n)$. Since $M(n) \cong R^{n^2}$ and $S(n) \cong \mathbb{R}^{n(n+1)/2}$ are diffeomorphic to Euclidean spaces, we have

$$T_A(M(n)) = M(n) \text{ and } T_{f(A)}(S(n)) = S(n).$$

Hence, given a matrix $C \in S(n)$, we need to show that there is a matrix $B \in M(n)$ with $df_A(B) = BA^t + AB^t = C$.

Since C is symmetric, we have $C = \frac{1}{2}(2C) = \frac{1}{2}(C + C^t)$. Since $AB^t = (BA^t)^t$, we set $B = \frac{1}{2}CA$. Then, using $AA^t = I$, we get

$$df_A(B) = (\frac{1}{2}CA)A^t + A(\frac{1}{2}CA)^t = \frac{1}{2}CAA^t + \frac{1}{2}AA^tC^t = \frac{1}{2}C + \frac{1}{2}C^t = C.$$

Thus I is a regular value, and $O(n)$ is a submanifold of $M(n)$. We can also calculate the dimension of $O(n)$:

$$\dim O(n) = \dim M(n) - \dim S(n) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

Lie groups

The manifold $O(n)$ is an example of a very important class of smooth manifolds. For, $O(n)$ is both a smooth manifold and a group such that the group operations are smooth. For both the multiplication map

$$O(n) \times O(n) \to O(n),\ (A,B) \mapsto AB$$

and the map of forming the inverse

$$O(n) \to O(n),\ A \mapsto A^{-1}$$

are smooth (for the latter note $A^{-1} = A^t$ for $A \in O(n)$, but taking inverse is also smooth for other matrix groups).

In general, a group which is also a manifold such that the group operations are smooth is called a **Lie group**.

Lie groups are extremely interesting and important and have a rich and exciting theory. For example, the tangent space at a Lie group at the identity element is a Lie algebra, a vector space with a certain additional operation. Such Lie algebras can be classified completely. Lie groups and Lie algebras play an important role in many different areas of mathematics and physics.

LECTURE 8

Milnor's proof of the Fundamental Theorem of Algebra

Last time, we forgot to mention a useful fact about tangent spaces of submanifold given as the preimage of a regular value. We remedy this sin of omission today before we move on.

Tangent space of regular fibers

Let Z be the preimage of a regular value $y \in Y$ under the smooth map $f\colon X \to Y$. Then the kernel of the derivative

$$df_x\colon T_x(X) \to T_y(Y)$$

at any point $x \in Z$ is the tangent space to $T_x(Z)$.

Proof: Since $f(Z) = y$, f is constant on Z. Therefore, df_x vanishes on the subspace $T_x(Z) \subset T_x(X)$. Hence df_x sends all of $T_x(Z)$ to zero. In other words, $T_x(Z) \subseteq \operatorname{Ker} df_x$.

But df_x is surjective, since f is a submersion at any regular point. Hence the dimension of the kernel of df_x is

$$\dim T_x(X) - \dim T_y(Y) = \dim X - \dim Y = \dim Z.$$

Hence $T_x(Z)$ is a subspace of the kernel of df_x of the same dimension as $\operatorname{Ker} df_x$. Thus $T_x(Z) = \operatorname{Ker} df_x$. **QED**

The Stack of Records Theorem

In order to make the final preparations for Milnor's proof, we have a closer look at a specific situation for regular values.

Suppose $f\colon X \to Y$ is a smooth map with $\dim X = \dim Y$ and X **compact**. Let $y \in Y$ **be a regular value** for f.

Let x be a point in $f^{-1}(y)$. Since y is a regular value, x is a regular point, i.e. df_x is surjective. But, since $\dim X = \dim Y$, this implies df_x is an isomorphism. Hence f **is a local diffeomorphism at** x.

Let $V \subset X$ and $U \subset Y$ be open neighborhoods around x and y, respectively, such that **$f_{|V} : V \to U$ is a diffeomorphism**.

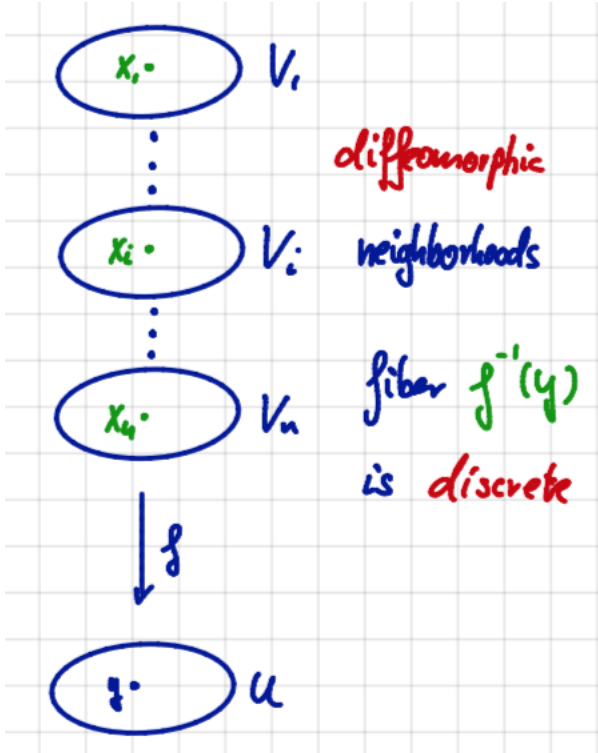
Now suppose x' is another point in $f^{-1}(y)$ with $x \neq x'$. Then $df_{x'}$ is an isomorphism as well, and we can choose an open neighborhood $V' \subset X$ around x' such that $f_{|V'}$ is a diffeomorphism onto an open subset $U' \subset Y$ containing y.

Then **V and V' are disjoint**. For, if $V \cap V' \neq \emptyset$, then f restricts to a diffeomorphism from $V \cap V'$ onto $U \cap U'$. Since $y \in U \cap U'$ and $f(x) = y = f(x')$, this would imply $x = x' \in V \cap V'$. So if $x \neq x'$, we must have $V \cap V' \emptyset$.

Hence all **the points in $f^{-1}(y)$ lie in pairwise disjoint open subsets of X**. We conclude that $f^{-1}(y)$ is discrete. Since the subset $\{y\}$ is closed in Y, the fiber $f^{-1}(y)$ is a closed subset of X. Since X is compact, this implies that **$f^{-1}(y)$ is compact** as well (closed subsets in compact spaces are compact). Hence as a compact and discrete space, **$f^{-1}(y)$ is a finite set**.

(For, given a compact discrete subset S in $\mathbb{R}^n$. Assume S was not finite. Since S is bounded, there is an $\epsilon > 0$ such that S is contained in the n-dimensional box with edges of length ϵ and center 0. Divide this box into 2^n n-dimensional boxes of equal size. The lenght of their edge is $\epsilon/2$. If S was infinite there must be at least one small box which still contains infinitely many points of S. We take this box and divide it into 2^n n-dimensional boxes of equal size. The lenght of their edges is now $\epsilon/4$. Again, if S was infinite there must be at least one of the smaller boxes which still contains infinitely many points of S.
By repeating the argument, we see that we can find an infinite sequence of points in S which converges. Since S is closed, any convergent infinite sequence of points in S must have a limit in S. Call this limit s. But then the subset $\{s\}$ would not be open in S, since every open subset of $\mathbb{R}^n$ containing s would also contain other points of S. Hence S would not be discrete. QED)

Let $f^{-1}(y) = \{x_1, \dots, x_n\}$. We can pick finitely many open subets $W_1, \dots, W_n$ in X with $x_i \in W_i$ which map diffeomorphically onto open subsets $U_1, \dots, U_n$ in Y each containing y. The finite intersection $U := U_1 \cap \cdots \cap U_n$ is open in Y and with $y \in U$. The inverse image $f^{-1}(U)$ is a disjoint union of open subsets $V_1, \dots, V_n$ and each V_i is mapped by f diffeomorphically onto U and $x_i \in V_i$.



Hence we have shown the following very useful result:

Stack of Records Theorem

Suppose $\dim X = \dim Y$, $f\colon X \to Y$ is a smooth map and X is compact. Let $y \in Y$ be a regular value for f. Then the set $f^{-1}(y)$ is a discrete finite subset $\{x_1, \ldots, x_n\}$ of X, and we can choose an open neighborhood $U \subset Y$ around y such that $f^{-1}(U) \subset X$ is the disjoint union $V_1 \cup \cdots \cup V_n$ of open subsets of X with $x_i \in V_i$ and f maps each V_i diffeomorphically onto U.

Aside

If in addition to the assumptions of the theorem all values in Y are regular, then $X \to Y$ is an example of a covering. In Topology, a continuous map $f\colon X \to Y$ is an (unramified) covering if every point in Y has an open neighborhood U such that $f^{-1}(U)$ is the disjoint union of open sets V_i such that f maps each V_i homeomorphically onto U. Coverings play an important role in Topology and Homotopy Theory.

Since $f^{-1}(y)$ is finite, it makes sense to talk about the number of elements in $f^{-1}(y)$ which we denote by $\#f^{-1}(y)$.

Locally constant fiber

The function $y \mapsto \#f^{-1}(y)$ on the set of regular points for f is **locally constant**, i.e. for every regular value y there is an open neighborhood $U \subset Y$ of y such that $\#f^{-1}(y) = \#f^{-1}(y')$ for all $y' \in U$.

Proof: Given a regular value y, let $x_1, \ldots, x_n$ be the points in $f^{-1}(y)$. We just learned that there is an open neighborhood U of y such that $f^{-1}(U) = V_1 \cup \cdots \cup V_n$ is the pairwise disjoint union of open neighborhoods V_i of x_i which all map diffeomorphically onto open subset U. This means that for every point $y' \in U$, there is exactly one point in V_i which maps to y'. And these are the only points which map onto y'. Hence $\#f^{-1}(y') = \#f^{-1}(y)$. **QED**

A short detour to general topology

To know that a function is locally constant can be very convenient in many situations. For example, locally constant functions on connected spaces are constant.

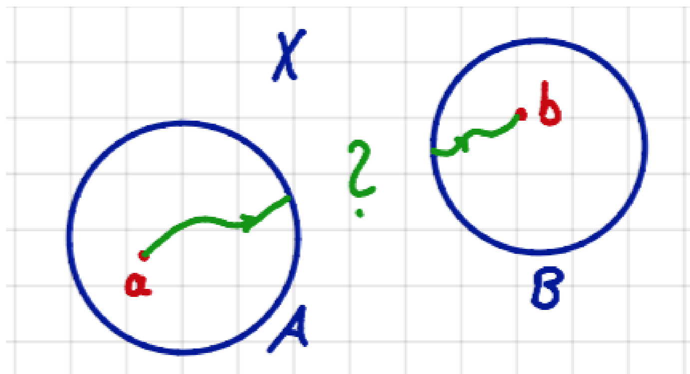
Recall that a topological space X is called **connected** if X cannot be written as the union of two nonempty disjoint open subsets; or equivalently, if X and $\emptyset$ are the only subsets which are both open and closed in X.

Connectedness is a "global" property of a topological space, i.e. it is invariant under homeomorphisms. In particular, two spaces cannot be homeomorphic if one is connected and the other is not. Familiar examples of connected spaces are intervals in $\mathbb{R}$. For example, the closed interval $[0,1]$ is connected.

The criterion for connectedness is rather elegant to state, but it does not tell us if we can actually "walk" from one point to another, as one would expect for a connected space. This is the point of a related and more concrete property. A topological space X is called **path-connected** if for any two points $x,y \in X$ there is a continuous map $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Again, path-connectedness is a topological property, i.e. it is preserved under homeomorphisms.

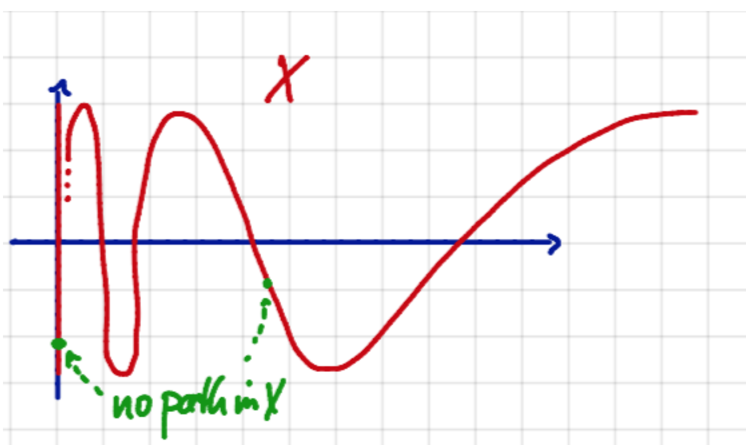
Path-connectedness is the stronger property, i.e. **if a space is path-connected, then it is also connected**. For, suppose X is path-connected. If X was not connected, then there would be two disjoint nonempty open subsets A and B with $X = A \cup B$. Since A and B are nonempty, we can choose two points $a \in A$

and $b \in B$. Since X is path-connected, there is a continuous map $\gamma\colon [0,1] \to X$ with $\gamma(0) = a$ and $\gamma(1) = b$. Hence $0 \in \gamma^{-1}(A) \subset [0,1]$ and $1 \in \gamma^{-1}(B) \subset [0,1]$. Since A and B are disjoint and open, both $\gamma^{-1}(A)$ and $\gamma^{-1}(B)$ are disjoint and open in $[0,1]$. Since $X = A \cup B$, we would have $[0,1] = \gamma^{-1}(A) \cup \gamma^{-1}(B)$ which contradicts the fact that $[0,1]$ is connected. Hence X must be connected.



But be aware that **there are connected spaces which are not path-connected**. A standard example is the subspace

$$X = \{(x, \sin(\log x)) \in \mathbb{R}^2 : x > 0\} \cup (0 \times [-1,1]).$$



Though the usual examples of connected spaces we will meet are path-connected. For example, every sphere is path-connected, and every sphere with finitely many points removed is still path-connected.

We conclude our detour with a lemma we will use in the next section. Given a map $f\colon X \to S$ from a topological space X to any set$S$. Recall that f is called

locally constant if for every $x \in X$ there is an open neighborhood $U_x \subset X$ such that $f_{|U_x}$ is constant.

A useful lemma

Let X be a connected space and $f\colon X \to S$ be locally constant. Then f is constant.

Proof: Let $s \in S$ be a value of f, i.e. $s = f(x)$ for some $x \in X$. We can write X as the **disjoint union** of the sets

$$A = \{x \in X : f(x) = s\} \text{ and } B = \{x \in X : f(x) \neq s\}.$$

Since f is locally constant, both A and B are open. For if $a \in A$, then there is an open neighborhood $U_a \subset A$ with $f(U_a) = \{s\}$, i.e. $U_a \subset A$. Similarly, if $b \in B$, then there is an open neighborhood $U_b \subset X$ with $f(U_b) = \{f(b)\}$, i.e. $U_b \subset B$. But since X is connected and $A \neq \emptyset$, we must have $A = X$, and f is constant. **QED**

Milnor's proof of the Fundamental Theorem of Algebra

Now we are ready to see how Milnor used the previous ideas for a simple proof of the following important result:

Fundamental Theorem of Algebra

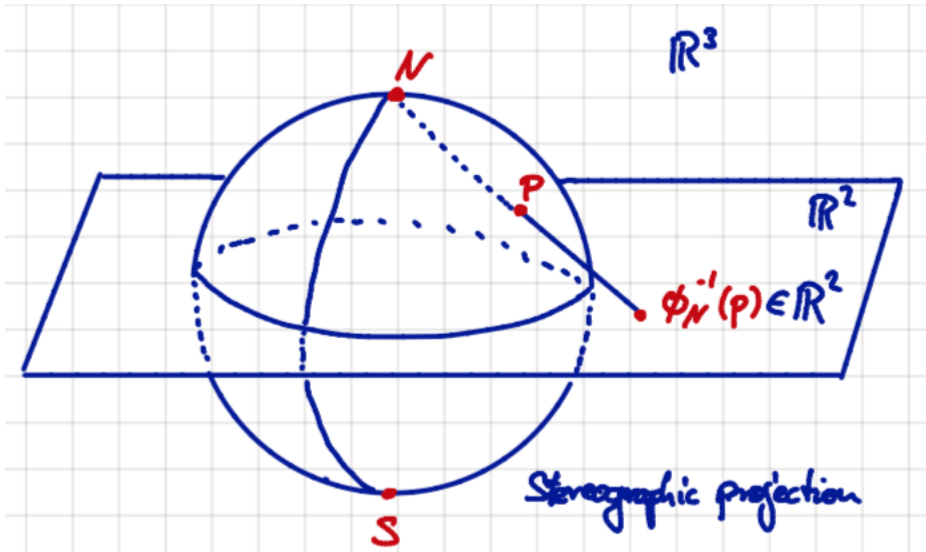
Every nonconstant complex polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

with $a_n \neq 0$ must have a zero.

As a consequence, **$P(z)$ must have exactly n zeroes** when we count them with multiplicities.

We are going to identify the complex numbers $\mathbb{C}$ with the points in real plane $\mathbb{R}^2$, but we keep in mind how that we can mutliply and form inverses for points in $\mathbb{C}$. To prove the theorem we need to extend the map $P\colon \mathbb{C} \to \mathbb{C}$ to a map on a **compact** space. Recall that S^2 is a compact subspace of $\mathbb{R}^3$ and that we can relate S^2 and the real plane $\mathbb{R}^2$ via stereographic projection:



The formulae for the projection from the northpole $N = (0,0,1) \in S^2$ are

$$\phi_N^{-1}\colon S^2 \setminus \{N\} \to \mathbb{R}^2,\ (x_1,x_2,x_3) \mapsto \frac{1}{1-x_3}(x_1,x_2) \text{ and}$$
$$\phi_N\colon \mathbb{R}^2 \to S^2 \setminus \{N\},\ (x_1,x_2) \mapsto \frac{1}{1+|x|^2}\left(2x_1,2x_2,|x|^2-1\right).$$

The formulae for the projection from the south pole $S = (0,0,-1) \in S^2$:

$$\phi_S^{-1}\colon S^2 \setminus \{S\} \to \mathbb{R}^2,\ (x_1,x_2,x_3) \mapsto \frac{1}{1+x_3}(x_1,x_2) \text{ and}$$
$$\phi_S\colon \mathbb{R}^2 \to S^2 \setminus \{S\},\ (x_1,x_2) \mapsto \frac{1}{1+|x|^2}\left(2x_1,2x_2,1-|x|^2\right).$$

Considering our polynomial P as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ we define a new map

$$\mathbf{f}\colon \mathbf{S^2} \to \mathbf{S^2},\ \begin{cases} f(x) := \phi_N \circ P \circ \phi_N^{-1}(x) & \text{for all } x \in S^2 \setminus \{N\} \\ f(N) := N & \text{for } x = N. \end{cases}$$

Claim: The map f is smooth.

Since ϕ_N and ϕ_N^{-1} are smooth and polynomials are smooth as well, it is clear that f is smooth at all points which are not the northpole. It remains to show that it is also smooth in a neighborhood of N.

In order to do this we use the projection from the south pole and define a map

$$Q\colon \mathbb{C} \to \mathbb{C} \text{ by } Q := \phi_S^{-1} \circ f \circ \phi_S.$$

Comparing the definitions of f and Q, we need to calculate

$$\begin{aligned}
\phi_N^{-1} \circ \phi_S(x_1,x_2) &= \phi_N^{-1}\left(\frac{1}{1+|x|^2}(2x_1,2x_2,|x|^2-1)\right) \\
&= \frac{1}{1-\frac{1}{1+|x|^2}}\left(\frac{2x_1}{1+|x|^2},\frac{2x_2}{1+|x|^2}\right) \\
&= \frac{1+|x|^2}{2|x|^2}\left(\frac{2x_1}{1+|x|^2},\frac{2x_2}{1+|x|^2}\right) \\
&= \frac{1}{|x|^2}(x_1,x_2).
\end{aligned}$$

Remembering complex conjugation $z \mapsto \bar{z}$ on $\mathbb{C}$, we can rewrite this as:

$$\phi_N^{-1} \circ \phi_S(z) = \frac{z}{|z|^2} = 1/\bar{z} \text{ for all } z \in \mathbb{C} \setminus \{0\}.$$

Similarly, we also get

$$\phi_S^{-1} \circ \phi_N(z) = \frac{z}{|z|^2} = 1/\bar{z} \text{ for all } z \in \mathbb{C} \setminus \{0\}.$$

Thus we get

$$\begin{aligned}
Q(z) &= \phi_S^{-1} \circ \phi_N \circ P \circ \phi_N^{-1} \circ \phi_S(z) \\
&= \phi_S^{-1} \circ \phi_N(P(1/\bar{z})) \\
&= \phi_S^{-1} \circ \phi_N(a_n\bar{z}^{-n} + a_{n-1}\bar{z}^{n-1} + \cdots + a_1\bar{z}^{-1} + a_0) \\
&= 1/\left(\bar{a}_n z^{-n} + \bar{a}_{n-1}z^{n-1} + \cdots + \bar{a}_1 z^{-1} + \bar{a}_0\right) \\
&= z^n/\left(\bar{a}_n + \bar{a}_{n-1}z + \cdots + \bar{a}_1 z^{n-1} + \bar{a}_0 z^n\right).
\end{aligned}$$

This shows that Q is smooth at $z = 0$ for

$$Q(0) = \phi_S^{-1}(f(\phi_S(0)) = \phi_S^{-1}(f(N)) = \phi_S^{-1}(N) = 0$$

and hence

$$\begin{aligned}
&\lim_{h\to 0} \frac{Q(h)-Q(0)}{h} \\
&= \lim_{h\to 0} \frac{h^n/\left(\bar{a}_n + \bar{a}_{n-1}h + \cdots + \bar{a}_0 h^n\right) - 0}{h} \\
&= \lim_{h\to 0} h^{n-1}/\left(\bar{a}_n + \bar{a}_{n-1}h + \cdots + \bar{a}_0 h^n\right) \\
&= 0.
\end{aligned}$$

Since smoothness is a local property, Q is smooth in a small open neighborhood of 0. Since ϕ_S and ϕ_S^{-1} are diffeomorphisms and since ϕ_S sends an open neighborhood of N in S^2 to an open neighborhood of 0 in $\mathbb{C}$, this implies that

$$f = \phi_S^{-1} \circ Q \circ \phi_S$$

f is smooth in an open neighborhhod of N.

Next, we observe that the smooth map $f\colon S^2 \to S^2$ has **only finitely many critical points**, i.e. points x where df_x fails to be surjective. For, since ϕ_N and ϕ_N^{-1} are diffeomorphisms, the only points that might be critical for f are the points where P fails to be a local diffeomorphism, and possibly N. But the derivative of P is given by the polynomial

$$dP_z = P'(z) = \sum_{j=1}^{n} j a_j z^{j-1}$$

which has at most $n-1$ zeroes. Hence there are only finitely many z where dP_z is not an isomorphism.

Thus the set R of regular values for f is S^2 with finitely many points removed and is therefore **connected**. This implies that the function

$$R \to \mathbb{Z},\ y \mapsto \#f^{-1}(y),$$

which we have seen is locally constant, must be **constant**.

This enables us to show:

Claim: f is onto.

For, assume there is a $y_0 \in S^2$ with $f^{-1}(y_0) = \emptyset$, i.e. $\#f^{-1}(y_0) = 0$. Then y_0 is a regular value for f by definition. Since the function $y \mapsto \#f^{-1}(y)$ is constant on the set of regular values, it would have to be zero for every regular value. Hence $\#f^{-1}(y)$ would be nonzero only for cricital values y. But that would mean that **f had only finitely many values**. Since f is continuous and S^2 connected, this would imply that f is constant. (If f had different values $y_1, \ldots, y_k \in S^2$, then $S^2 = f^{-1}(y_1) \cup \cdots \cup f^{-1}(y_k)$ with $f^{-1}(y_i) \cap f^{-1}(y_j) \neq \emptyset$ and each $f^{-1}(y_1)$ would be nonempty and open (and closed), since f is continuous. That is not possible, since S^2 is connected.) But P is not constant, and ϕ_N and ϕ_N^{-1} are diffeomorphisms. Thus **f is not constant**. We conclude that **f must be onto**.

Conclusion: In particular, $f^{-1}(S) \neq \emptyset$ and there must be at least one point $p \in S^2$ with $f(p) = S$. Since ϕ_N is a diffeomorphism and $\phi_N(0) = S$, p must satisfy $P(\phi_N^{-1}(p)) = 0$. Hence **$z := \phi_N^{-1}(p) \in \mathbb{C}$ is a zero of P. QED**

LECTURE 9

A brief excursion into Lie groups - Part 1

Lie groups

A **Lie group** is a group G which is also a smooth manifold such that the two maps

$$\mu\colon G \times G \to G,\ (g,h) \mapsto g \cdot h$$

and

$$\iota\colon G \to G,\ g \mapsto g^{-1}$$

corresponding to the two group operations of multiplication and taking inverses, respectively, are both smooth. (We usually omit the dot and just write gh instead of $g \cdot h$.)
In fact, we can summarize the condition that μ and ι are smooth by requiring that

$$G \times G \to G,\ (g,h) \mapsto gh^{-1}$$

is smooth.

If G is a Lie group, then any element $g \in G$ defines maps

$$L_g \text{ and } R_g\colon G \to G,$$

called **left translation** and **right translation**, respectively, by

$$L_g(h) = gh \text{ and } R_g(h) = hg.$$

Since L_g can be expressed as the composition of smooth maps

$$G \xrightarrow{i_g} G \times G \xrightarrow{\mu} G,$$

with $i_g(h) = (g,h)$, it follows that L_g is smooth. It is actually a **diffeomorphism of G**, because $L_{g^{-1}}$ is a smooth inverse for it. Similarly, $R_g\colon G \to G$ is a diffeomorphism. In fact, many of the important properties of Lie groups follow from the fact that we can systematically map any point to any other by such a global diffeomorphism. This translation makes the study of Lie groups much more accessible compared to arbitrary smooth manifolds. In particular, we can move an open neighborhood around any point in G to make it an open neighborhood of the identity element. Hence, in a Lie group, we basically only need to study neighborhoods of the identity element.

Here are some simple examples of Lie groups:

- The real numbers $\mathbb{R}$ and Euclidean space $\mathbb{R}^n$ are Lie groups under addition, because the coordinates of $x - y$ are linear and therefore smooth functions of (x,y).
- Similarly, $\mathbb{C}$ and $\mathbb{C}^n$ are Lie groups under addition.
- Any finite group with the discrete topology is a (compact) Lie group.
- Suppose G is a Lie group and $H \subseteq G$ is an open subgroup (i.e. a subgroup which is also an open subspace). Then H is a Lie group as well.
- The set $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ of nonzero real numbers is a 1-dimensional Lie group under multiplication. The subset $\mathbb{R}^+$ of positive real numbers is an open subgroup, and is thus itself a 1-dimensional Lie group (still under multiplication).
- The set $\mathbb{C}^*$ of nonzero complex numbers is a 2-dimensional Lie group under complex multiplication.
- The unit circle $S^1 \subset \mathbb{C}^*$ is a Lie group under the operations induced by multiplication of complex numbers.
- A finite product of k copies of S^1 is a Lie group. We denote it by $\mathbb{T}^k$. In particular, the 2-dimensional torus $\mathbb{T}^2 = S^1 \times S^1$ is a Lie group.
- More generally, the product of Lie groups is again a Lie group.

We will see more examples below. But before, we introduce the notion of maps between Lie groups which respect the Lie group structure.

Lie group homomorphisms

If G and H are Lie groups, a **Lie group homomorphism** from G to H is a smooth map $F: G \to H$ that is also a group homomorphism. It is called a **Lie group isomorphism** if it is also a diffeomorphism, which implies that it has an inverse that is also a Lie group homomorphism. In this case, we say that G and H are isomorphic Lie groups.

Here are some examples of Lie group homomorphisms:

- The inclusion map $S^1 \hookrightarrow \mathbb{C}$ is a Lie group homomorphism.
- Considering $\mathbb{R}$ as a Lie group under addition, and R^* as a Lie group under multiplication, the map

$$\exp\colon \mathbb{R} \to \mathbb{R}^*,\ t \mapsto e^t$$

is smooth, and is a Lie group homomorphism, since $e^{s+t} = e^s e^t$. The image of exp is the open subgroup R^+ consisting of positive real numbers. In fact, $\exp\colon \mathbb{R} \to \mathbb{R}^+$ is a Lie group isomorphism with inverse $\log\colon \mathbb{R}^+ \to \mathbb{R}$.

- Similarly, $\exp\colon \mathbb{C} \to \mathbb{C}^*$ given by $\exp(z) = e^z$ is a Lie group homomorphism. It is surjective but not injective, because its kernel consists of the complex numbers of the form $2\pi ik$, where k is an integer.
- The map
$$\epsilon\colon \mathbb{R} \to S^1,\, t \mapsto e^{2\pi it}$$
is a Lie group homomorphism whose kernel is the set $\mathbb{Z}$ of integers.
- Similarly, the map
$$\epsilon^n\colon \mathbb{R}^n \to \mathbb{T}^n,\, (t_1,\dots,t_n) \mapsto (e^{2\pi it_1},\dots,e^{2\pi it_n})$$
is a Lie group homomorphism whose kernel is $\mathbb{Z}^n$.
- If G is a Lie group and $g \in G$, **conjugation by** g is the map $C_g\colon G \to G$ given by $C_g(h) = ghg^{-1}$. Because group multiplication and inversion are smooth, C_g is smooth and it is a group homomorphism:
$$C_g(hh') = gh_1hh'g^{-1} = (ghg^{-1})(gh'g^{-1}) = C_g(h)C_g(h').$$
In fact, it is a **Lie group isomorphism**, because it has $C_{g^{-1}}$ as an inverse. A subgroup $H \subseteq G$ is said to be **normal** if $C_g(H) = H$ for every $g \in G$.

Here is an important theorem about Lie group homomorphisms:

Constant Rank Theorem

Let $f\colon G \to H$ be a Lie group homomorphism. Then the derivative df_g has the same rank (as a linear map) for all $g \in G$.

Proof: Let e_G and e_H denote the identity elements in G and H, respectively. Suppose g_0 is an arbitrary element of G. We will show that df_{g_0} has the same rank as df_e. The fact that f is a homomorphism means that for all $g \in G$,
$$f(L_{g_0}(g)) = f(g_0g) = f(g_0)f(g) = L_{f(g_0)}(f(g));$$
or in other words, $f \circ L_{g_0} = L_{f(g_0)} \circ f$. Taking differentials of both sides at the identity and using the chain rule yields
$$df_{g_0} \circ d(L_{g_0})_{e_G} = d(L_{f(g_0)})_{e_H} \circ df_{e_G}.$$

Recall that left multiplication by any element of a Lie group is a diffeomorphism, so both $d(L_{g_0})_{e_G}$ and $d(L_{f(g_0)})_{e_H}$ are isomorphisms. Because composing with an isomorphism does not change the rank of a linear map, it follows that df_{g_0} and df_{e_G} have the same rank. **QED**

Lie group isomorphisms revisited

Every bijective Lie group homomorphism $f\colon G \to H$ is automatically a Lie group isomorphism.
For, there must be a point $g \in G$ where df_g is an isomorphism. Otherwise the Local Immersion and Submersion Theorems would imply that f looked like the canonical immersion or submersion, respectively, and f would not be bijective. By the previous theorem, this implies that df_g is an isomorphism for all $g \in G$. Hence it is a bijective local diffeomorphism everywhere. Bijective local diffeomorphisms are global diffeomorphisms. Since the map is a Lie group homomorphism, it is a Lie group isomorphism.

Now let us study some more interesting examples:

The General Linear Group

The general linear group

$$GL(n) = \{A \in M(n) : \det A \neq 0\}$$

of all invertible $n \times n$-matrices with entries in $\mathbb{R}$, is a smooth manifold of dimension n^2, since it is an **open** subset of $M(n) \cong \mathbb{R}^{n^2}$. To check that it is open, look at its complement

$$M(n) \setminus GL(n) = \{A \in M(n) : \det A = 0\} = \det^{-1}(0).$$

Since $\det\colon M(n) \to \mathbb{R}$ is continuous (it is a polynomial in the entries of the matrix) and since $\{0\}$ is a closed subset of $\mathbb{R}$, $\det^{-1}(0)$ is closed in $M(n)$.

We claim that $GL(n)$ is a Lie group. To show this we need to check that multiplication and taking inverses are smooth operations. Given two matrices A and B in $GL(n)$, the entry in position (i,j) in AB is given by

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Hence $(AB)_{ij}$ is a polynomial in the coordinates of A and B. Thus matrix multiplication

$$\mu\colon GL(n) \times GL(n) \to GL(n)$$

is a smooth map.

Recall that the (i,j)-minor of a matrix A is the determinant of the submatrix of A obtained by deleting the ith row and the jth column of A. By **Cramer's rule** from linear algebra, the (i,j)-entry of A^{-1} is

$$(A^{-1})_{ij} = \frac{1}{\det A} \cdot (1)^{i+j}((j,i)\text{-minor of } A),$$

which is a smooth function of the a_{ij}'s provided $\det A \neq 0$, i.e. the map

$$M(n) \to \mathbb{R},\ A \mapsto (A^{-1})_{ij}$$

is smooth because it depends smoothly on the entries of A. Therefore, the map of taking inverses

$$\iota \colon GL(n) \to GL(n)$$

is also smooth.

$GL(n)$ exists over many bases

In fact, we can matrices with entries in any ring K. We denote the corresponding matrix groups by $M(n,K), GL(n,K), \ldots$. Since $K = \mathbb{R}$ is the most important case for us, we omit mentioning the base when it is clear that we work over $\mathbb{R}$.
Another very important case is $K = \mathbb{C}$. The complex general linear group $GL(n,\mathbb{C})$ is also a Lie group. It is a group under matrix multiplication, and it is an open submanifold of $M(n,\mathbb{C})$ and thus a $2n^2$-dimensional smooth manifold. It is a Lie group, since matrix products and inverses are smooth functions of the real and imaginary parts of the matrix entries.
Note that the determinant is a Lie group homomorphism for both $\mathbb{R}$ and $\mathbb{C}$:

$$\det \colon GL(n,\mathbb{R}) \to \mathbb{R}^* \text{ and } \det \colon GL(n,\mathbb{C}) \to \mathbb{C}^*.$$

For $n = 1$, we just have $GL(1,\mathbb{R}) = \mathbb{R}^*$ and $GL(1,\mathbb{C}) = \mathbb{C}^*$.

The Special Linear Group

Another example of a Lie group is the **special linear group**

$$SL(n) = \{A \in M(n) : \det A = 1\}.$$

Note that $SL(n)$ consists of all transformations of $\mathbb{R}^n$ into itself which preserve volumes and orientations. (We will discuss orientations later.)

In order to show that $SL(n)$ is a manifold, we would like to use the preimage theorem for regular values of the map

$$\det \colon M(n) \to \mathbb{R}.$$

For $SL(n) = \det^{-1}(1)$. To do this, we need to show that 1 is a regular value of det. In fact, we are going to show that 0 is the only critical value of det.

As a preparation, we are going to look at the following general situation.

Euler's identity for homogeneous polynomials

Let $P(x_1, \dots, x_k)$ be a homogeneous polynomial of degree m in k variables. First, we are going to show Euler's identity

$$\sum_i x_i \partial P/\partial x_i = mP. \tag{1}$$

Define a new function Q by

$$Q(x_1, \dots, x_k, t) := P(tx_1, \dots, tx_k) - t^m P(x_1, \dots, x_k).$$

Since P is homogeneous, we know Q is always 0. Hence its derivative with respect to t is zero as well. Hence we get

$$0 = \partial Q/\partial t = \sum_i x_i \partial P/\partial x_i(tx_1, \dots, tx_k) - mt^{m-1} P(tx_1, \dots, tx_k) \tag{2}$$

where we apply the chain rule to the first summand of Q which is the composite $t \mapsto tx \mapsto P(tx)$. Setting $t = 1$ in (2) yields (1).

Fibers of homogeneous polynomials form manifolds

Now we consider our homogeneous polynomial P as a map

$$\mathbb{R}^k \to \mathbb{R}, \ (x_1, \dots, x_k) \mapsto P(x_1, \dots, x_k).$$

We claim that **0 is the only critical value of P**.

The derivative of P at a point $(x_1, \dots, x_k)$ is

$$dP_x \colon \mathbb{R}^k \to \mathbb{R}, \ (z_1, \dots, z_k) \mapsto (\partial P/\partial x_1(x) \dots \partial P/\partial x_k(x)) \cdot \begin{pmatrix} z_1 \\ \vdots \\ z_k \end{pmatrix}$$

$$= \sum_i z_i \partial P/\partial x_i(x).$$

To show that dP_x is nonsingluar, i.e. surjective, it suffices to show that dP_x is nontrivial. But applying dP_x to x and using Euler's identity yields

$$dP_x(x) = \sum_i x_i \partial P/\partial x_i(x_1, \dots, x_k) = mP(x_1, \dots, x_k).$$

Hence **if $x = (x_1, \dots, x_k)$ is not a zero of P, then $dP_x(x)$ is nonzero**. Hence only points in the fiber over 0 might be critical points, and all nonzero real numebrs are regular values of P. This shows that $P^{-1}(a)$ is a $k-1$-dimensional submanifold of $\mathbb{R}^k$ for all $a \neq 0$.

Given two real numbers $a,b > 0$, then $(b/a)^{1/m}$ exists and we if $P(x) = a$, we have

$$P((b/a)^{1/m}x_1, \ldots, (b/a)^{1/m}x_k) = b/aP(x_1, \ldots, x_k) = b.$$

Multiplying each coordinate with $(b/a)^{1/m}$ corresponds to multiplicatin with the diagonal matrix with $(b/a)^{1/m}$ on the diagonal. This map is a linear isomorphism of $\mathbb{R}^k$ to itself. Hence we have the diffeomorphism

$$P^{-1}(a) \to P^{-1}(b),\ (x_1, \ldots, x_k) \mapsto ((b/a)^{1/m}x_1, \ldots, (b/a)^{1/m}x_k).$$

Similarly, if both $a,b < 0$ are negative, then $(b/a)^{1/m}$ exists and the same argument shows that $P^{-1}(a)$ and $P^{-1}(b)$ are diffeomorphic.

Algebraic Geometry in a nutshell

The study of the zeroes of polynomials is the central theme in Algebraic Geometry. This is a classical and fascinating part of pure mathematics. In the past 2-3 decades, strong and fascinating conncetions between Algebraic Geometry and Homotopy Theory have been developed, summarized in the field of Motivic Homotopy Theory. Just ask to learn more about it.

Back to matrices: If we think of the entries in an $n \times n$-matrix A as variables, then $\det A$ is a **homogeneous polynomial of degree n**. It is given by Leibniz' formula

$$\det(A) = \sum_{\sigma} (\operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}) \tag{3}$$

where the sum runs over all permutations of the set $\{1, \ldots, n\}$ and $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation σ. Hence we can apply the previous argument to

$$P = \det \colon M(n) = \mathbb{R}^{n^2} \to \mathbb{R}$$

and get that 0 is the only critical value of det. **Thus the special linear group $SL(n) = \det^{-1}(1)$ is a smooth submanifold of dimension $n^2 - 1$ in $M(n)$.**

LECTURE 10

A brief excursion into Lie groups - Part 2

The Special Linear Group

We continue our study of the **special linear group**

$$SL(n) = \{A \in M(n) : \det A = 1\}.$$

Last time, we learned that $SL(n)$ is a smooth manifold of dimension $n^2 - 1$. The same argument as for $GL(n)$ shows that it even is a Lie group. We will see another argument for that today.

But first we would like to calculate the **tangent space of $SL(n)$ at the identity matrix**.

This space plays a special role for any Lie group. In fact, the translation property of Lie groups implies that the tangent to a Lie group G at any matrix in G is isomorphic to tangent space to G at the identity element. It carries an additional structure and is an example of a **Lie algebra**.

To determine the tangent space at the identity, we use a result we proved last week which said: if $Z = f^{-1}(y) \subseteq X$ is a submanifold defined by a regular value y of a smooth map $f\colon X \to Y$, then $T_x(Z) = \operatorname{Ker}(df_x) \subseteq T_x(X)$.

Hence we need to calculate the **derivative of $\det$ at the identity**.

Recall that the determinant of a matrix A is given by Leibniz' formula

$$\det(B) = \sum_{\sigma} (\operatorname{sgn}(\sigma) \prod_{i=1}^{n} b_{i\sigma(i)}) \tag{4}$$

where the sum runs over all permutations of the set $\{1, \ldots, n\}$ and $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation σ.

Given a matrix A, in the determinant of $B := I + sA$, every summand contains at least a factor s^2 unless it is the product of at least $n-1$ diagonal entries $b_{ii} = 1 + sa_{ii}$ (because we need $n-1$ factors **not containing s** which is only possible when we multiply $n-1$ times 1). But if a permutation $\{1, \ldots, n\}$ leaves

$n-1$ numbers fixed, it also has to leave the remaining one fixed. Hence the only summand in (4) which does not contain a factor s^2 is the summand

$$\prod_{i=1}^{n}(1+sa_{ii}) = (1+sa_{11})\cdots(1+sa_{nn}) = 1 + s\cdot \mathrm{tr}\,(A) + O(s^2).$$

The derivative of the determninant at the identity

$$d(\det)_I \colon T_I(M(n)) = M(n) \to T_1(\mathbb{R}) = \mathbb{R}$$

is then given by

$$\begin{aligned} d(\det)_I(A) &= \lim_{s\to 0} \frac{\det(I+sA) - \det I}{s} \\ &= \lim_{s\to 0} \frac{1 + s\cdot \mathrm{tr}\,(A) + O(s^2) - 1}{s} \\ &= \lim_{s\to 0} \frac{s\cdot \mathrm{tr}\,(A) + O(s^2)}{s} \\ &= \lim_{s\to 0} \mathrm{tr}\,(A) + O(s) \\ &= \mathrm{tr}\,(A). \end{aligned}$$

By the result from the previous lecture, we get

$$T_I(SL(n)) = \mathrm{Ker}\,(d(\det)_I) = \{A \in M(n) : \mathrm{tr}\,(A) = 0\}.$$

In other words, the tangent space to $SL(n)$ at the identity is the space of matrices whose trace vanishes.

The Special Orthogonal Group

Recall that the orthogonal group $O(n)$ is defined as the subset of matrices A in $M(n)$ such $AA^t = I$. This equation implies, in particular, that every $A \in O(n)$ is invertible with $A^{-1} = A^t$. Hence the determinant of an $A \in O(n)$ must satisfy $(\det A)^2 = 1$, i.e. $\det A = \pm 1$. Thus, $O(n)$ splits into two disjoint parts, the subset of matrices with determinant $+1$ and the subset of matrices with determinant -1.

If A and B have determinant -1, then their product AB has determinant $+1$. Hence the subset of matrices with determinant -1 is not closed under multiplication and therefore not a subgroup of $O(n)$. But the other part is a Lie subgroup of $O(n)$ and is called the **Special Orthogonal Group** $SO(n)$:

$$SO(n) = \{A \in O(n) : \det A = 1\} \subset O(n).$$

Unitary and Special Unitary Groups

The **unitary group** $U(n)$ is defined to be

$$U(n) := \{A \in GL(n,\mathbb{C}) : \bar{A}^t A = I\},$$

where $\bar{A}$ denotes the complex conjugate of A, the matrix obtained from A by conjugating every entry of A. A similar argument as for $O(n)$ shows that $U(n)$ is a submanifold of $GL(n,\mathbb{C})$ and that $\dim U(n) = n^2$.

The **special unitary group** $SU(n)$ is defined to be the subgroup of $U(n)$ of matrices of determinant 1.

Some identities

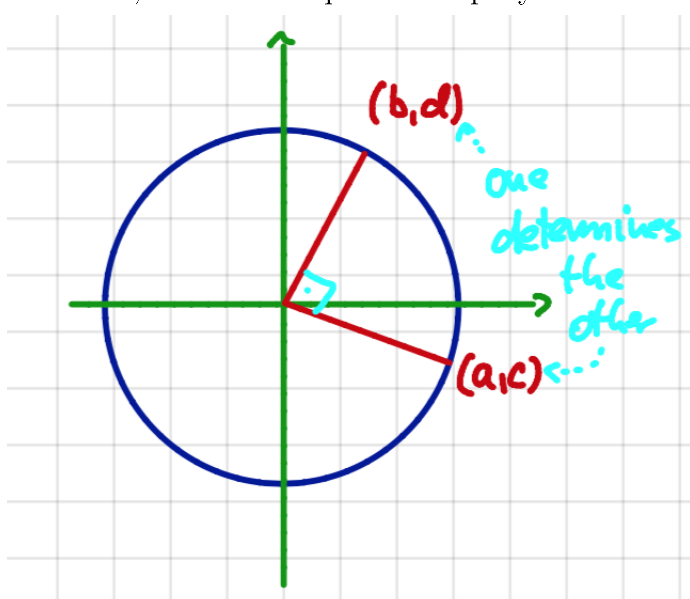
There are a couple of identities, most of which are incidental and do not reflect any deeper pattern. They are interesting nevertheless. For example:

(a) For $n = 1$, $O(1)$ consists of just two points: $O(1) = \{-1, +1\}$.
(b) For $n = 2$, $SO(2)$ is diffeomorphic to S^1:

For, any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SO(2)$ satisfies

$$A^t A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence A corresponds to two points (a,c) and (b,d) on $S^1 \subset \mathbb{R}^2$ whose corresponding vectors are orthongonal to each other. Since we also know $\det A = ad - bc = 1$, one of these points uniquely determines the other



and we can write A as $\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ for some real number t. Now one can check that the map

$$S^1 \to SO(2),\ (\cos t, \sin t) \mapsto \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

is a diffeomorphism and Lie group isomorphism.

(c) For $n = 2$, $SU(2)$ is diffeomorphic to S^3: Any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$ satisfies

$$\bar{A}^t A = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \bar{a}a + \bar{c}c & \bar{a}b + \bar{c}d \\ \bar{b}a + \bar{d}c & \bar{b}b + \bar{d}d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Together with $\det A = ad - bc = 1$ we get four linear equations for the complex numbers a, b, c, d, and their complex conjugates. Unraveling these equations shows that we can write A as

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \text{ with } a\bar{a} + b\bar{b} = 1.$$

Hence A corresponds uniquely to a pair of complex numbers (a,b) which satisfies $a\bar{a} + b\bar{b} = 1$. Since this is exactly the defining condition for elements of $S^3 \subset \mathbb{C}^2$, we see that

$$S^3 \to SU(2), (a,b) \mapsto \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

is a diffeomorphism.

Spin groups

There are other important examples of Lie groups which, in general, do not arise as closed subgroups of $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$. For example, the nth **Spin group** $\mathrm{Spin}(n)$ is the n-dimensional Lie group which is a double cover of $SO(n)$. The latter means that $\mathrm{Spin}(n)$ is equipped with a smooth surjective map $\pi\colon \mathrm{Spin}(n) \to SO(n)$ such that each point in $SO(n)$ has an open neighborhood U such that $\pi^{-1}(U)$ is a disjoint union of open subsets in $Spin(n)$ each of which is mapped diffeomorphically onto U by π. (We have seen covering spaces when we discussed the Stack of Records Theorem.) The map π is part of a short exact sequence of groups

$$1 \to \mathbb{Z}/2 \to \mathrm{Spin}(n) \to SO(n) \to 1.$$

Spin groups can be constructed for example via Clifford algebras. However, there are some exceptional isomorphisms in low dimensions which we can write down:

$$\begin{aligned}\mathrm{Spin}(1) &\cong O(1),\\ \mathrm{Spin}(2) &\cong SO(2),\\ \mathrm{Spin}(3) &\cong SU(2),\\ \mathrm{Spin}(4) &\cong SU(2)\times SU(2),\\ \mathrm{Spin}(6) &\cong SU(4).\end{aligned}$$

Topology of Lie groups

Just as $O(n)$ (this was an exercise), $SO(n)$ is compact (whereas $GL(n)$ is not compact as an open subset of $M(n)$). Similarly, $U(n)$ and $SU(n)$ are compact.

Moreover, note that both $SO(n)$ and its complement are both open and closed in $O(n)$. They are the **two connected components of $O(n)$**. In particular, there is no continuous path in $O(n)$ from a matrix with determinant $+1$ to one with determinant -1. In fact, there is no such path in $GL(n)$:

The real general linear group is not connected

Let γ be a path in $GL(n)$, i.e. a continuous map

$$\gamma\colon [0,1] \to GL(n).$$

Since γ and det are continuous, so is their composite

$$\det\circ\gamma\colon [0,1] \xrightarrow{\gamma} GL(n) \xrightarrow{\det} \mathbb{R}.$$

Hence if $\det(\gamma(0)) > 0$ and $\det(\gamma(1)) < 0$, then the Intermediate Value Theorem from Calculus implies that there must be a real number $t_0 \in (0,1)$ such that $\det(\gamma(t_0)) = 0 \notin GL(n)$. Hence γ would have to leave $GL(n)$.

Thus also $GL(n)$ has two connected components, one of which is an open subgroup consisting to all matrices A with $\det A > 0$. The other one is just an open subset consisting to all matrices A with $\det A < 0$.

The complex general linear group is connected

However, $GL(n,\mathbb{C})$ is path-connected. We see the difference between $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$ most clearly for the case $n = 1$: $GL(1,\mathbb{R}) = \mathbb{R}^*$

is not path-connected, since we cannot cross 0; whereas $GL(1,\mathbb{C}) = \mathbb{C}^*$ is path-connected, since we can just walk around 0 in the plane.
More generally, to show that $GL(n,\mathbb{C})$ is path-connected, it suffices to show that there is path from any matrix $A \in GL(n,\mathbb{C})$ to the identity matrix $I \in GL(n,\mathbb{C})$. Therefore, we define first the function

$$P\colon \mathbb{C} \to \mathbb{C},\ z \mapsto \det(A + z(I - A)).$$

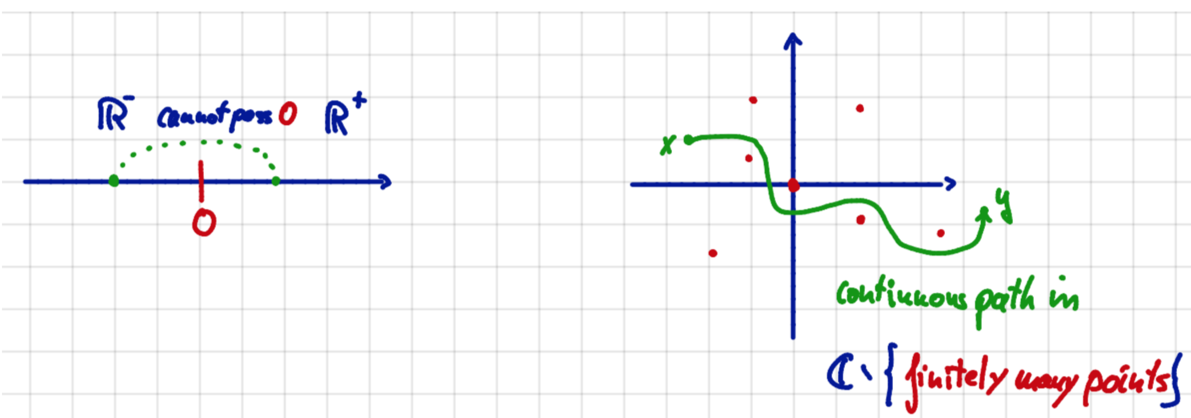
Then we have $P(0) = \det A \neq 0$ and $P(1) = \det I = 1 \neq 0$. Since P is a polynomial of degree n, it has only finitely many zeroes. Since $\mathbb{C} \setminus \{\text{set of finitely many points}\}$ is path-connected, we can find a path $\gamma\colon [0,1] \to \mathbb{C}$ with $\gamma(0) = 1$, $\gamma(1)$ and which avoids the zeroes of P, i.e.

$$P(\gamma(t)) \neq 0 \text{ for all } t.$$

Then the continuous map

$$\Gamma = P \circ \gamma\colon [0,1] \to GL(n,\mathbb{C}),\ t \mapsto A + \gamma(t)(I - A)$$

is the desired path from A to I.



The fact that GL$(n,\mathbb{C})$ is connected while $GL(n,\mathbb{R})$ is not plays a crucial role for orientations of vector spaces, vector bundles, manifolds etc. For, every complex vector space, complex vector bundle, complex manifold, etc has a **natural orientation**. We will get back to this later.

Open neighborhoods of the identity.

Recall that if G is a group and $S \subset G$ is a subset, the **subgroup generated by S** is the smallest subgroup containing S, i.e., the intersection of all subgroups containing S. One can check that the subgroup generated by S is equal to the set of all elements of G that can be expressed as finite products of elements of S and their inverses.

Neighborhoods of the identity

Suppose G is a Lie group, and $W \subset G$ is any neighborhood of the identity. Then

(a) W generates an open subgroup of G.

(b) If G is connected, then W generates G. In particular, an open subgroup in a connected Lie group must be equal to the whole group.

Proof: Let $W \subset G$ be any neighborhood of the identity, and let H be the subgroup generated by W. To simplify notation, if A and B are subsets of G, we write

$$AB := \{ab : a \in A, b \in B\}, \text{ and } A^{-1} := \{a^{-1} : a \in A\}.$$

For each positive integer k, let W_k denote the set of all elements of G that can be expressed as products of k or fewer elements of $W \cup W^{-1}$. As mentioned above, H is the union of all the sets W_k as k ranges over the positive integers.

Now, $W^{-1}1$ is open because it is the image of W under the inversion map, which is a diffeomorphism. Thus, $W_1 = W \cup W^{-1}$ is open, and, for each $k > 1$, we have

$$W_k = W_1 W_{k-1} = \cup_{g \in W_1} L_g(W_{k-1}).$$

Because each L_g is a diffeomorphism, it follows by induction that each W_k is open, and thus H is open as a union of open subsets.

(b) Assume G is connected. We just showed that H is an open subgroup of G. It is an exercise to show that an open subgroup in a connected Lie group is equal to the whole group. **QED**

Lie subgroups

In the previous paragraph we talked about subgroups of a Lie group. But we did not disucss how the subgroup structure relates to the structure as a smooth manifold. Actually, this is a subtle and interesting point that illustrates the importance of the distinction between immersions and embeddings once again. So here is the definition of a Lie subgroup:

Definition of Lie subgroups

A **Lie subgroup** of a Lie group G is an abstract subgroup H such that if there exists a smooth manifold X and an **immersion** $f\colon X \to G$ from X to G such that $H = \operatorname{Im}(f) \subseteq G$ is the image of f, and the group operations on H are smooth, in the sense that $X \times X \xrightarrow{f\times f} G \times G \xrightarrow{\mu} G$ and $X \xrightarrow{f} G \xrightarrow{\iota} G$ are smooth.

Let us have a closer look at this rather complicated definition:

An “abstract subgroup simply means a subgroup in the algebraic sense. The group operations on the subgroup H are the restrictions of the multiplication map μ and the inverse map ι from G to H.

If H were defined to be a submanifold of G, then the multiplication map $H \times H \to H$ and similarly the inverse map $H \to H$ would automatically be smooth, and the definition would be much shorter. But since a Lie subgroup is defined to be an “immersed submanifold”, it is necessary to impose the last condition.

If H is in fact also a submanifold, then life is easier:

Embedded Lie subgroups

If H is an abstract subgroup and a submanifold of a Lie group G, then it is a Lie subgroup of G. In this case, the inclusion map $H \hookrightarrow G$ is an embedding, and we call H an **embedded subgroup**.

Proof: Since H is a subgroup, multiplication and taking inverses in H are just the restrictions of multiplication and taking inverses in G and both have image in H. Since H is a submanifold we can take $X = H$ in the above definiton, the restrcitions of smooth maps to H are again smooth. **QED**

For example, the subgroups $SL(n)$ and $O(n)$ of $GL(n)$ are both submanifolds, and therefore embedded Lie subgroups. Another example is given as follows:

Complex vs Real

One easily verifies that

$$\mathbb{C} \to M(2,\mathbb{R}),\ z = x + iy \mapsto \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

is an embedding. More generally, this map induces an embedding

$$GL(n,\mathbb{C}) \hookrightarrow GL(2n,\mathbb{R})$$

by replacing each entry $z = x + iy$ in $A \in GL(n,\mathbb{C})$ by the block $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$:

$$\begin{pmatrix} x_{11} + iy_{11} & \dots & x_{1n} + iy_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} + iy_{n1} & \dots & x_{nn} + iy_{nn} \end{pmatrix} \mapsto \begin{pmatrix} x_{11} & -y_{11} & \dots & x_{1n} & -y_{1n} \\ y_{11} & x_{11} & & y_{1n} & x_{1n} \\ \vdots & & \ddots & & \vdots \\ x_{n1} & -y_{n1} & \dots & x_{nn} & -y_{nn} \\ y_{n1} & x_{n1} & & y_{nn} & x_{nn} \end{pmatrix}$$

This way, $GL(n,\mathbb{C})$ is an embedded Lie subgroup of $GL(2n,\mathbb{R})$.

Now let us get back to understanding the definition of a Lie subgroup. The subleties of immersed and embedded subgroups can be illustrated by a familiar example:

Example of an immersed but not embedded Lie subgroup

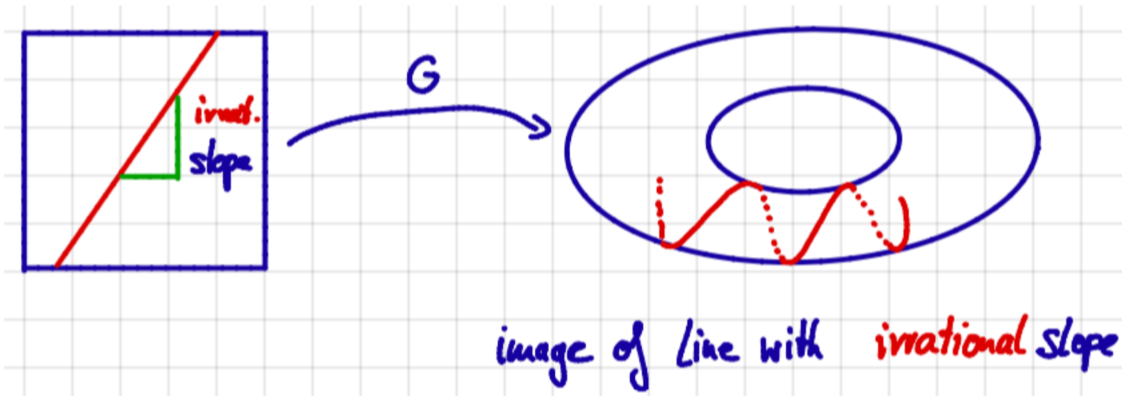
Recall the maps $g\colon \mathbb{R} \to S^1$, $t \mapsto (\cos(2\pi t), \sin(2\pi t))$, and

$$G\colon \mathbb{R}^2 \to S^1 \times S^1 = \mathbb{T}^2,\ G(x,y) = (g(x),g(y))$$

The map G is a local diffeomorphism from the plane onto the torus T^2.
Given a real number α, we defined the map γ_α by

$$\gamma_\alpha\colon \mathbb{R} \to \mathbb{T}^2,\ \gamma(t) = (g(t),g(\alpha \cdot t)).$$

We learned that γ_α is always an immersion, but its image is **not a submanifold** of $\mathbb{T}^2$ ifα is an **irrational** number. However, when α is rational, then $\gamma_\alpha(\mathbb{R})$ is a submanifold of T^2.
After checking that $\gamma_\alpha(\mathbb{R})$ is an abstract subgroup, we see that $\gamma_\alpha(\mathbb{R})$ is in fact a **Lie subgroup of $\mathbb{T}^2$** for every real number α. (Note that, in this example, the smooth manifold X and the smooth map $f\colon X \to G$ in the definition of Lie subgroups is $X = \mathbb{R}$, $f = \gamma_\alpha$, and $H = \gamma_\alpha(\mathbb{R})$.)



For an explanation of why a Lie subgroup is defined in such a complicated way, we refer to a fact we will only be able to appreciate later when we learn more about Lie theory:

Why so complicated?

A fundamental theorem in Lie group theory asserts the existence of a **one-to-one correspondence** between the connected Lie subgroups of a Lie group G and the Lie subalgebras of its Lie algebra $\mathfrak{g}$ (tangent space at the identity with its Lie bracket):

$$\{\text{connected Lie subgroups in } G\} \overset{1-1}{\longleftrightarrow} \{\text{Lie subalgebras in } \mathfrak{g}\}.$$

In the previous example, the Lie algebra of $\mathbb{T}^2$ has $\mathbb{R}^2$ as the underlying vector space, and the one-dimensional Lie subalgebras are all the lines through the origin (with addtion as group operation). Such a line is determined by its slope α. Hence **every** α should correspond to a **Lie subgroup** $\gamma_\alpha(\mathbb{R})$ in $\mathbb{T}^2$.
However, if a Lie subgroup had been defined as a subgroup that is also a submanifold, then one would have to exclude all the lines with irrational slopes as Lie subgroups of the torus. In this case it would not be possible to have a one-to-one correspondence between the connected subgroups of a Lie group and the Lie subalgebras of its Lie algebra. But this correspondence is extremely useful in Lie theory.

The following theorem is a very useful fact which we state here without proof (you can find it in Lee's book, Chapter 7, Theorem 7.21):

Closed Subgroup Theorem

Suppose G is a Lie group and $H \subseteq G$ is a Lie subgroup. Then H is closed in G if and only if it is an embedded Lie subgroup.

LECTURE 11

Transversality

Cut out submanifolds as zeros of functions

In order to prepare the following discussion of transversality, let us have another look at the conditions of when preimages are submanifolds.

Question

Suppose that $g_1, \ldots, g_k$ are smooth, real-valued functions on a manifold X of dimension $n > k$ (each g_i is a smooth function $X \to \mathbb{R}$). Under what conditions is the set Z of common zeros a reasonable geometric object? In particular, when is Z a manifold?

We have seen an answer to this question. Collect the n functions to define the map

$$g = (g_1, \ldots, g_k) \colon X \to \mathbb{R}^k.$$

Then we know that $Z = g^{-1}(0)$ is a submanifold of X **if 0 is a regular value of g**.

Remark

Historically, the study of zero sets of collections of functions has been of considerable mathematical interest. For, think of the zeroes as solutions to equations. Solving equations is a fundamental goal in mathematics (though not the only one!). In classical algebraic geometry, for example, one studies sets cut out in (complex) Euclidean space as the zero sets of polynomials (in several complex variables).

In order to make it easier to find an answer to our question, we would like to reformulate the regularity condition for 0 directly in terms of the functions g_i. Since each g_i is a smooth map of X into $\mathbb{R}$, its derivative at a point x is a linear map

$$d(g_i)x : T_x(X) \to \mathbb{R}.$$

We call such a map a **linear functional** on the vector space $T_x(X)$. The set

$$T_x(X)^* := \mathrm{Hom}_{\mathbb{R}}(T_x(X),\mathbb{R})$$

of all linear functionals on $T_x(X)$ is a vector space with pointwise addition and scalar multiplication.

The derivative of g

$$dg_x \colon T_x(X) \to \mathbb{R}^k$$

equals the k-tuple of the linear functionals $(d(g_1)_x, \dots, d(g_k)_x)$. For, each $(d(g_i)_x$ is a $(1 \times n)$-matrix which is the ith row of the matrix representing dg_x.

Hence, as a linear map to a k-dimensional vector space, we see that

$$\begin{aligned} & dg_x \text{ is surjective} \\ \iff & dg_x \text{ has full rank} \\ \iff & \text{the row vectors } d(g_1)_x, \dots, d(g_k)_x \text{ are linearly independent.} \end{aligned}$$

This is the same as to say that $d(g_1)_x, \dots, d(g_k)_x$ **are linearly independent** in the vector space $T_x(X)^*$ of linear functionals on $T_x(X)$. We are going to rephrase this condition by saying that the k functions $g_1, \dots, g_k$ **are independent at** x.

This yields another way of stating the Preimage Theorem:

Preimage Theorem revisited

If the smooth, real-valued functions $g_1, \dots, g_k$ on X are **independent at each point** x where they all vanish (i.e. $g_1(x) = \cdots = g_k(x) = 0$), then the set Z of **common zeros is a submanifold** of X with dimension equal to $\dim X - k$.

It is convenient here to define the **codimension** of an arbitrary submanifold Z of X by the formula

$$\mathrm{codim}\, Z = \dim X - \dim Z.$$

We can think of the codimension as a **measure of how much bigger** X **is compared to** Z. In particular, note that the codimension depends not only on Z, but also on the surrounding manifold X. Hence we should always speak of the **codimension of** Z **in** X. However, the number of functions we use to cut out a submanifold determines the codimension, independently of the size of X:

Cut out manfiolds

Thus k **independent functions** on X cut out a submanifold of **codimension k**.

Once again, a natural question arises:

Question

Can every submanifold Z of X be "cut out" by independent functions?

Answer

The answer is **no, in general**.

However, there are two useful partial converses:

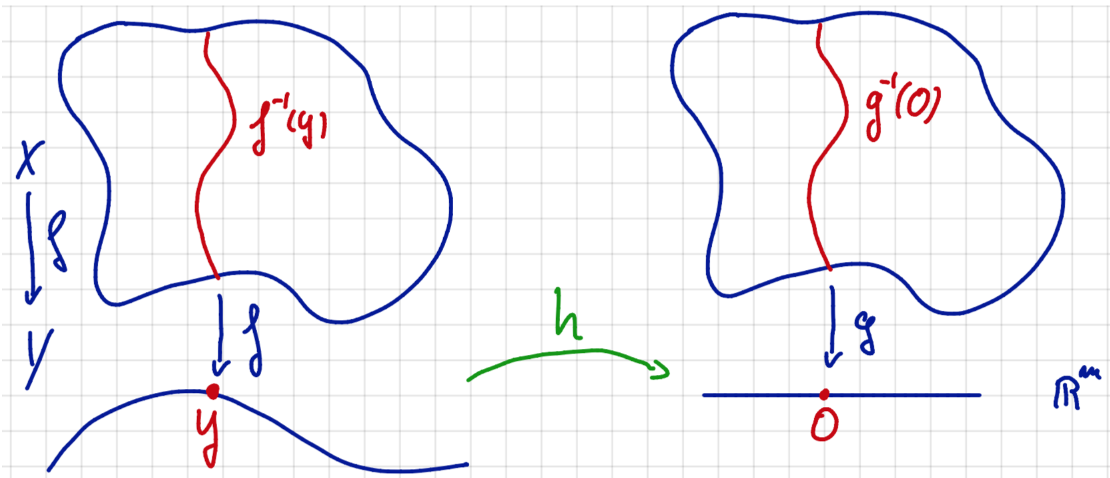
Cut out manfiolds: Partial Converse 1

If y is a regular value of a smooth map $f\colon X \to Y$, then the preimage submanifold $f^{-1}(y)$ can be cut out by independent functions.

Note that **the point** here is that we express $f^{-1}(y)$ as the set of **common zeros** for some function, not just as the preimage of some value in Y.

Proof: Assuming $\dim Y = m$, we just need to choose local coordinates around y, i.e. a diffeomorphism $h\colon W \to V$ with $W \subset Y$ and $V \subset \mathbb{R}^m$ open and $h(y) = 0$. Then we define the new map

$$g = h \circ f\colon f^{-1}(W) \to \mathbb{R}^m \text{ with } g^{-1}(0) = f^{-1}(h^{-1}(0)) = f^{-1}(y) \subseteq X.$$



The origin $0 \in \mathbb{R}^m$ is a regular value for g, for if $x \in g^{-1}(0)$ then

$$dg_x = dh_{f(x)} \circ df_x$$

is surjective, since $dh_{f(x)}$ is an isomorphism and df_x is surjective (x being a regular point for f). Hence every point in $g^{-1}(0)$ is regular, and 0 is a regular value for g. Thus the components $g_1, \ldots, g_m$ of g with $g_i \colon X \to \mathbb{R}$ are independent functions which cut out $f^{-1}(y)$. **QED**

Simple Example

In many cases, the result does not tell us too much new. It is just convenient to know that we can choose 0 as the regular value.
A simple example is given by defining S^n as $f^{-1}(0)$ of the map

$$g \colon \mathbb{R}^{n+1} \to \mathbb{R},\ x_1^2 + \cdots + x_{n+1}^2 \mathbf{-1}.$$

As we pointed out, it is not possible to write every submanifold as the zero set of some map. But locally we can!

Cut out manfiolds: Partial Converse 2

Every submanifold Z of X is **locally cut out by independent functions**. More specifically, let m be the codimension of Z in X, and let z be any point of Z. Then there exist m independent functions $g_1.\ldots, g_m$ defined on some open neighborhood W of z in X such that $Z \cap W$ is the common vanishing set of the g_i. In other words, $Z \cap W$ is cut out be independent functions in W.

Proof: This is just Exercise 5 of Exercise Set 3 applied to the immersion $Z \to W$. The idea is to use the Local Immersion Theorem and pick local coordinate functions $g_1, \ldots, g_n$ ($n = \dim X$) defined on W such that $Z \cap W$ is the set of common zeros of the m functions $g_{n-m+1}, \ldots, g_n$, i.e.

$$Z \cap W = \{x \in W : g_{n-m+1}(x) = 0, \ldots, g_n(x) = 0\}.$$

QED

As a consequence we see that every manifold can be cut out locally by independent functions on Euclidean space (but not globally in general!)

Cut out manifolds by smooth conditions

Now we would like to understand what happens when we do not take the preimage of just a single point, but of a whole submanifold (not an arbitrary subset, since we need some control).

Given a smooth map $f \colon X \to Y$ between smooth manifolds. Assume that $Z \subseteq Y$ is a submanifold of Y. We would like to understand:

Question

Under which conditions is the subset $f^{-1}(Z) \subseteq X$ an interesting geometric object? In particular, when is $f^{-1}(Z)$ a manifold, and therefore a submanifold of X?

Note that $f^{-1}(Z)$ is the set of all $x \in X$ such that $f(x) \in Z$. In other words, it is the colllection of all the fibers $f^{-1}(z)$ for all $z \in Z$. This gives us a hint to how we can answer the question. We look at the points $z \in Z$ each at a time. This fits nicely into our general strategy: whether a space is a manifold or not is determined by the neighborhoods of points.

Strategy

More precisely, in order to check that $f^{-1}(Z)$ is a manifold, it suffices to check that for each point $x \in f^{-1}(Z)$ there is an open neighborhood $U \subset X$ of x in X such that $f^{-1}(Z) \cap U$ is a manifold. For then $f^{-1}(Z) \cap U$ **inherits the local coordinate functions** from U (by restricting them to the subset $f^{-1}(Z) \cap U$).

So let us pick a point $z \in Z$ and let $x \in X$ satisfy $f(x) = z$. We have just learned that we can write Z in a neighborhood $W \subseteq Y$ around z as the **zero set of independent functions** $g_1, \dots, g_k$, where k denotes the codimension of Z in Y. This means:

$$W \cap Z = \{w \in W : g_1(w) = \cdots = g_k(w) = 0\} \tag{5}$$

and $d(g_1)_w, \dots, d(g_k)_w$ are linearly independent in $T_w(Y)^*$ for all $w \in W \cap Z$.

We set $U := f^{-1}(W)$ which is an open neighborhood of x in X. Since taking preimages of sets commmutes with intersecting sets, we have

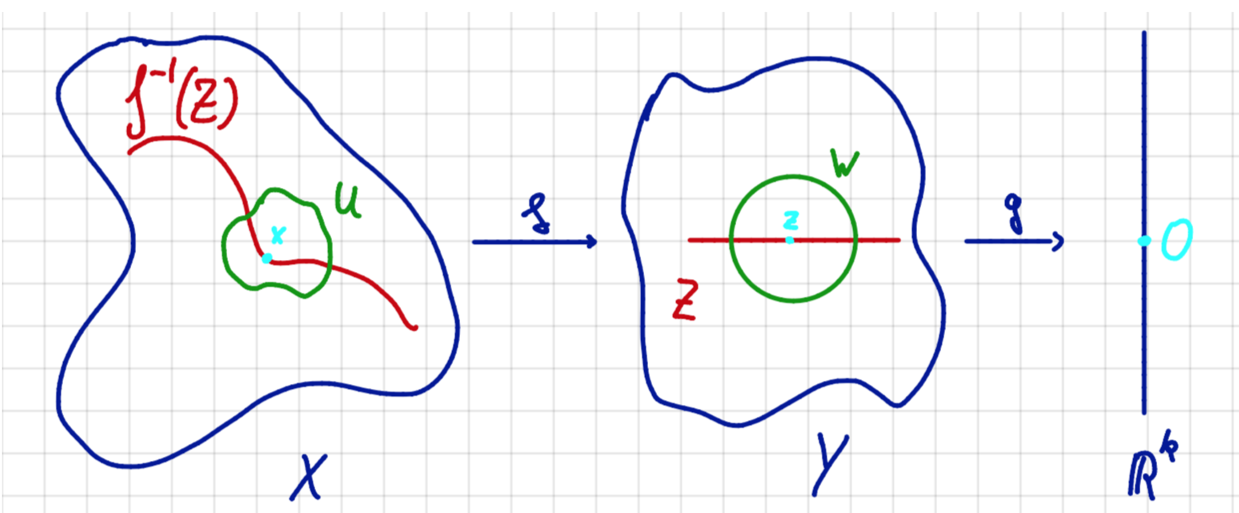
$$f^{-1}(W \cap Z) = f^{-1}(W) \cap f^{-1}(Z) = U \cap f^{-1}(Z).$$

Hence equation (5) implies that, near x, the preimage $f^{-1}(Z)$ **is the zero set of the functions** $g_1 \circ f, \dots, g_k \circ f$ in U:

$$U \cap f^{-1}(Z) = \{u \in U : (g_1 \circ f)(u) = \cdots = (g_k \circ f)(u) = 0\}.$$

Let $g \colon W \to \mathbb{R}^k$ denote the submersion $(g_1, \dots, g_k)$ defined around z. Then the Preimage Theorem applied to the composite smooth map $g \circ f \colon U \to \mathbb{R}^k$ gives us:

$$U \cap f^{-1}(Z) = (g \circ f)^{-1}(0) \text{ is a manifold } \textbf{if } \mathbf{0} \text{ is a regular value of } \mathbf{g \circ f}.$$



Hence in order to show that $f^{-1}(Z)$ is a manifold we need to understand when 0 is a regular value of $g \circ f$.

So what does it mean that 0 is a regular value of the composite $g \circ f$? The chain rule tells us

$$d(g \circ f)_x = dg_z \circ df_x.$$

Thus, the linear map

$$d(g \circ f)_x \colon T_x(X) \to \mathbb{R}^k \text{ is surjective}$$
$$\iff dg_z \text{ maps the image of } df_x \text{ onto } \mathbb{R}^k.$$

We know that $dg_z \colon T_z(Y) \to \mathbb{R}^k$, on the **whole** tangent space to Y at z, is a surjective linear map whose kernel is the subspace $T_z(Z)$. Thus dg_z induces an isomorphism

$$d\bar{g}_z \colon T_z(Y)/T_z(Z) \xrightarrow{\cong} \mathbb{R}^k.$$

In particular, $(dg_z)_{|\mathrm{Im}\,(df_x)}$ can only be **surjective if** $\mathrm{Im}\,(df_x)$ **and** $T_z(Z)$ **together span all of** $T_z(Y)$.

We conclude that $g \circ f$ **is a submersion** at $x \in f^{-1}(Z)$ **if and only if**

$$\mathrm{Im}\,(df_x) + T_z(Z) = T_z(Y).$$

We give this condition a name:

Transversality

Let $f \colon X \to Y$ be a smooth map and $Z \subseteq Y$ a submanifold. Then f is said to be **transversal to the submanifold** Z, denoted $f \pitchfork Z$, if

$$\mathrm{Im}\,(\mathbf{df_x}) + \mathbf{T_{f(x)}(Z)} = \mathbf{T_{f(x)}(Y)}$$

at each point $x \in f^{-1}(Z)$ in the preimage of Z.

The above discussion then shows

Transversality Theorem

If the smooth map $f \colon X \to Y$ is **transversal** to a submanifold $Z \subseteq Y$, then $f^{-1}(Z)$ **is a submanifold of** X. Moreover, the codimension of $f^{-1}(Z)$ in X equals the codimension of Z in Y.

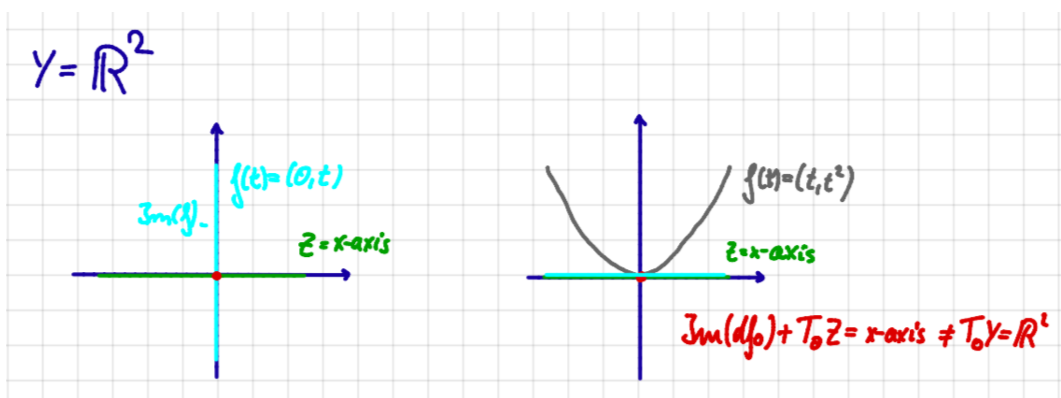
The number of independent functions $g_1, \ldots, g_k$ we needed to locally write Z as a zero set in Y, is the same as the number of independent functions $g_1 \circ f, \ldots, g_k \circ f$ we needed to locally write $f^{-1}(Z)$ as a zero set in X. Therefore the **codimension of** $f^{-1}(Z)$ **in** X is **equal** the **codimension of** Z **in** Y.

Transversality revisited

To make this explicit, note that our discussion showed that $f \pitchfork Z$ equivalent to: for every $x \in X$ with $f(x) \in Z$, there is an open neighborhood W around $f(x)$ in Y and a submersion $g\colon W \to \mathbb{R}^k$, with $k = \operatorname{codim} Z$, such that $W \cap Z = g^{-1}(0)$ and 0 is a regular value of $g \circ f$.

For some very simple examples of transversality and non-transversality, consider $Y = \mathbb{R}^2$ with the submanifold Z being the x-axis. Then

- The map $f\colon \mathbb{R}^1 \to \mathbb{R}^2$ defined by $f(t) = (0,t)$ is transversal to Z, with $f^{-1}(Z) = \{(0,0)\}$.
- The map $f\colon \mathbb{R}^1 \to \mathbb{R}^2$ defined by $f(t) = (t,t^2)$, however, is **not** transversal to Z, with $f^{-1}(Z) = \{(0,0)\}$.



- The map $f\colon \mathbb{R}^1 \to \mathbb{R}^2$ defined by $f(t) = (t,t^2 - 1)$ is transversal to Z, with $f^{-1}(Z) = \{(-1,0),(1,0)\}$.
- The map $f\colon \mathbb{R}^1 \to \mathbb{R}^2$ defined by $f(t) = (t, \cos t - 1)$ is **not** transversal to Z, with $f^{-1}(Z) = \{(0,0)\}$.

LECTURE 12

Transversality of submanifolds

Today, we are going to study some important special cases of transversality.

First, transversality is in fact a generalization of Regularity:

Regular vs Transversal

When Z is just a single point z, its tangent space is the zero subspace of $T_z(Y)$. Thus f is transversal to $\{z\}$ if $df_x(T_x(X)) = T_z(Y)$ for all $x \in f^{-1}(z)$. This is exactly what it means to say that z is a regular value of f. So transversality includes the notion of regularity as a special case.

The second one tells us how we should actually think of and visualize transversality. Roughly speaking, we want to know how the image of f and Z meet in Y:

Intersection of submanifolds

The most important situation is the transversality of the inclusion map i of one submanifold $X \subset Y$ with another submanifold $Z \subset Y$.
To say a point $x \in X$ belongs to the preimage $i^{-1}(Z)$ simply means that x belongs to the intersection $X \cap Z$. Also, the derivative $di_x \colon T_x(X) \to T_x(Y)$ is merely the inclusion map of $T_x(X)$ into $T_x(Y)$. So $i \pitchfork Z$ **if and only if**, for every $y \in X \cap Z$,

$$\mathbf{T_y(X) + T_y(Z) = T_y(Y)}. \tag{6}$$

Notice that this equation is symmetric in X and Z. When it holds, we shall say that the **two submanifolds X and Z are transversal**, and write $X \pitchfork Z$.

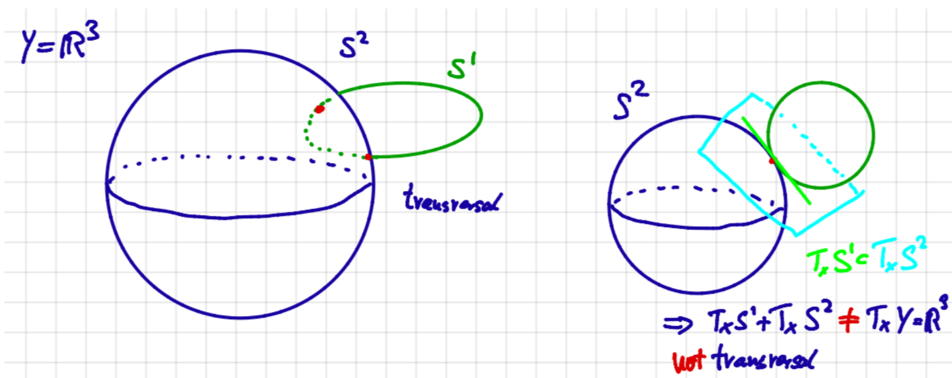
Warning: For equation (6) to be true, it is **not sufficient** that $\dim T_x(X) + \dim T_x(Z) = \dim T_x(Y)$. The two subspaces must span together all of $T_x(Y)$.

The transversality theorem for this specialize case then says:

Intersection of transversal submanifolds

The intersection of **two transversal submanifolds** X and Z of Y is again a submanifold. Moreover, the codimensions in Y satisfy

$$\operatorname{codim}(X \cap Z) = \operatorname{codim} X + \operatorname{codim} Z.$$



The additivity of codimensions follows from the codimension formula of the Transversality Theorem:

$$\begin{aligned} \operatorname{codim} i^{-1}(Z) \text{ in } X &= \operatorname{codim} Z \text{ in } Y \\ \Rightarrow \dim X - \dim X \cap Z &= \dim Y - \dim Z \\ \Rightarrow \dim Y - \dim X \cap Z &= (\dim Y - \dim Z) + (\dim Y - \dim X) \\ \Rightarrow \operatorname{codim} X \cap Z &= \operatorname{codim} Z + \operatorname{codim} X. \end{aligned}$$

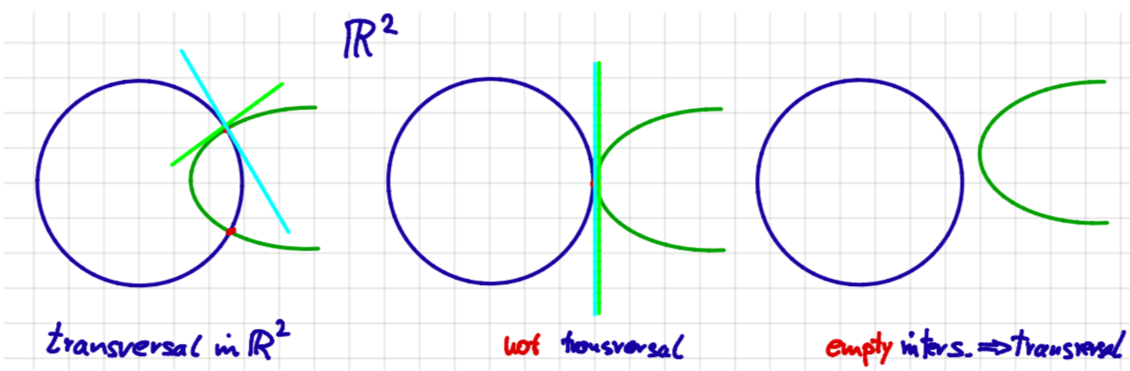
Intersect as a little as possible

We have just seen that two manifolds intersect transversally if their tangent spaces together span the whole ambient space. A different way to think of transversality is: Two manifolds intersect transversally if they intersect as little as possible at every point. And we **measure the degree of intersection in terms of tangent spaces**: If two submanifolds intersect, then they transversally if the intersection of their tangent spaces in the ambient space is minimal.

Note that the **converse of the Transversality Theorem is not true**. Weh ave seen a simple example last time: the submanifolds $X\{(x,y) \in \mathbb{R}^2 : y = x^2\}$ and $Z = \{(x,y) \in \mathbb{R}^2 : y = 0\}$ do **not intersect transversally at** 0 in $Y = \mathbb{R}^2$, but their intersection $X \cap Z = \{0\}$ is **a zero-dimensional manifold**. However, there do, of course, exist intersections which are not transversal and where the intersection is not a manifold. See the example below!

Empty intersections are transversal

It is useful to note that any smooth map $f\colon X \to Y$ whose image does not meet a submanifold Z of Y, i.e. $f^{-1}(Z) = \emptyset$, is transversal to Z for trivial reasons. For in this case **there is no condition to be satisfied**. In particular, two submanifolds which do not intersect at all, are transversal. Moreover, if $f\colon X \to Y$ is a **submersion**, then f is transversal to any submanifold Z of Y, since then $\operatorname{Im}(df_x) = T_{f(x)}(Y)$ for every x.



The ambient space matters

It is important to note that the transversality of X and Z also depends on the ambient space Y. For example, the two coordinate axes intersect transversally in $\mathbb{R}^2$, but not when considered to be submanifolds of $\mathbb{R}^3$. In general, if the dimensions of X and Z do not add up to at least the dimension of Y, then they can only intersect transversally by not intersecting at all. For example, if X and Z are curves in $\mathbb{R}^3$, then $X \pitchfork Y$ if and only if $X \cap Y = \emptyset$.

Let us have a look at an example:

Example

In $Y = \mathbb{R}^3$, we consider the two submanifolds

$$X = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\}$$

and the sphere

$$Z_a = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = a\}.$$

We would like to understand for which a these two submanifolds intersect transversally in Y.
Therefore, we need to determine the tangent space of X and Z_a at points where they intersect. We observe that $X = f^{-1}(0)$ for the map

$$f\colon \mathbb{R}^3 \to \mathbb{R},\ (x,y,z) \mapsto x^2 + y^2 - z^2 - 1$$

and $Z_a = g^{-1}(0)$ for the map

$$g\colon \mathbb{R}^3 \to \mathbb{R},\ (x,y,z) \mapsto x^2 + y^2 + z^2 - a.$$

Since 0 is a regular value of f, the tangent space to X at a point $p = (x,y,z)$ is the kernel of the derivative of f at p (expressed as a matrix in the standard basis)

$$df_p = (2x,2y,-2z)\colon \mathbb{R}^3 \to \mathbb{R}.$$

Hence the tangent space to X at $p = (x,y,z)$ is

$$T_p(X) = \mathrm{Ker}\,(df_p) = \mathrm{span}(\{(z,0,x),(0,z,y)\}) \subset \mathbb{R}^3.$$

Similarly, since 0 is a regular value of g, the tangent space to Z_a at a point $p = (x,y,z)$ is the kernel of the derivative of g at p (expressed as a matrix in the standard basis)

$$dg_p = (2x,2y,2z)\colon \mathbb{R}^3 \to \mathbb{R}.$$

Hence the tangent space to Z_a at $p = (x,y,z)$ is

$$T_p(Z_a) = \mathrm{Ker}\,(dg_p) = \mathrm{span}(\{(-z,0,x),(0,-z,y)\}) \subset \mathbb{R}^3.$$

Now X and Z intersect in the points $p = (x,y,z)$ which satisfy

$$x^2 + y^2 - z^2 - 1 = 0 = x^2 + y^2 + z^2 - a.$$

Subtracting both equations yields the condition

$$2z^2 = a - 1. \tag{7}$$

This gives us three cases for the intersection $X \cap Z_a$:

- **If** $a < 1$, then X and Z_a do not intersect, since there is no z which can satisfy condition (7): $X \cap Z_a = \emptyset$.

- **If** $a = 1$, then X and Z_1 intersect in the circle with radius 1 in the xy-plane in $\mathbb{R}^3$ with the origin as center, i.e.

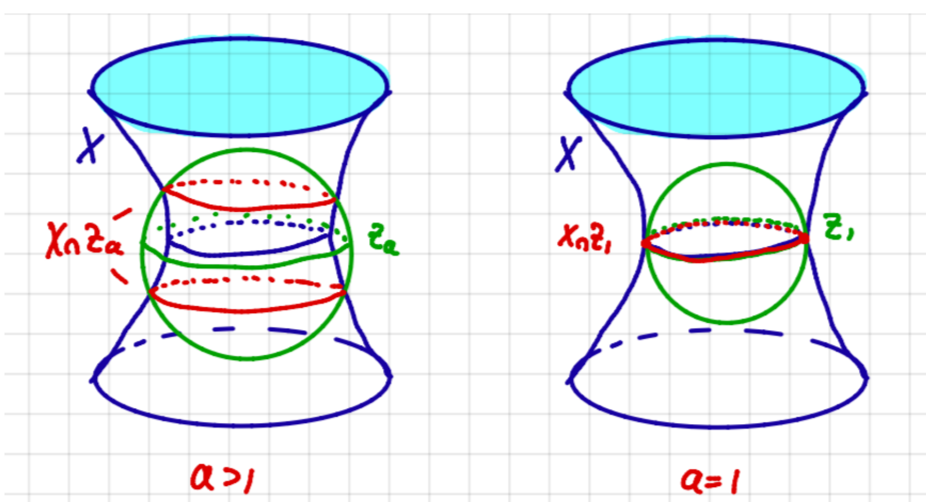
$$X \cap Z_1 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1 \text{ and } z = 0\}.$$

- **If** $a > 1$, then X and Z_a intersect in two disjoint circles with lie in the planes parallel to the xy-plane in $\mathbb{R}^3$ with z-coordinate $z = \pm\sqrt{(a-1)/2}$:

$$X \cap Z_a = \{(x,y,z) \in \mathbb{R}^3 :$$
$$x^2 + y^2 = \frac{a+1}{2} \text{ and } z = \pm\sqrt{(a-1)/2}\}.$$

Now we need to check transversality (recall $T_p(\mathbb{R}^3) = \mathbb{R}^3$ at every p):

- **If** $a < 1$, then the intersection is empty and therefore **transversal**.
- **If** $a = 1$, then $T_p(X)$ and $T_p(Z_1)$ span the xy-plane in $\mathbb{R}^3$, and not all of $\mathbb{R}^3$, at every $p \in X \cap Z_1$. Thus the intersection is **not transversal**.
- **If** $a > 1$, let $p = (x,y,z) \in X \cap Z_a$. Then $T_p(X)$ and $T_p(Z_a)$ together span all of $\mathbb{R}^3$, for the vector $(-z,0,x) \in T_p(X)$ is not a linear combination of $(z,0,x)$ and $(0,z,y)$ ($z \neq 0$). Since $T_p(Z_a)$ is 2-dimensional, this shows $Tp(X) + T_p(Z_a) = \mathbb{R}^3$ at every $p \in X \cap Z_a$. Thus the intersection is **transversal**.



Here is an example of an intersection which is not transversal and where the intersection is not a manifold:

Non-transversal intersection which is not a manifold

Let $Y = \mathbb{R}^3$ and let Z be the hyperplane defined by

$$Z = \{(x,y,z) \in \mathbb{R}^3 : x = 1\}$$

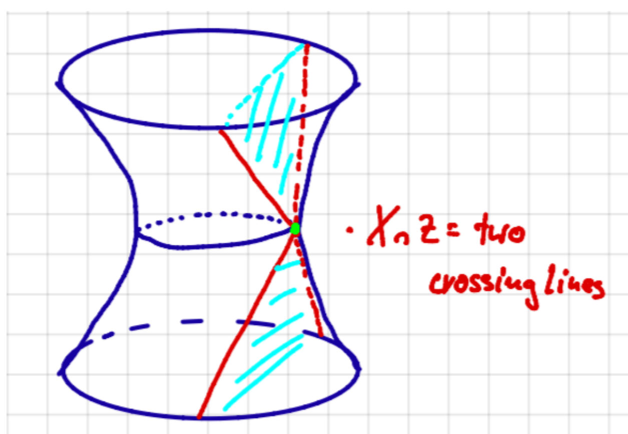
and let X be the hyperboloid defined by

$$X = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\}.$$

The intersection of X and Z is given by the points satisfying $x = 1$ and $x^2 + y^2 - z^2 = 1$, i.e. all points such that $x = 1$ and $y^2 = z^2$. This means

$$X \cap Z = \{(x,y,z) \in \mathbb{R}^3 : x = 1,\ y = \pm z\}.$$

We have seen in one of the first lectures that a space consisting of two lines crossing each other is not a manifold. The intersection point, here the point $p = (1,0,0)$ does not have a neighborhood in $X \cap Z$ which is diffeomorphic to an open subset in Euclidean space. **Thus $X \cap Z$ is not a manifold**.
As a reality check, let us look at the tangent spaces to X and Z at p: Since Z is a parallel translate of a vector subspace of $\mathbb{R}^3$, we see that $T_p(Z)$ is the yz-plane in $\mathbb{R}^3$ (all points with $x = 0$). The tangent space to X was calculated in the previous example (and in an exercise). At $p = (1,0,0)$, $T_p(X)$ is the vector subspace in $\mathbb{R}^3$ spanned by the vectors $(0,1,0)$ and $(0,0,1)$. In other words, $T_p(X)$ is the xy-plane in $\mathbb{R}^3$. Thus $T_p(Z)$ and $T_p(X)$ do **not span** $T_p(Y) = \mathbb{R}^3$. (The problem here is that Z "is" the tangent plane to X at p.)



Codimension Formula revisited

Another way to rephrase the **codimension formula** is to say that when X is locally cut out by k independent functions and Z is locally cut out by

l independent functions, then $X \cap Z$ is locally cut out by $k + l$ independent functions.

In fact, we can reprove the theorem by using independent functions:

Let y be a point in $X \cap Z \subseteq Y$. Around y, the submanifold X is cut out of Y by $k = \operatorname{codim} X$ independent functions, i.e. there is an open neighborhood $U \subseteq Y$ around y and k independent functions

$$f_1, \ldots, f_k \colon U \to \mathbb{R}$$

such that $X \cap U$ is defined by the vanishing of the f_i:

$$X \cap U = \{u \in U : f_1(u) = \cdots = f_k(u) = 0\}.$$

The independence of the f_i implies that 0 is a regular value of $f = (f_1, \ldots, f_k) \colon U \to \mathbb{R}^k$. In particular,

$$df_x \colon T_x(Y) \to \mathbb{R}^k \text{ is surjective.} \tag{8}$$

By the corollary to the Preimage Theorem we know

$$T_y(X) = \operatorname{Ker}(df_y) \subseteq T_Y(Y).$$

Then (8) implies

$$\dim \operatorname{Ker}(df_y) = \dim T_x(X) = \dim T_x(Y) - k.$$

Similarly, around y, the submanifold Z is cut out by $l = \operatorname{codim} Z$ independent functions, i.e. there is an open neighborhood $V \subseteq Y$ around y and l independent functions

$$g_1, \ldots, g_l \colon V \to \mathbb{R}$$

such that $Z \cap V$ is defined by the vanishing of the g_i:

$$Z \cap V = \{v \in V : g_1(v) = \cdots = g_l(v) = 0\}.$$

The independence of the g_i means that 0 is a regular value of $g = (g_1, \ldots, g_l) \colon V \to \mathbb{R}^l$. In particular,

$$dg_y \colon T_y(Y) \to \mathbb{R}^l \text{ is surjective.} \tag{9}$$

The tangent space to Z at y is

$$T_y(Z) = \operatorname{Ker}(dg_y) \subseteq T_Y(Y).$$

Then (9) implies

$$\dim \operatorname{Ker}(dg_y) = \dim T_y(Z) = \dim T_y(Y) - l.$$

We set $W := U \cap V$ which is an open neighborhood of y. Then, arond y, $X \cap Z$ is locally cut out by the combined collection of $k+l$ functions $f_1, \ldots, f_k, g_1, \ldots, g_l$, i.e.

$$\begin{aligned}&(X \cap Z) \cap W \\ &= \{w \in W : f_1(w) = \cdots = f_k(w) = g_1(w) = \cdots = g_l(w) = 0\}.\end{aligned}$$

We write h for the collection of functions f and g:

$$h = (f_1, \ldots, f_k, g_1, \ldots, g_l)\colon W \to \mathbb{R}^{k+l}.$$

The derivative of h at y is

$$dh_y\colon T_y(Y)\colon \mathbb{R}^{k+l},\ v \mapsto dh_y(v) = (df_y(v), dg_y(v)).$$

Now we want to relate the independence of the f_i's and g_i's to transversality:

As vector subspaces of $T_y(Y)$, $\operatorname{Ker}(df_y)$ and $\operatorname{Ker}(dg_y)$ satisfy the dimension formula

$$\begin{aligned}&\dim \operatorname{Ker}(df_y) + \dim \operatorname{Ker}(dg_y) \\ &= \dim(\operatorname{Ker}(df_y) + \operatorname{Ker}(dg_y)) + \dim(\operatorname{Ker}(df_y) \cap \operatorname{Ker}(dg_y)).\end{aligned}$$

From (8) and (9) we get that this equation is equivalent to

$$\begin{aligned}&\dim T_y(Y) - k + \dim T_y(Y) - l \\ &= \dim(\operatorname{Ker}(df_y) + \operatorname{Ker}(dg_y)) + \dim(\operatorname{Ker}(df_y) \cap \operatorname{Ker}(dg_y)).\end{aligned} \tag{10}$$

Hence the left hand side is $2 \dim T_y(Y) - (k+l)$. For the right hand side, we have

$$\dim(\operatorname{Ker}(df_y) + \operatorname{Ker}(dg_y)) \leq \dim T_y(Y) \tag{11}$$

and

$$\begin{aligned}&\dim T_y(Y) - \dim(\operatorname{Ker}(df_y) \cap \operatorname{Ker}(dg_y)) \leq k + l, \\ &\text{i.e. } \dim(\operatorname{Ker}(df_y) \cap \operatorname{Ker}(dg_y)) \geq \dim T_y(Y) - (k+l).\end{aligned} \tag{12}$$

Hence, given (10), the two inequalities (11) and (12) imply

$$\dim(\operatorname{Ker}(df_y) + \operatorname{Ker}(dg_y)) = \dim T_y(Y) \tag{13}$$

$$\iff \dim(\operatorname{Ker}(df_y) \cap \operatorname{Ker}(dg_y)) = \dim T_y(Y) - (k+l). \tag{14}$$

Now the first equation (13) means exactly that X and Z are **transversal** in Y, while the second equation (14) is true if and only if $d(h)_y$ is surjective, i.e. if and only if the $k+l$ functions $f_1, \ldots, f_k, g_1, \ldots, g_l$ are **independent**.

We are going to exploit what we just observed a bit further. Let us keep the above notation. **Now we assume again that X and Z meet transversally in Y.** Then 0 is a regular value of h. This implies that the tangent space to $X \cap Z$ at y equals $\operatorname{Ker}(dh_y)$. For $v \in T_y(Y)$, we have $dh_y(v) = 0$ if and only if both $df_y(v) = 0$ and $dg_y(v) = 0$. Thus $\operatorname{Ker}(dh_y)$ is the intersection of the kernel of $\operatorname{Ker}(df_y)$ and $\operatorname{Ker}(dg_y)$ in $T_y(Y)$:

$$\operatorname{Ker}(dh_y) = \operatorname{Ker}(df_y) \cap \operatorname{Ker}(dg_y) \text{ in } T_y(Y).$$

Thus we have proved the following useful fact:

Tangent space of intersections

If X and Z are submanifolds which meet transversally in Y, then the tangent space to the intersection $X \cap Z$ is the intersection of the tangent spaces, i.e.

$$T_y(X \cap Z)) = T_y(X) \cap T_y(Z) \text{ for all } y \in X \cap Z.$$

In the exercises for this week we prove a generalization of this fact to the preimage of a submanifold Z under a smooth map f when $f \pitchfork Z$:

Tangent space of preimages

Let $f\colon X \to Y$ be a map transversal to a submanifold Z in Y. Then $T_x(f^{-1}(Z))$ is the preimage of $T_{f(x)}(Z)$ under the linear map $df_x\colon T_x(X) \to T_{f(x)}(Y)$:

$$T_x(f^{-1}(Z)) = (df_x)^{-1}(T_{f(x)}(Z)).$$

A famous example of transversal intersections is given by Brieskorn Manifolds.

Exotic Spheres

Consider the following intersections in $\mathbb{C}^5 \setminus \{0\}$:

$$\begin{aligned} S^7_k = &\{z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1} = 0\} \\ &\cap \{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 = 1\}. \end{aligned}$$

In this week's exercises, we show that this is a transversal intersection. One can show that, for each value $k = 1, \ldots, 28$, S^7_k is a smooth manifold which is homeomorphic to S^7. But none of these manifolds are diffeomorphic. These are so called **exotic 7-spheres** were constructed by **Brieskorn** and represent each of the 28 diffeomorphism classes on S^7. That such exotic 7-spheres is a famous and groundbreaking result of **Milnor**. Milnor's work started an amazing story about the diffeomorphic structures on spheres which culminated in the solution of the **Kerviare Invariant One Problem** by **Hill, Hopkins and Ravenel** in 2009.

LECTURE 13

Homotopy and Stability

Today we are going to introduce one of the most important concepts in topology. Actually, the idea of studying objects **up to homotopy** has turned out be extremely influential and successful in many areas in mathematics.

Homotopy

Let I denote the unit interval $[0,1]$ in $\mathbb{R}$. We say that two smooth maps f_0 and f_1 from X to Y are **homotopic**, denoted $f_0 \sim f_1$, if there exists a smooth map $F\colon X \times I \to Y$ such that

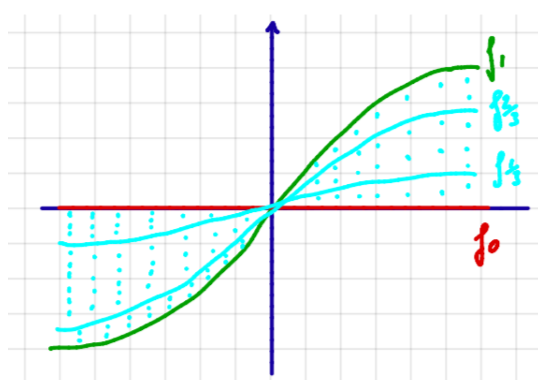
$$F(x,0) = f_0(x) \text{ and } F(x,1) = f_1(x).$$

F is called a **homotopy** between f_0 and f_1. We also write $f_t(x)$ for $F(x,t)$. In other words, a homotopy is a **family of smooth functions** f_t which smoothly interpolates between f_0 and f_1

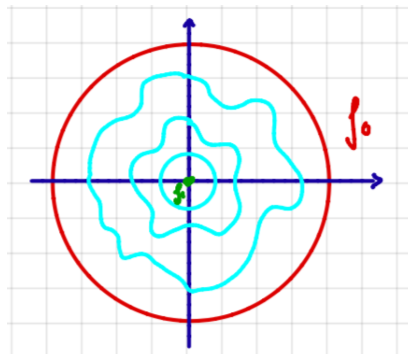
To require that F is **smooth** is necessary because we are working with smooth manifolds. For general topological spaces, one just requires that F is **continuous**.

Some examples:

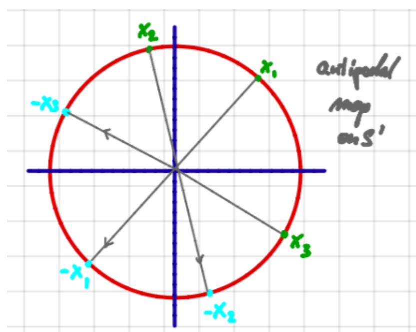
- $f_0\colon \mathbb{R} \to \mathbb{R}^2$, $x \mapsto (x,0)$ and $f_1\colon \mathbb{R} \to \mathbb{R}^2$, $x \mapsto (x, \sin x)$ with homotopy $F\colon \mathbb{R} \times [0,1] \to \mathbb{R}^2$, $(x,t) \mapsto (x, t\sin x)$.



- Let $\gamma\colon S^1 \to \mathbb{R}^2$ be a smooth loop (a smooth path where start and end points agree). Then γ is homotopic to the constant map $S^1 \to \{0\} \subset \mathbb{R}^2$. In fact, this is true when we replace $\mathbb{R}^2$ with any $\mathbb{R}^k$, since $\mathbb{R}^k$ is contractible (see the exercises).



- In the exercises, we will show that the **antipodal map** on the k-sphere $S^k \to S^k$, $x \mapsto -x$ (which sends a point to the point on "the other side" of the sphere) is homotopic to the identity on S^k.



- An important example of two maps which are **not homotopic**: The constant map $f\colon S^1 \to \mathbb{R}^2 \setminus \{0\}$, $p \mapsto (1,0)$ and the map $g\colon S^1 \to \mathbb{R}^2 \setminus \{0\}$, $p \mapsto p$ are not homotopic. We will learn more about this later, and there are much better conceptual arguments in algebraic topology which explain this fact. Here is a first, hands-on argument:

  Assume there were a smooth homotopy $F\colon S^1 \times [0,1] \to \mathbb{R}^2 \setminus \{0\}$ from f to g. For every fixed point $p \in S^1$, $F(p,t)$ defines a path from p to $(1,0)$ in $\mathbb{R}^2 \setminus \{0\}$. Let Z be the subspace of S^1 of points with negative x-coordinate:

$$Z := \{p = (x,y) \in S^1 : x \leq 0\}.$$

  Then by the **Intermediate Value Theorem**, for every $p \in Z$, there is t such that the x-coordinate of $F(p,t)$ is 0. Since $[0,1]$ is **compact**, there is in fact a **minimal** such t for each $p \in Z$. We denote this minimum by

$t_0(p)$ and write

$$F(p,t_0(p)) = (0,y_0(p)).$$

As (0,0) is not a point of $\mathbb{R}^2 \setminus \{0\}$, for each p, we have either $y_0(p) > 0$ or $y_0(p) < 0$.

Since F is smooth in both variables, $y_0(p)$ depends smoothly on p as well. Thus, if $y_0(p) > 0$ for some p, then there is an open neighborhood $U \subset S^1$ around p such that $y_0(q) > 0$ for all $q \in U$. In other words, the subset

$$U_{>0} := \{p = (x,y) \in Z : y_0(p) > 0\} \text{ is open in } Z.$$

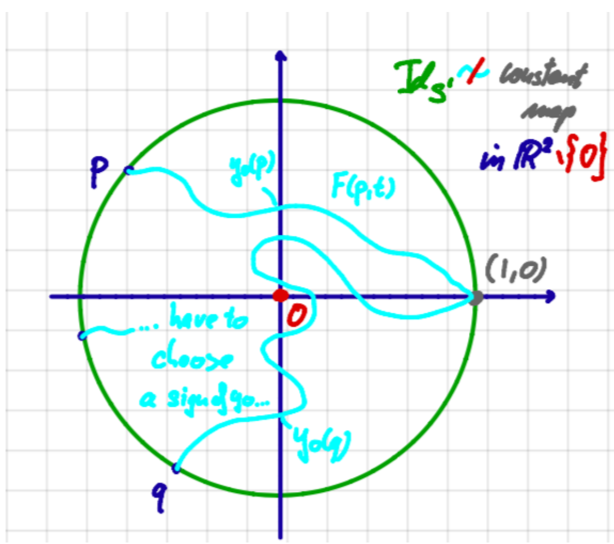
Similarly, the subset

$$U_{<0} := \{p = (x,y) \in Z : y_0(p) < 0\} \text{ is open in } Z.$$

Both spaces are nonempty, since $(0,1) \in U_{>0}$ and $(0,-1) \in U_{<0}$. Moreover, they are disjoint and mutual complements of each other in Z, i.e.

$$U_{>0} = Z \setminus U_{<0} \text{ and } U_{<0} = Z \setminus U_{>0}.$$

Thus, Z is the disjoint union of the two nonempty and both open and closed subsets $U_{>0}$ and $U_{<0}$. Since Z is connected (being the continuous image of a closed interval), this would imply either $Z = U_{>0}$ or $Z = U_{<0}$. But this is impossible. Thus the smooth homotopy F cannot exist.



Homotopy is an equivalence relation

Given two smooth manifolds X and Y, homotopy is an equivalence relation on smooth maps from X to Y. The equivalence class to which a mapping belongs is its **homotopy class**.

Proof:

We need to check that $\sim$ is reflexive, symmetric, and transitive:

Reflexivity is clear as every map is homotopic to itself via the homotopy $f_t = f$ for all t.

For symmetry, suppose $f \sim g$ and let F be a homotopy. Then the map defined by $(x,t) \mapsto F(x,1-t)$ is a homotopy from g to f. Hence $g \sim f$ as well.

For transitivity, we need to introduce a smart technique first:

Smooth bump functions

An extremely useful tool in differential topology are smooth bump functions which allow smooth transitions. We start with the function

$$f\colon \mathbb{R} \to \mathbb{R},\ f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

We observe that **f is smooth**: We only need to think about $x \geq 0$. Since the ith derivative has the form e^{-1/x^2} times a rational polynomial. Such a product is differentiable and

$$\lim_{x \to 0} f^{(i)}(x) = 0,$$

since e^{-1/x^2} goes to 0 faster than any rational polynomial can go to $\pm\infty$. Now, for any given real numbers $a < b$, we define a function

$$g(x) := f(x-a)f(b-x)$$

As a product of two smooth functions, g is smooth, and

$$\begin{cases} g(x) = 0 & x \leq a \text{ (since } f(x-a) = 0) \\ g(x) > 0 & a < x < b \\ g(x) = 0 & x \geq b \text{ (since } f(b-x) = 0) \end{cases}$$

Next we define yet another function

$$h\colon \mathbb{R} \to \mathbb{R},\ h(x) := \frac{\int_{-\infty}^{x} g(t)dt}{\int_{-\infty}^{\infty} g(t)dt}.$$

By the Fundamental Theorem of Calculus, h is smooth, nondecreasing, and

$$\begin{cases} h(x) = 0 & x \leq a \\ 0 < h(x) < 1 & a < x < b \\ h(x) = 1 & x \geq b \end{cases}$$

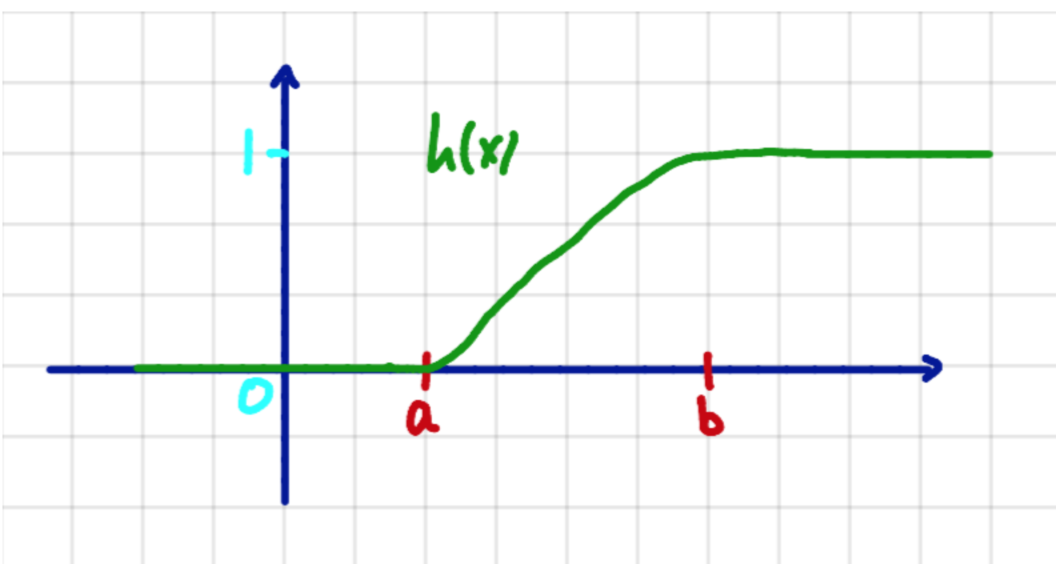
Then h is a **smooth bump function**.

Finally, we can also define higher dimensional smooth bump functions by setting

$$H\colon \mathbb{R}^k \to \mathbb{R},\ H(x) := 1 - h(|x|).$$

Then $H(x)$ is equal 1 on the closed ball around the origin with radius a, is 0 outside the open ball with radius b, and between 0 and 1 on the intermediate points:

$$\begin{cases} H(x) = 1 & x \in \bar{B}_a(0) \\ 0 < H(x) < 1 & a < |x| < b \\ H(x) = 0 & x \in \mathbb{R}^k \setminus B_b(0) \end{cases}$$



Back to the proof:

Suppose $f \sim g$ and $g \sim h$, and let F be a homotopy from f to g and G be a homotopy from g to h. We would like to compose F and G to get a homotopy from f to h. Since we require our homotopies to be **smooth**, we need to make sure that the transition from F to G is smooth.

In order to this, we need to manipulate F and G a bit. And here we are lucky that we have our smooth bump functions at our disposal. So let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$\varphi(t) = \begin{cases} 0 & x \leq 1/4 \\ 1 & x \geq 3/4 \end{cases}$$

and define new homotopies $\tilde{F}$ from f to g and $\tilde{H}$ from g to h by

$$\tilde{F}(x,t) := F(x,\varphi(t)) \text{ and } \tilde{G}(x,t) := G(x,\varphi(t)).$$

Now we can define the map

$$H\colon X \times [0,1] \to Y,\ H(x,t) = \begin{cases} \tilde{F}(x,2t) & t \in [0,1/2] \\ \tilde{G}(x,2t-1) & t \in [1/2,1]. \end{cases}$$

This is map well-defined and smooth, since $\tilde{F}(x,2t) = \tilde{G}(x,2t-1)$ for $t \in [3/8,5/8]$. Thus H is a smooth homotopy from f to h. Hence $\sim$ is also transitive and an equivalence relation. **QED**

Homotopy is one of the most crucial notions in topology. In fact, a lot of properties in topology are **invariant under homotopy**. Therefore, they can be studied by considering maps only "up to homotopy". This led to the construction of the **homotopy category of spaces** in which morphisms are continuous maps modulo homotopy, i.e. $f \sim g$ if and only if f and g are homotopic. To be able to pass to the homotopy category is a very powerful method which has had great influences in many areas of mathematics. We will not be able to fully appreciate the homotopy category this semester.

However, we would like to start to exploit homotopy for our purposes. Despite the above remark, there also a lot of properties of maps which are not invariant under homotopy.

In fact, **many of the properties** we have studied so far are **not invariant**, i.e. if f_0 has a property P and f_t is a homotopy from f_0 to f_1, then it is often not true that f_1 has property P. For example, we could start with an embedding f_0 and end up with a constant map.

So let us ask **a more modest question**: given f_0 has property P, is there always a small $\epsilon > 0$ such that f_t has property P for all $t \in [0,\epsilon)$? For example, if f_0 is an embedding there is always a small $\epsilon > 0$ such that f_t remains an embedding for $0 \leq t < \epsilon$. In other words, embeddings are a so called stable class:

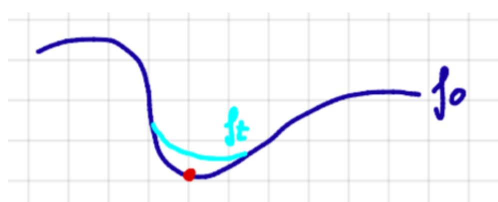
Stable properties

A property P is **stable** provided that whenever $f_0 \colon X \to Y$ possesses the property and $f_t \colon X \to Y$ is a homotopy of f_0 then, for some $\epsilon > 0$, each f_t with $t < \epsilon$ also possesses the property.
We also call the maps which have a stable property, a **stable class**. Examples are the classes of embeddings, local diffeomorphisms, submersions,... as we will learn soon.

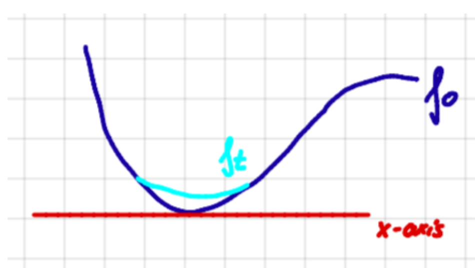
Note that stability is a very natural condition to ask for. For real-world measurements, only stable properties are interesting, since any tiny perturbation of the data would make an unstable property appear or disappear.

In order to get a better idea of stability, let us look at the difference between requiring that things merely intersect or that they intersect transversally:

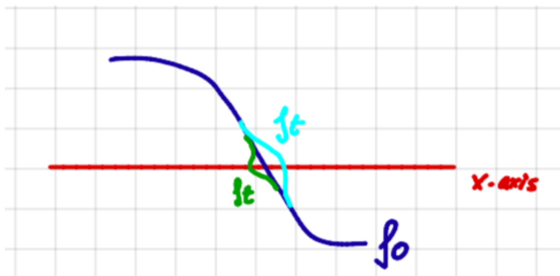
- That a smooth map $f_0 \colon \mathbb{R} \to \mathbb{R}^2$ passes through a fixed point in $\mathbb{R}^2$ is **not** a stable property. It disappears immediately.



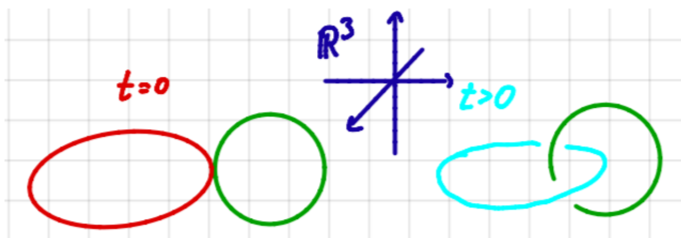
- That a smooth map $f_0 \colon \mathbb{R} \to \mathbb{R}^2$ merely intersects the x-axis is **not** a stable property. It disappears immediately.



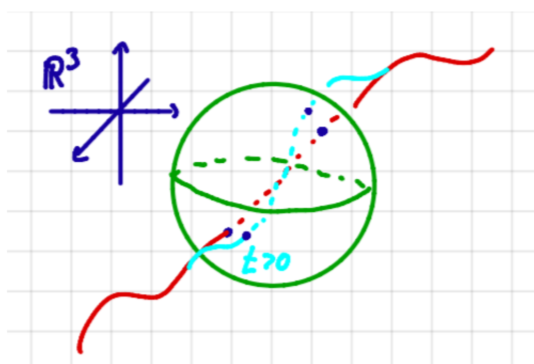
- However, that a smooth map $f_0 \colon \mathbb{R} \to \mathbb{R}^2$ intersects the x-axis transversally **is** a stable property. It persists after a small perturbartion.



- That two smooth curves (connected 1-dimensional manifolds) meet in $\mathbb{R}^3$ is **not** a stable property. It disappears immediately.



- That a smooth curve and a smooth surface (2-dimensional manifold) intersect transversally in $\mathbb{R}^3$ **is** a stable property. It persists after a small perturbartion.



This reveals yet another very important feature of transversality. The following theorem tells us that the properties which turned out to be useful for us so far are all stable.

Stability Theorem

The following classes of smooth maps from a **compact** manifold X to a manifold Y are **stable classes**:

(a) local diffeomorphisms.
(b) immersions.

(c) submersions.
(d) maps which are transversal to any specified closed submanifold $Z \subset Y$.
(e) embeddings.
(f) diffeomorphisms.

Proof:

(a) First we note that **local diffeomorphisms** are just immersions in the special case when $\dim X = \dim Y$, so (a) follows from (b).

(b) Assume $f_0 \colon X \to Y$ is an **immersion** and $\dim X = m$. Letf_t be a homotopy of f_0. That f_0 is an immersion means that $d(f_0)_x$ is injective for all $x \in X$. We need to show that there is an $\epsilon > 0$ such that $d(f_t)_x$ is injective for all points (x,t) in $X \times [0,\epsilon) \subset X \times I$.

Given a point $x_0 \in X$, that $d(f_0)_{x_0}$ is injective implies that the matrix representing $d(f_0)_{x_0}$ (in local coordinates) has an $m \times m$-submatrix $A(x_0,0)$ with nonvanishing determinant. Since the determinant is continuous, this submatrix will have **nonvanishing determinant** in an open neighborhood of $(x_0,0)$ in $X \times [0,1]$. Since X **is compact**, **finitely many** such neighborhoods suffice to cover all of $X \times \{0\}$. Hence there is a small $\epsilon > 0$ (it is the minimum for the open intervals $[0,\epsilon_i)$ covering $\{0\}$) such that the intersection of these finitely many neighborhoods contains $X \times [0,\epsilon)$. This is what we needed.

(c) If f_0 is a **submersion**, almost the same argument works. We just need to choose an $n \times n$-submatrix of the surjective map $d(f_0)_x$ with $n = \dim Y$.

(d) Let $Z \subset Y$ be a **closed** submanifold, and assume that f_0 is a map which is **transversal to** Z. Then we have shown that, for every point $x \in X$, there is a smooth function g which sends a neighborhood of $f(x)$ to $0 \in \mathbb{R}^{\operatorname{codim} Z}$ and such that $g \circ f_0$ **is a submersion**. Since Z is closed in Y, $f^{-1}(Z)$ **is closed in** X and therefore **also compact**. Therefore, by (c), there is an $\epsilon > 0$ such that $g \circ f_t$ **is still a submersion** for all $t < \epsilon$. This is means that f_t is still transversal to Z for all $t < \epsilon$.

(e) Assume that f_0 is an **embedding**, and let f_t be a homotopy of f_0. Since X **is compact**, f_0 and each f_t are automaticlly proper maps. Hence we need to show that when f_0 is a one-to-one immersion, then so is f_t in a small neighborhood. We just checked that being an immersion is stable. Hence it remains to show that f_t **is still one-to-one if** t **is small enough**.

Therefor we define a smooth map

$$G\colon X \times I \to Y \times I,\ G(x,t) := (f_t(x),t).$$

Then **if (e) is false**, i.e. if f_t **not one-to-one** in some small neighborhood of 0, then, for every $\epsilon > 0$, we can find a t with $0 < t < \epsilon$ and $x,y \in X$ such that $f_t(x) = f_t(y)$. For example, for every $\epsilon_i = 1/i$, we could find such a t_i, x_i and y_i. Thus there is an **infinite sequence** $t_i \to 0$, and an infinite sequence of points $x_i \neq y_i \in X$ where f_{t_i} fails to be injective, i.e. such that

$$f_{t_i}(x_i) = G(x_i,t_i) = G(y_i,t_i) = f_{t_i}(y_i).$$

Since X **is compact**, we may pass to **subsequences which converges** $x_i \to x_0$ and $y_i \to y_0$. Then

$$G(x_0,0) = \lim_i G(x_i,t_i) = \lim_i G(y_i,t_i) = G(y_0,0).$$

But $G(x_0,0) = f_0(x_0)$ and $G(y_0,0) = f_0(x_0)$. By assumption, f_0 is injective, and **hence** $x_0 = y_0$.

Now, **after choosing local coordinates**, we can express the derivative of G at $(x_0,0)$ by the matrix

$$dG_{(x_0,0)} = \begin{pmatrix} & * \\ d(f_0)_{x_0} & \vdots \\ & * \\ 0 \cdots 0 & 1 \end{pmatrix}$$

where the 0's in the lowest row arise from the fact that the first coordinates do not depend on t, and the 1 is the derivative of the function $t \mapsto t$.

Since f_0 is an **immersion**, $d(f_0)_{x_0}$ has $k = \dim X$ independent rows. Thus the matrix of $dG_{(x_0,0)}$ has $k+1$ independent rows, and hence $dG_{(x_0,0)}$ is an injective linear map. Thus, G is an **immersion around** $(x_0,0)$ and hence G must be **one-to-one on some neighborhood of** $(x_0,0)$. **But**, since the sequences (x_i,t_i) and (y_i,t_i) both **converge to** $(x_0,0)$, for large i, both (x_i,t_i) and (y_i,t_i) belong to this neighborhood. This contradicts the injectivity of G.

(f) Assume that $f_0\colon X \to Y$ is a diffeomorphism. Since X is compact, this implies that Y **is compact as well**. Let f_t be a homotopy of f_0. We need to show that there is an $\epsilon > 0$ such that f_t is diffeomorphism for all $t < \epsilon$.

Since X **is compact**, X has only finitely many connected components, and so does Y. Hence we can check the statement for each of these connected components separately. For, this gives us an ϵ_i for each component. Since there are finitely many components, we can just take the minimum of the ϵ_i's as the ϵ for all of X and Y.

Thus we may assume that **X and Y are connceted**. By (a) and (e), we know that being a local diffeomorphism and being an embedding is a stable property. Thus there is a $\epsilon > 0$ such that f_t is a local diffeomorphism and an embedding. For f_t being a diffeomorphism, it remains to show that f_t is surjective.

We fix a $t < \epsilon$. Since f_t is a local diffeomorphism, it is open and hence **$f_t(X)$ is open in Y**. But $f_t(X)$ is also closed, since it is compact being the image of a compact space. Since Y is connected, this implies $f_t(X) = Y$. **QED**

Note that the condition that Z is **closed in Y** in point (d) is **necessary**. For a simple example, in $Y = \mathbb{R}^2$ we consider the subspace

$$Z = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1, y = 0\} \subset \mathbb{R}^2$$

(which is just image of an interval in $\mathbb{R}^2$). It a subspace which is neither open nor closed in $\mathbb{R}^2$. But Z is a one-dimensional submanifold of $\mathbb{R}^2$. Now, for $X = [-1,0]$, we define f_0 to be the smooth map

$$f_0 \colon [-1,0] \to \mathbb{R}^2,\ x \mapsto (x,0).$$

Since $f_0^{-1}(Z) = \emptyset$, **f_0 is transversal to Z**. But, for the homotopy f_t, given by

$$f_t \colon [-1,0] \times [0,1] \to \mathbb{R}^2,\ (x,t) \mapsto (x+t,0),$$

we have $f_t^{-1}(Z) \neq \emptyset$ **for every $t > 0$**. But both $\mathrm{Im}\,(df_x)$ and $T_{f(x)}(Z)$ are just $\mathbb{R}$ embedded as the x-axis in $\mathbb{R}^2 = T_{f(x)}(Y)$. Hence f_t is **not transversal** to Z **for any $t > 0$**. Note that this would not have happened if Z had been the closed submanifold $\{(x,0) : 0 \leq x \leq 1\}$.

An even more important assumption we made in the theorem is that X is **compact**. The next example will show that we cannot drop this assumption for any of the properties in theorem.

Compactness matters

The Stability Theorem fails when X is **not compact**. For a simple example, let $\rho \colon \mathbb{R} \to \mathbb{R}$ be a smooth function with $\rho(s) = 1$ for $|s| < 1$ and $\rho(s) = 0$ for $|s| > 2$. Then we define

$$f_t \colon \mathbb{R} \to \mathbb{R},\ f_t(x) = x\rho(tx).$$

For $t = 0$, $f_0(x) = x$ for all x, i.e. $f_0 = \mathrm{Id}$. Hence f_0 is a local diffeomorphism, an immersion, a submersion, an embedding, a diffeomorphism and transversal to every submanifold of $\mathbb{R}$.

But for **any fixed** $t > 0$, we have $|tx| > 2$ when $x > 2/|t|$. Hence, for this fixed t, $f_t(x) = 0$ **for all** $x > 2/|t|$.
Thus f_t **is neither** a local diffeomorphism, an immersion, a submersion, an embedding, nor a diffeomorphism, and is not transversal to $\{0\} \subset \mathbb{R}$.
We see what is going wrong when we replace the domain with a closed interval, i.e. a compact subspace of $\mathbb{R}$. Say $X = [a,b]$ with $b > 0$. Then we can choose $\epsilon > 0$ which is small enough such that $1/\epsilon > \max(|a|,|b|)$, and it would not be possible to choose x bigger than $1/|t|$. Then we had $f_t(x) = x$ for all x and all $t < \epsilon$.

LECTURE 14

Sard's Theorem and Morse functions

Now we are going to shift perspectives and ask:

Given a map f which does not have a property P. Is it possible to **bump f a little bit** such that it gets **property P**?

If this is possible for every map, P is a particularly nice property:

Generic properties

A property P of maps is called **generic** if, for any f_0, there is a homotopy F for f_0 and an $\epsilon > 0$ such that f_t has property P for all $t \in (0, \epsilon)$.

If we look back at the images we used to illustrate stable and unstable properties, we see that non-transversal intersections are rather the exception than the norm. Now we have a way to give this feeling a precise meaning: **Transversality is generic**.

We are not going to prove this statement for the moment, but content ourselves with looking at an important special case. Recall that transversality is a generalization of regularity:

$$f \pitchfork \{y\} \iff y \text{ is a regular value of } f.$$

An analog, though not equivalent, version of the above question is now: Given a smooth map $f\colon X \to Y$ and a critical value y. Is it possible to **bump y a little bit** such that it gets **regular**?

The answer is yes and is the content of a famous theorem:

Sard's Theorem

If $f\colon X \to Y$ is any smooth map of manifolds, then almost every point in Y is a regular value of f.

To say that "almost every point" is a regular value of f sounds sloppy, but is a well-defined term in measure theory. It means by definition that the complement of regular values in Y has **measure zero**. Since the complement of the regular values are the critical values, Sard's theorem says that the set of critical values of a smooth map of manifolds has measure zero.

Sard's Theorem for manifolds follows from Sard's Theorem in Calculus. We are not going to prove either of them, since the required techniques are not so interesting for this course.

Measure zero in a measure zero box

A rectangular solid in $\mathbb{R}^n$ is just a cartesian product of n intervals in $\mathbb{R}^n$, and its volume is the product of the lengths of the n intervals. An arbitrary set A in $\mathbb{R}^n$ is said to have **(Lebesgue) measure zero** if, for every $\epsilon > 0$, there exists a **countable** collection $\{S_1, S_2, \ldots\}$ of rectangular solids in $\mathbb{R}^n$, such that A is contained in the union of the S_i, and

$$\sum_{i=1}^{\infty} \mathrm{vol}\,(S_i) < \epsilon.$$

Then in a manifold X, an arbitrary subset $C \subset X$ has **measure zero** if, for every local parametrization ϕ of X, the preimage $\phi^{-1}(C)$ has measure zero in Euclidean space.
(Note that measure and volume depend on the ambient space.)
An example of a **measure zero** subset is given by the set of **rational numbers** in $\mathbb{R}$. Hence for measure theorists, "almost every" real number is irrational. This example illustrates that something that happens almost never, can still happen often enough to be noticed.

We learn from the previous box: By definition, no nonempty rectangular solid in $\mathbb{R}^n$ has measure zero. Hence it cannot be contained in a set of measure zero. Now, every nonempty open subset of $\mathbb{R}^n$ contains some nonempty rectangular solid. Thus, no nonempty open subset of $\mathbb{R}^n$ has measure zero. Hence, no nonempty open subset of a manifold Y has measure zero. In other words, no set of measure zero in a manifold Y can contain a nonempty open subset of Y.

In view of Sard's Theorem, this tells us that the set of critical values of a smooth map $f\colon X \to Y$ cannot contain any nonempty open subset of Y. Thus, its complement, the set of regular values, must have a **nonempty intersection with every nonempty open subset** of Y. A subset of a topological space with this property, i.e. having a nonempty intersection with every nonempty open subset, is called **dense**.

Hence we can rephrase Sard's Theorem in more topological terms by:

Sard's Theorem in dense form

The set of regular values of any smooth map $f\colon X \to Y$ is **dense** in Y. More generally, if $f_i\colon X_i \to Y$ are any countable number of smooth maps, then the points of Y that are simultaneously regular values for all of the f_i, are dense.

Morse Functions

Before we study a very interesting application of Sard's Theorem, we recall some terminology (we have already used these terms in the proof of the Fundamental Theorem of Algebra, but did not make a fuzz about it).

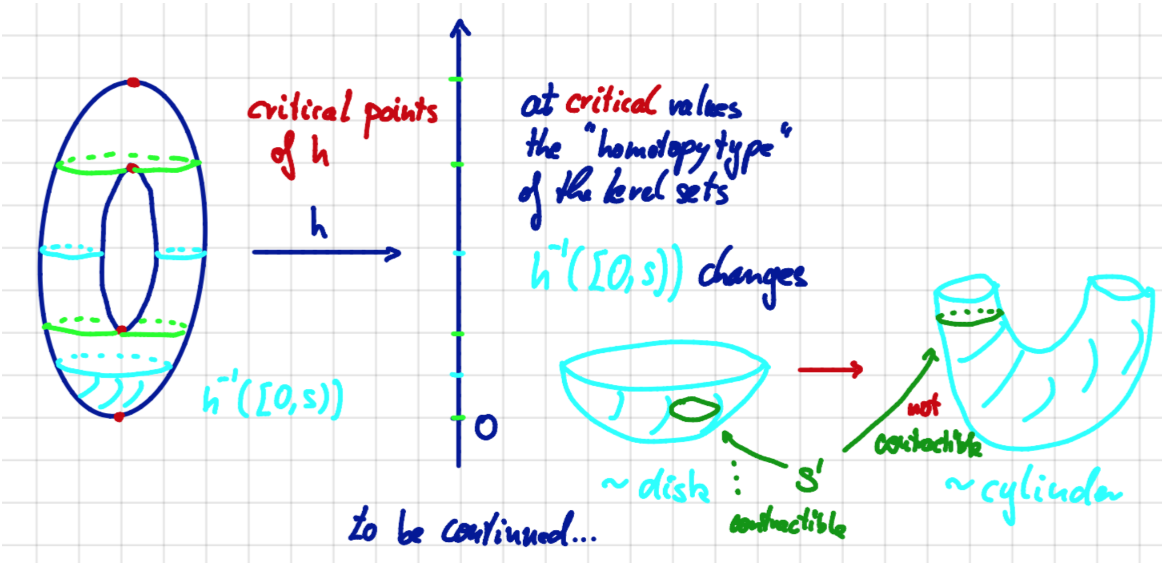
If $f\colon X \to Y$ is a smooth map, a regular value of f is a point $y \in Y$ such that df_x is surjective for every $x \in X$ with $f(x) = y$. We call such an $x \in X$ also a regular point of f. Note that this is the same as to say that f is regular at x. Hence y is a **regular value** of f if **every** $x \in f^{-1}(y)$ is a **regular point**.

On the other hand, if df_x is not surjective, we call x a critical point of f. Hence $y \in Y$ is a **critical value** if **at least one** of the points $x \in f^{-1}(y)$ is a **critical point**.

We understand the local behavior of smooth maps at regular points by the Local Submersion Theorem (up to diffeomorphism look like the canonical submersion). But what about the local behavior at critical points? In fact, it is often at critical points that the interesting stuff happens. It is often at critical points that the topology of a manifold can change.

For example, for a smooth map $f\colon X \to \mathbb{R}$, if X is **compact**, then we know that f must have a **maximum** and a **minimum**. At a point $x \in X$ where $f(x)$ is either a maximal or a minimal value, f cannot change in any direction in X. In other words, the derivative df_x must vanish (recall $df_x(h)$ is a measure for the change of f in direction h). Hence x is a critical point in our terminology.

A standard example is given by the height function on a torus:



So let us stick to smooth functions, i.e. smooth maps to $\mathbb{R}$. We want to understand how critical points look like locally. Let us look a smooth function $f\colon \mathbb{R}^k \to \mathbb{R}$. Locally around a point $c \in X$, we can describe f by

$$f(x) = f(c) + \sum_{i=1}^{k} \frac{\partial f}{\partial x_i}(c) \cdot (x_i - c_i) + \frac{1}{2} \sum_{i,j=1}^{k} \frac{\partial^2 f}{\partial x_i \partial x_j}(c) \cdot (x_i - c_i)(x_j - c_j) + o(|x|^3).$$

If c is a critical point, then by definition

$$df_c = (\partial f/\partial x_1(c), \dots, \partial f/\partial x_k(c)) = 0$$

(otherwise df_c was surjective as a linear map $\mathbb{R}^k \to \mathbb{R}$). Hence the best possible measure for the local behavior of f at c is the Hessian matrix of the second partial derivatives. Critical points where the Hessian matrix is invertible is the best we can hope for.

Nondegenerate critical points and Morse functions

For a smooth function $f\colon \mathbb{R}^k \to \mathbb{R}$, a point $c \in \mathbb{R}^k$ where df_c vanishes, but the Hessian matrix $H(f)_c = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(c)\right)$ is invertible at c, is called a **nondegenerate critical point**.
A smooth function $f\colon \mathbb{R}^k \to \mathbb{R}$ for which all critical points are nondegenerate is called a **Morse function**.

Nondegenerate critical points are much easier to study than arbitrary critical points, since they are **isolated from the other critical points**, i.e. there is an open neighborhood which does not contain any other critical points. Hence Morse functions are easier to understand than arbitrary smooth functions.

To see that nondegenerate critical points are isolated, we define a map $g\colon \mathbb{R}^k \to \mathbb{R}^k$ by the formula

$$g = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_k}\right). \tag{15}$$

$$\text{Then } df_x = 0 \iff g(x) = 0.$$

Moreover, the matrix representing the derivative dg_x is the Hessian of f at x. So if x is nondegenerate, then not only is $g(x) = 0$, but **g maps a neighborhood of x diffeomorphically onto a neighborhood of 0** as well. In particular, g is injective in that neighborhood of x. Thus g can be zero at no other points in this neighborhood, and f has no other critical point in this neighborhood.

Another reason to be interested in Morse functions is the fact that there are a lot of them.

Morse functions on $\mathbb{R}^k$ are generic

Let $f\colon U \to \mathbb{R}$ be a smooth function defined on some open $U \subseteq \mathbb{R}^k$ and $a \in \mathbb{R}^k$, define

$$f_a(x) = f(x) + a \cdot x.$$

Then, **for almost all** $a \in \mathbb{R}^k$, f_a is a Morse function.

Proof: We us again the function g from (15).The derivative of f_a at a point $p \in U$ then satisfies

$$(df_a)_p = \left(\frac{\partial f_a}{\partial x_1}(p), \dots, \frac{\partial f_a}{\partial x_k}(p)\right) = g(p) + a.$$

Hence the **critical points of f_a** are the points $p \in U$ with $g(p)+a = 0$. Moreover, the **Hessian of f_a** at p is the matrix dg_p, i.e.

$$H(f_a)_p = H(f)_p = dg_p.$$

Hence

$$\begin{aligned} f_a \text{ is Morse} &\iff \det(H(f_a)_p) \neq 0 \text{ at all critical points } p \\ &\iff \det(dg_p) \neq 0 \text{ at all } p \text{ with } g(p) + a = 0 \\ &\iff -a \text{ is a regular value of } g. \end{aligned}$$

By **Sard's Theorem**, $-a$ is a regular value of g for almost all $a \in \mathbb{R}^k$. Therefore almost every f_a is a Morse function. **QED**

Now we would like to transport the concept of nondegenerate critical points to **manifolds**. So let X be a smooth manifold. Suppose that $f\colon X \to \mathbb{R}$ has a **critical point at x** and that $\phi\colon U \to X$ is a local parametrization with $\phi(0) = x$. Then

$$d(f \circ \phi)_0 = df_x \circ d\phi_0$$

and hence 0 is a critical point for the function $f \circ \phi$. We call x a **nondegenerate critical point for f** if 0 is a nondegenerate critical point for $f \circ \phi$.

Independence of choice

Since we made a choice of a local parametrization for this definition, we need to make sure that the criterion is independent of the choice.
So let $\psi\colon V \to X$ be another local parametrization with $\psi(0) = x$. We define $\theta := \psi^{-1} \circ \phi\colon U \to V$. Since θ is a diffeomorphism, the critical points of $f \circ \phi$ and $f \circ \psi \circ \theta$ are the same.
Assuming that x is a critical point of f, i.e. $df_x = 0$, the chain rule implies for the two Hessian matrices at 0:

$$H(f \circ \phi)_0 = (d\theta_0)^t H(f \circ \psi)_0 d\theta_0.$$

Since $d\theta_0$ is invertible, we see

$$H(f \circ \phi)_0 \text{ is invertible} \iff H(f \circ \psi)_0 \text{ is invertible.}$$

An important result on Morse functions is that they can be described in some sort of canonical form. It extends our understanding of the local behavior of smooth maps.

Morse Lemma

Let X be a smooth manifold and $f\colon X \to \mathbb{R}$. Suppose that $a \in X$ is a **nondegenerate critical point** of f. Then there is a local parametrization $\phi\colon U \to X$ with $\phi(0) = a$ and local coordinate functions $\phi^{-1} = (x_1, \ldots, x_k)$ around a such that

$$f(x) = f(a) + \sum_{ij} h_{ij} x_i x_j$$

for all $x \in \phi(U)$ where the h_{ij} are the entries of the Hessian of f at a:

$$h_{ij} = (H(f \circ \phi))_0)_{ij} = \frac{\partial^2 (f \circ \phi)}{\partial x_i \partial x_j}(0).$$

(Note that the h_{ij} depend on the chosen coordinate system.)

We are not going to discuss the proof of this classical result. However, we are going to show that it applies to many functions.

In fact, we can generalize the fact that "almost all" functions are Morse to the level of **manifolds**: Suppose $X \subset \mathbb{R}^N$, and let $x_1, \ldots, x_N \in \mathbb{R}^N$ be the usual coordinate functions on R^N. If $f\colon X \to \mathbb{R}$ is a smooth function on X and $a = (a_1, \ldots, a_N)$ is an N-tuple of numbers, we define again a new function $f_a\colon X \to \mathbb{R}$ by

$$f_a := f + a_1 x_1 + \cdots + a_N x_N.$$

Morse functions on any manifold are generic

For every smooth function $f\colon X \to \mathbb{R}$ and for **almost every** $a \in \mathbb{R}^N$, f_a is a Morse function on X, i.e. all its critical points are nondegenerate.

Proof: We would like to use the above result for $U \subset \mathbb{R}^k$ open. Since $X \subset \mathbb{R}^N$ is in general **not open** (in fact, it is never open if $\dim X < N$), the **strategy is to cover X** by open subsets and then try to lift the k-dimensional result to open sets in $\mathbb{R}^N$.

So let x be any point in X. First we are going to choose a suitable local coordinate system around x. Let $v_1, \ldots, v_k \in \mathbb{R}^N$ be a basis of $T_x(X)$ (for $k = \dim X$). Then the matrix $[v_1 \cdots v_k]$, having the v_i's as columns, has rank k. Hence it has k linearly independent rows, say $i_1, \ldots, i_k$. Let $\pi\colon \mathbb{R}^N \to \mathbb{R}^k$ be projection defined by $(x_1, \ldots, x_N) \mapsto (x_{i_1}, \ldots, x_{i_k})$ where the $x_1, \ldots, x_N$ denote the standard coordinates on $\mathbb{R}^N$. Then

$$(d\pi_x)_{|T_x(X)}\colon T_x(X) \to \mathbb{R}^k \text{ is an isomorphism}$$

by construction. Hence, by the Inverse Function Theorem,

$$\pi_{|X}\colon X \to \mathbb{R}^k \text{ is a local diffeomorphism.}$$

Hence we can take the k-tuple of functions $(x_{i_1}, \ldots, x_{i_k})\colon X \to \mathbb{R}^k$ to define a **local coordinate system around x**.

Therefore we can cover X with open subsets $U_\alpha \subseteq \mathbb{R}^N$ such that on each U_α **some k-tuple** of the functions $x_1, \ldots, x_N$ on $\mathbb{R}^N$ form a coordinate system. Moreover, it is always possible to choose a **countable** subfamily of the U_α's. Hence we may assume there are only **countably many U_α**.

Let $S \subset \mathbb{R}^N$ be the subset of a such that f_a is **not Morse**. Since the countable union of sets with measure zero has measure zero, it suffices to show that **for each U_α** the set S_α of a's such that $f_a \colon U_\alpha \to \mathbb{R}$ is **not Morse**, has **measure zero**.

So let us look at one of the U_α's. We want to show that S_α has measure zero in $\mathbb{R}^N$.

For simplicity, assume $x_1, \ldots, x_k$ form a coordinate system around x on U_α. We can write any $a \in \mathbb{R}^N$ as $a = (b,c)$, where b denotes the **first k** coordinates and c denotes the **last $N-k$** coordinates. Around a given point x, we can thus write

$$f_a(x) = f(x) + c \cdot (x_{k+1}, \ldots, x_N) + b \cdot (x_1, \ldots, x_k).$$

The function $x \mapsto f(x) + c \cdot (x_{k+1}, \ldots, x_N)$ is smooth. Hence we can apply our previous result on genericity of Morse functions on open subsets in $\mathbb{R}^k$ to this function and get that **f_a is a Morse function for almost every $b \in \mathbb{R}^k$**.

Thus, for a fixed c, the subset of all $b \in \mathbb{R}^k$ where f_a is not Morse, has measure zero in $\mathbb{R}^k$. Hence $S_\alpha \cap (\mathbb{R}^k \times \{0\})$ has measure zero in $\mathbb{R}^N$. It is a calssical result in Measure Theory, called **Fubini's Theorem**, which then implies that the set S_α of all $a = (b,c)$ where a does not yield a Morse function has measure zero in $\mathbb{R}^N$. Hence f_a is a Morse function for almost every a. **QED**

Finally, we can also show that being a Morse function is a stable property. In order to prove stability, we start with a little lemma:

First Lemma

Let f be a smooth function on an open set $U \subset \mathbb{R}^k$. For each $x \in U$, let $H(f)_x$ be the Hessian matrix of f at x. Then f is a Morse function if and only if

$$(\det(H(f)_x))^2 + \sum_{i=1}^{k} \left(\frac{\partial f}{\partial x_i}(x) \right)^2 > 0 \text{ for all } x \in U. \tag{16}$$

Proof: A point x is regular if $df_x = (\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_k}(x)) \neq 0$, and x is a nondegenerate critical point if $df_x = 0$ and $\det(H(f)_x) \neq 0$. Hence f is Morse if and only if (16) is satisfied. **QED**

Second Lemma

Suppose that f_t is a homotopic family of functions on $\mathbb{R}^k$. If f_0 is a Morse function on some oepn subset $U \subset \mathbb{R}^k$ containing a **compact** set $K \subset \mathbb{R}^k$, then so is every f_t for t sufficiently small.

Proof: We define the map

$$F: U \times [0,1] \to \mathbb{R},\ (x,t) \mapsto (\det(H(f_t)_x))^2 + \sum_{i=1}^{k} \left(\frac{\partial f}{\partial x_i}(x) \right)^2.$$

Since f is smooth, F depends smoothly on both variables. By the First Lemma and the assumption, we know $F(x,0) > 0$ for all $x \in U \times \{0\}$. Since $K \subset U$ is compact, F has a minimum on $K \times \{0\}$, i.e. there is a $\delta > 0$ such that $F(x,0) \geq 2\delta$ for all $x \in K$. Since F is continuous, there an open neighborhood $W \subset U \times [0,1]$ containing $K \times \{0\}$ such that $F(x,t) > \delta$ for all $(x,t) \in W$. In fact, we can cover $K \times \{0\}$ by open subsets $W_i \subset U \times [0,1]$ such that $F(x,t) > \delta$ for all $(x,t) \in W_i$. Each such open subset W_i has the form $V_i \times [0,\epsilon_i)$ for some open $V_i \subset U$ and $\epsilon_i > 0$. Since K is compact, **finitely many** such open W_i suffice to cover $K \times \{0\}$. Let ϵ be the **minimum** of the finitely many ϵ_i. Then we have $F(x,t) > \delta$ for all $(x,t) \in K \times [0,\epsilon)$. Since F is continuous, for any fixed $t \in [0,\epsilon)$, there is again an open subset $V\mathbb{R}^k$ containing K such that $F(x,t) > 0$ for all $(x,t) \in V \times \{t\}$. Thus f_t is Morse in a neighborhood of K for all sufficiently small t. **QED**

Finally, we are ready to prove stability of Morse functions.

Stability of Morse functions

Let X be a **compact** smooth manifold, let $f_0 \colon X \to \mathbb{R}$ be a smooth function and f_t be a homotopy of f_0. If f_0 is Morse, then there is an $\epsilon > 0$ such that f_t is a Morse function for all $t \in [0,\epsilon)$.

Proof: For $x \in X$, let $\phi_x \colon U_x \to X$ be a local parametrization around x. Then $f_0 \circ \phi_x$ is a Morse function on U. Since $\{0\}$ is a compact subset of U, the Second Lemma above implies that there is an open subet $V_x \subset U_x$ containing $\{0\}$ and an $\epsilon(x) > 0$ such that f_t is Morse on V_x for all $t \in [0,\epsilon(x))$. The images $\phi_x(V_x)$ are open in X and cover X. Since X is compact, finitely many suffice to cover X, say

$$X = \phi_{x_1}(V_{x_1}) \cup \cdots \cup \phi_{x_n}(V_{x_n})$$

Then we can set $\epsilon :=$ minimum of $\epsilon(x_1), \ldots, \epsilon(x_n)$. Then $f_t \colon X \to \mathbb{R}$ is a Morse function for all $t \in [0,\epsilon)$. **QED**

LECTURE 15

Embedding Manifolds in Euclidean Space

We have two objectives today. The first one is to study how manifolds can be embedded into Euclidean space. In particular, given a k-dimensional manifold, what is the minimal N such that we can be sure that there is an embedding $X \subset \mathbb{R}^N$? The second one is to give an intrinsic definition of manifolds. In the next lecture, we are going to relate these two objectives and show that every abstract smooth manifold can be embedded into some Euclidean space.

To address the first question we need a useful new device, the tangent bundle.

The Tangent Bundle

Let $X \subset \mathbb{R}^N$ be a smooth manifold. For every $x \in X$, the tangent space $T_x(X)$ to X at x is a vector subspace of $\mathbb{R}^N$. If we let x vary, these tangent space will in general overlap in $\mathbb{R}^N$. (For example, if X is a vector space, they will all be equal.)

Hence, in order to be able to keep track of the information contained in all the different tangent spaces, we need a smart device that keeps those spaces apart:

Tangent bundles

The **tangent bundle** of X, denoted $T(X)$, is the subset of $X \times \mathbb{R}^N$ defined by

$$T(X) := \{(x,v) \in X \times \mathbb{R}^N : v \in T_x(X)\}.$$

In particular, $T(X)$ contains a natural copy X_0 of X, consisting of the points $(x,0)$. In the direction perpendicular to X_0, it contains copies of each tangent space $Tx(X)$ embedded as the sets

$$\{(x,v) \in T(X) : \text{ for a fixed } x\}.$$

There is a natural projection map

$$\pi\colon T(X) \to X,\ (x,v) \mapsto x.$$

Any smooth map $f\colon X \to Y$ induces a **global derivative map**

$$df\colon T(X) \to T(Y),\ (x,v) \mapsto (f(x), dfx(v)).$$

Note that, since $X \subset \mathbb{R}^N$ and $T_x(X) \subset \mathbb{R}^N$ for every x, $T(X)$ is also a subset of Euclidean space:

$$T(X) \subset \mathbb{R}^N \times \mathbb{R}^N.$$

Therefore, if $Y \subset \mathbb{R}^M$, then df maps a subset of R^{2N} into $\mathbb{R}^{2M}$.

We claim that **df is smooth**. For since $f\colon X \to \mathbb{R}^M$ is smooth, it extends by definition around any point $x \in X$ to a smooth map $F\colon U \to \mathbb{R}^M$, where U is an open set of R^N. Then $dF\colon T(U) \to \mathbb{R}^{2M}$ **locally extends df**. But, since $U \subset \mathbb{R}^N$ is open and hence $T_u(U) = \mathbb{R}^N$ for every $u \in U$, $T(U)$ is all of $U \times \mathbb{R}^N$. Since $U \times \mathbb{R}^N$ is an open set in $\mathbb{R}^{2N}$, dF is a linear and hence smooth map defined on an open subset of $\mathbb{R}^{2N}$. This shows that $df\colon T(X) \to \mathbb{R}^{2M}$ may be locally extended to a smooth map on an open subset of R^{2N}, meaning that df is smooth.

Given smooth maps $f\colon X \to Y$ and $g\colon Y \to Z$, the global derivative of the composite is equal to the composite of global derivatives:

$$d(g \circ f) = dg \circ df\colon T(X) \to T(Z).$$

For, the chain rule implies that, for any $(x,v) \in T(X)$,

$$\begin{aligned} d(g \circ f)(x,v) &= ((g \circ f)(x), d(g \circ f)_x(v) \\ &= ((g(f(x)), (dg_{f(x)} \circ df_x)(v)) \\ &= dg(df(x,v)) \\ &= dg \circ df(x,v). \end{aligned}$$

As a consequence we get:

Tangent bundles are intrinsic

If $f\colon X \to Y$ is a diffeomorphism, so is $df\colon T(X) \to T(Y)$. For the chain rule implies that $df^{-1} \circ df$ is the identity map of $T(X)$ and $df \circ df^{-1}$ is the identity map of $T(Y)$. Thus diffeomorphic manifolds have diffeomorphic tangent bundles. As a result, $T(X)$ is an object intrinsically associated to X.

Finally, we are going to show that $T(X)$ is in fact itself a smooth manifold. Let W be an open set of X. In particular, W is also a manifold, and we can consider its tangent bundle $T(W)$. Since $T_x(W) = T_x(X)$ for every $x \in W$, $T(W)$ is by definition

$$T(W) = \{(x,v) \in T(X) : x \in W\} = T(X) \cap (W \times \mathbb{R}^N) \subset T(X).$$

Since $W \times \mathbb{R}^N$ is open in $X \times \mathbb{R}^N$, **$T(W)$ is open in $T(X)$**.

Now suppose that W is the image of a **local parametrization** $\phi\colon U \to W$, where U is an open set in $\mathbb{R}^k$. Then the global derivative $d\phi\colon T(U) \to T(W)$ is a diffeomorphism. But $T(U) = U \times \mathbb{R}^k$ is an open subset of R^{2k}, so **$d\phi$ is a parametrization** of the open set $T(W)$ in $T(X)$. Since every point of $T(X)$ sits in such a neighborhood, we have proved the following useful result:

Tangent bundles are manifolds

The tangent bundle of a manifold X is a smooth manifold of dimension $\dim T(X) = 2 \dim X$.

Whitney's Theorem

Whitney's Theorem

Every k-dimensional manifold admits a one-to-one immersion in $\mathbb{R}^{2k+1}$.

Proof: Let $X \subset \mathbb{R}^N$ be k-dimensional manifold which is a subset in $\mathbb{R}^N$ for some $N > 2k+1$. In particular, we are given an **injective immersion $X \to \mathbb{R}^N$**. Our goal is to show that we can choose N to be $2k+1$ and still have an injective immerison. Therefor we are going to construct a linear projection $\mathbb{R}^N \to \mathbb{R}^{2k+1}$ that restricts to a one-to-one immersion $X \to \mathbb{R}^{2k+1}$ on X.

The construction works by induction: Whenever we are given an injective immersion $f\colon X \to \mathbb{R}^M$ with $M > 2k+1$, then there exists a unit vector $a \in \mathbb{R}^M$ such that the composition of f with the projection map carrying $\mathbb{R}^M$ onto the orthogonal complement of a is still an injective immersion. The complement $H := \{b \in \mathbb{R}^M : b \perp a\}$ is an $M-1$-dimensional vector subspace of $\mathbb{R}^M$, hence isomorphic to R^{M-1}. Thus, after choosing a basis for H, we obtain an injective immersion into $\mathbb{R}^{M-1}$.

Continuing this procedure yields a chain of linear maps

$$\mathbb{R}^N \to \mathbb{R}^{N-1} \to \cdots \to \mathbb{R}^{2k+1}$$

such that the composition $X \to \mathbb{R}^N \to \mathbb{R}^{2k+1}$ is still an injective immersion.

So let us **assume we have an injective immersion**

$$f\colon X \to \mathbb{R}^M \text{ with } M > 2k+1.$$

We define two smooth maps

$$\begin{array}{ccc} & & X \times X \times \mathbb{R} \\ & & \downarrow h \\ T(X) & \xrightarrow[g]{} & \mathbb{R}^M \end{array}$$

by

$$h\colon X \times X \times \mathbb{R} \to \mathbb{R}^M,\ (x,y,t) \mapsto t(f(x) - f(y)).$$

and

$$g\colon T(X) \to \mathbb{R}^M,\ (x,v) \mapsto df_x(v).$$

By **Sard's theorem**, the sets S_g and $_h$ of critical values of g and h, respectively, have measrure zero in $\mathbb{R}^M$. Hence the union of S_g and S_h still has measure zero in $\mathbb{R}^M$. Thus the intersection of the sets of regular values of g and h, which is the complement of $S_g \cup S_h$, is nonempty.

Since $\dim T(X) = 2k$, $\dim X \times X \times \mathbb{R} = 2k+1$, but $M > 2k+1$, the only regular values of g and h are the points in $\mathbb{R}^M$ which are not in the image of g or h. Hence there exists a point $a \in \mathbb{R}^M$ which is neither in the image of g nor in the image of h. Note that, since 0 belongs to both images, we must have $a \neq 0$.

Let π be the projection of $\mathbb{R}^M$ onto the orthogonal complement H of a.

First claim: $\pi \circ f\colon X \to H$ is **injective**.

For suppose that $\pi \circ f(x) = \pi \circ f(y)$. Then, since π is **linear**, we have $\pi(f(x) - f(y)) = 0$, i.e.

$$\begin{aligned} f(x) - f(y) \in \operatorname{Ker}(\pi) &= \operatorname{span}(a) \text{ in } \mathbb{R}^M \\ &= \{w \in \mathbb{R}^M : w = t \cdot a \text{ for some } t \in \mathbb{R}\}. \end{aligned}$$

Thus there is a $t \in \mathbb{R}$ with $f(x) - f(y) = ta$. If $x \neq y$ then $t \neq 0$, since f **is injective**. But then

$$a = 1/t(f(x) - f(y)) = h(x,y,1/t)$$

which contradicts the choice of a.

Second claim: $\pi \circ f\colon X \to H$ is an **immersion**.

For suppose there was a nonzero vector v in $T_x(X)$ for which $d(\pi \circ f)_x = 0$. Because π is **linear**, the chain rule yields

$$d(\pi \circ f)_x = \pi \circ df_x.$$

Thus $\pi(df_x(v)) = 0$, so $df_x(v) = ta$ for some $t \in \mathbb{R}$. Because **f is an immersion**, we must have $ta \neq 0$. But since we know $a \neq 0$, this implies $t \neq 0$. Thus, since df_x is linear,

$$a = \frac{1}{t} df_x(v) = df_x(\frac{1}{t}v) = g(x, \frac{1}{t}v)$$

which again contradicts the choice of a. **QED**

For compact manifolds, one-to-one immersions are the same as embeddings. So we have just proved the embedding theorem in the compact case.

Whitney's Embedding for compact manifolds

Every compact k-dimensional manifold admits an embedding in $\mathbb{R}^{2k+1}$.

Note that Whitney's result does **not** give us the minimal N for an individual manifold. For example, we know that S^n is embedded in $\mathbb{R}^{n+1}$ for every n. The result tells us that, in general, **$N = 2k + 1$ will always work**. In fact, Whitney showed that $N = 2k$ always works. But the proof is much harder, and we will not discuss it in this course.

In order to extend Whitney's theorem (for $N = 2k + 1$) to noncompact manifolds, we must modify the immersion to make it proper. This is a topological, not a differential problem.

Before we develop the necessary tools to address this problem, we are going to contemplate a bit on a way to define manifolds without referring to a given embedding into some $\mathbb{R}^N$. The key idea that should be preserved in any new definition should be that **a manifold is a space which locally looks like Euclidean space**.

Abstract smooth manifolds

Hausdorff spaces

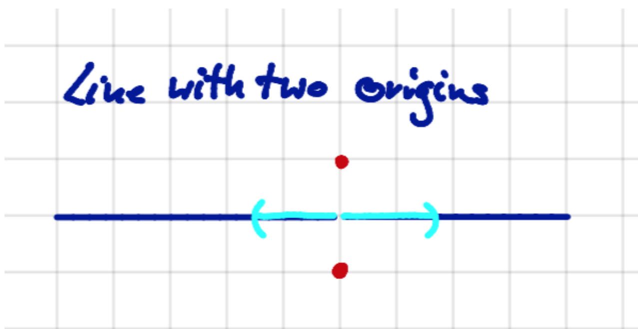
A topological space X is called **Hausdorff** if, for any two distinct points $x,y \in X$, there are two **disjoint** open subsets $U,V \subset X$ such that $x \in U$ and $y \in V$.

In other words, in a Hausdorff space we can separate points by open neighborhoods.

Every subspace of $\mathbb{R}^N$ (with the relative topology) is a Hausdorff space. However, there are spaces which are not Hausdorff.

For a typical example, consider two copies of the real line $Y_1 := \mathbb{R} \times \{1\}$ and $Y_2 := \mathbb{R} \times \{2\}$ as subspaces of $\mathbb{R}^2$. On $Y_1 \cup Y_2$, we define the equivalence realtion $(x,1) \sim (x,2)$ for all $x \neq 0$.

Let X be the set of equivalence classes. The topology on X is the quotient topology defined as follows: a subset $W \subset X$ is open in X if and only if both its preimages in $\mathbb{R} \times \{1\}$ and $\mathbb{R} \times \{2\}$ are open.



Then X looks like the real line except that the origin is replaced with two different copies of the origin. Away from the double origin, X looks perfectly nice like a one-dimensional manfiold. But every neighborhood of one of the origins contains the other. Hence we cannot separate the two origins by open subsets, and X is **not Hausdorff**.

For our definition of an abstract manifold, we want to avoid such pathological spaces.

Abstract manifolds

Let X be a topological space.
A **chart** on X is a pair (V,ϕ) where $V \subset X$ is an open subset and $\phi\colon V \to U$ is a homeomorphism from V to an open subset $U \subset \mathbb{R}^k$ of $\mathbb{R}^k$.
An **abstract smooth k-manifold** is a Hausdorff space X together with a (countable) collection of charts (V_α,ϕ_α) on X such that

(1) every point in X is in the domain of some chart, and

(2) for every pair of overlapping charts ϕ_α and ϕ_β, i.e.

$$V_{\alpha\beta} := V_\alpha \cap V_\beta \neq \emptyset,$$

the **change-of-coordinates map**

$$\phi_\beta \circ \phi_\alpha^{-1}\colon \phi_\alpha(V_{\alpha\beta}) \to \phi_\beta(V_{\alpha\beta})$$

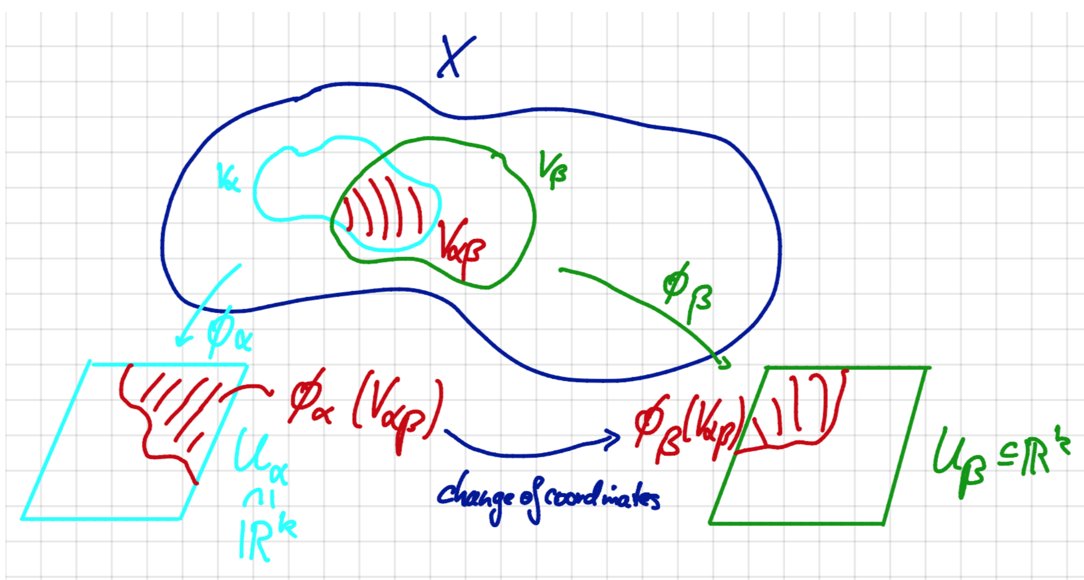
is **smooth** (as a map between open subsets of $\mathbb{R}^k$). In fact, this means that the change-of-coordinates maps are diffeomorphisms, since they are mutual smooth inverses to each other.

Let X be an abstract smooth k-manifold and $f\colon X \to \mathbb{R}^n$ be a continuous map. Then f is smooth if for every chart $\phi_\alpha\colon V_\alpha \to U_\alpha$, the composition $f \circ \phi_\alpha^{-1}\colon U_\alpha \to \mathbb{R}^n$ is smooth.

More generally, let X be an abstract smooth k-manifold and Y an abstract smooth m-manifold and $f\colon X \to Y$ a continuous map. Then **f is smooth at $x \in X$** if, for every chart $\phi\colon V \to U$ on X around x and every chart $\psi\colon V' \to U'$ on Y around $f(x)$, the map

$$\psi \circ f_{|V \cap f^{-1}(V')} \circ \phi^{-1}_{|U \cap \phi(f^{-1}(V'))}\colon U \cap \phi(f^{-1}(V')) \to U'$$

is a smooth map as a map from an open subset of $\mathbb{R}^k$ to an open subset of $\mathbb{R}^m$. We call f smooth if it is smooth at every $x \in X$.



Note that the smooth k-dimensional manifolds $X \subset \mathbb{R}^N$ we have been studying so far are examples of abstract smooth k-manifolds:

- The **Hausdorff** property is satisfied in $\mathbb{R}^N$ and therefore also for every subspace of $\mathbb{R}^N$ (with relative topology we have been using).
- Moreover, every open cover $\{U_\alpha\}$ of $\mathbb{R}^N$ has a **countable refinement**. For, we can take the collection of all open balls which are contained in some U_α, which have rational radii, and which are centered at points having only rational coordinates.
- For an open cover $\{V_\alpha\}$ of a subset $X \subset \mathbb{R}^N$, we can write $V_\alpha = U_\alpha \cap X$ for some open subsets U_α of $\mathbb{R}^N$. Then let $\{\tilde{U}_i\}$ be a **countable refinement** of $\{U_\alpha\}$ in R^N, and define $\tilde{V}_i = \tilde{U}_i \cap X$.
- The **charts** are just what we called **local coordinates** and the inverses of charts are what we called **local parametrizations**. One difference is that we required local parametrizations to be diffeomorphisms. For an abstract manifold X, we need the charts to **define what smoothness means** for a map on X. Hence a priori it makes only sense to talk about the smoothness of the change of coordinate maps. A posteriori we can then check that charts are in fact diffeomorphisms.
- Similarly for smooth maps between manifolds. We only know what smoothness of maps between Euclidean spaces means. Hence we need to use the charts to first translate the maps into maps between Euclidean spaces.
- In the abstract definition, we take care of the fact that the images of the charts/local parametrizations overlap. In fact, we use the overlap to define the smooth structure.
- Finally, a chosen collection of charts is called an **atlas** on the manifold. One can show that every manifold has a maximal atlas, i.e. the images of the charts are as "big as possible".

Here is an important example which we can easily be described with the new definition of an abstract manifold, but for which it is not obvious how we can embedd it into $\mathbb{R}^N$.

(Actually, it is a difficult question how to embedd these guys into $\mathbb{R}^N$ with N as small as possible. In fact, if $n = 2^m$ for some m and if there is an immersion $\mathbb{R}\mathrm{P}^n \to \mathbb{R}^N$, then N must be at least $2^m - 1$. You will learn about the techniques to show this in the Algebraic Topology course.)

Real Projective Space

The **real projective n-space $\mathbb{R}\mathrm{P}^n$** is the set of all straight lines through the origin in R^{n+1}. As a topological space, $\mathbb{R}\mathrm{P}^n$ is the quotient space

$$\mathbb{R}\mathrm{P}^n = (\mathbb{R}^{n+1} \setminus \{0\}) / \sim$$

where the equivalence relation is given by $x \sim y$ if there is a nonzero real number λ such that $x = \lambda y$. This means that a subset V is open in $\mathbb{R}\mathrm{P}^n$ if and only if its preimage $U = \{x \in \mathbb{R}^{n+1} \setminus \{0\} : [x] \in V\}$ is open in $\mathbb{R}^{n+1} \setminus \{0\}$.

Note that since each line through the origin intersects the unit sphere in two (antipodal) points, $\mathbb{R}\mathrm{P}^n$ can alternatively be described as

$$S^n / \sim$$

where the equivalence relation is $x \sim -x$. As a quotient of S^n, we see that $\mathbb{R}\mathrm{P}^n$ is **compact**.

We claim that $\mathbb{R}\mathrm{P}^n$ is an **abstract n-dimensional smooth manifold**.

If $x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$, we write $[x]$ for its equivalence class considered as a point in $\mathbb{R}\mathrm{P}^n$. One also often writes $[x] = [x_0 : \ldots : x_n]$.

For $0 \leq i \leq n$, let

$$V_i := \{[x] \in \mathbb{R}P^n : x_i \neq 0\}.$$

The preimage of V_i in $\mathbb{R}^{n+1}$ is the open subset $\{x \in \mathbb{R}^{n+1} : x_i \neq 0\}$. Hence each V_i is open in $\mathbb{R}\mathrm{P}^n$. By varying i, this gives an open cover of $\mathbb{R}\mathrm{P}^n$ because every representative $(x_0, \ldots, x_n)$ of a point $[x] \in \mathbb{R}\mathrm{P}^n$ must have at least one coordinate $\neq 0$ (otherwise it would be the origin which is excluded).

For each i, we have the maps $\phi_i \colon \mathbb{R}^n \to V_i$

$$(x_0, \ldots, \widehat{x_i}, \ldots, x_n) \mapsto [x_0 : \ldots : x_{i-1} : 1 : x_{i+1} : \ldots : x_n].$$

and $\phi_i^{-1} \colon V_i \to \mathbb{R}^n$

$$[x_0 : \ldots : x_i : \ldots : x_n] \mapsto \frac{1}{x_i}(x_0, \ldots, \widehat{x_i}, \ldots, x_n)$$

where $\widehat{x_i}$ means that x_i is omitted.

Since we use a representative of an equivalence class for the definition of ϕ_i^{-1}, we need to check that the definition is independent of the chosen representative. But if $[x_0 : \ldots : x_i : \ldots : x_n] = [\lambda x_0 : \ldots : \lambda x_i : \ldots : \lambda x_n]$ for some $\lambda \neq 0$, then

$$\begin{aligned}\phi_i^{-1}([\lambda x]) &= \frac{1}{\lambda x_i}(\lambda x_0, \ldots, \lambda x_{i-1}, \lambda x_{i+1}, \ldots, \lambda x_n) \\ &= \frac{1}{x_i}(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = \phi_k^{-1}([x]).\end{aligned}$$

It is easy to see that ϕ and ϕ_i^{-1} are mutual inverses which are both continuous.

Finally, the change-of-coordinate maps are smooth: For

$$\phi_i^{-1}(V_i \cap V_j) \xrightarrow{\phi_i} V_i \cap V_j \xrightarrow{\phi_j^{-1}} \phi_j^{-1}(V_i \cap V_j)$$

is just

$$(x_0, \ldots, \widehat{x_i}, \ldots, x_n) \mapsto \frac{1}{x_j}(x_0, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, \widehat{x_j}, \ldots, x_n)$$

which is **smooth** whenever $x_j \neq 0$.

To have such an intrinsic definition of a manifold is important and nice. However, the definition is quite abstract indeed. And, in fact, we are going to show that every abstract smooth manifold can be embedded into Euclidean space and is therefore a manifold for our previous definition. Hence all the machinery we have developed can be applied to abstract manifolds.

LECTURE 16

Embedding Abstract Manifolds in Euclidean Space

We start with some general facts and some terminology.

The closure of a subset

Let X be a topological space and A be an arbitrary subset. The **closure of A** in X, denoted $\overline{A}$, is the intersection of all closed subsets on X which contain A.
For example, the closure of an open ball $B_\epsilon(0)$ in $\mathbb{R}^N$ is just the closed ball

$$\overline{B_\epsilon(0)} = \{x \in \mathbb{R}^N : |x| \leq \epsilon\}.$$

We need the closure of a subset for example when we want to talk about the support of a function:

Support of a function

Let X be a smooth manifold and $f\colon X \to \mathbb{R}$ be a smooth function $f\colon X \to \mathbb{R}$. The **closed** subset

$$\operatorname{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}}$$

is called the **support of f**.

We are now going to introduce a fundamental tool for studying manifolds.

Partition of unity

Let X be an abstract smooth k-manifold and let $\{U_\alpha\}$ be an open cover, i.e. a collection of open subsets in X which cover X. A sequence of smooth functions $\{\rho_i\colon X \to \mathbb{R}\}$ is called a **partition of unity subordinate to the open cover $\{U_\alpha\}$** if it has the following properties:

(a) $0 \leq \rho_i(x) \leq 1$ for all $x \in X$ and all i.

(b) Each $x \in X$ has a neighborhood on which all but finitely many functions ρ_i are identically zero.
(c) For each i, $\text{supp}(\rho_i) \subset U_\alpha$ for some α.
(d) For each $x \in X$, $\sum_i \rho_i(x) = 1$. (Note that according to (b), this sum is always finite.)

The most general existence result for partitions of unity (without assuming that each ρ_i is smooth) is that they exist on every paracompact space, i.e. spaces on which every open cover has a locally finite refinement (every point has a neighborhood that intersects only finitely many sets in the cover).

Before we prove that partictions of unity exist on manifolds, we need some preparation.

Separating closed subsets

Let A and C be disjoint closed subsets in $\mathbb{R}^N$. Then there are disjoint open subsets U and V such that $A \subset U$ and $C \subset V$.

Proof: For each $a \in A$, choose an $\epsilon_a > 0$ such that $B_{2\epsilon_a}(a) \cap C = \emptyset$. This is possible since C is closed. Similarly, for each $c \in C$, choose an $\epsilon_c > 0$ such that $B_{2\epsilon_c}(c) \cap A = \emptyset$. We define

$$U := \cup_{a \in A} B_{\epsilon_a}(a) \text{ and } V := \cup_{c \in C} B_{\epsilon_c}(c).$$

Then U and V are open subsets with $A \subset U$ and $C \subset V$. We claim that U and V are disjoint.

For, if $x \in U \cap V$, then

$$x \in B_{\epsilon_a}(a) \cap B_{\epsilon_c}(c)$$

for some $a \in A$ and $c \in C$. By the triangle inequality, this implies

$$|a - c| < \epsilon_a + \epsilon_c.$$

But, if $\epsilon_a \leq \epsilon_c$, then $|a - c| < 2\epsilon_c$ and $a \in B_{2\epsilon_c}(c)$. And, if $\epsilon_c \leq \epsilon_a$, then $|a - c| < 2\epsilon_a$ and $c \in B_{2\epsilon_a}(a)$. Both cases are impossible. **QED**

Another important tool that we wil need are smooth bump functions. We have met them in a previous lecture. Today we will need them in a slightly more interesting form:

Smooth bump functions revisited

Let $U \subset \mathbb{R}^N$ be open and $K \subset U$ be compact. Then there is a smooth function $\varphi\colon \mathbb{R}^N \to \mathbb{R}$ with $\varphi(x) = 1$ for all $x \in K$ and $\varphi(x) = 0$ for all $x \in \mathbb{R}^N \setminus C$ for some closed subset C with $K \subset C \subset U$.

Proof: Recall the smooth function

$$f\colon \mathbb{R} \to \mathbb{R},\ f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

For any given $\epsilon > 0$, we define a function

$$f_\epsilon(x) := f(x)f(x-\epsilon).$$

As a product of two smooth functions, f_ϵ is smooth.

Next we define yet another function

$$g_\epsilon\colon \mathbb{R} \to \mathbb{R},\ g_\epsilon(x) := \frac{\int_0^x f_\epsilon(t)dt}{\int_0^\epsilon f_\epsilon(t)dt}.$$

By the Fundamental Theorem of Calculus, g_ϵ is smooth, nondecreasing, and

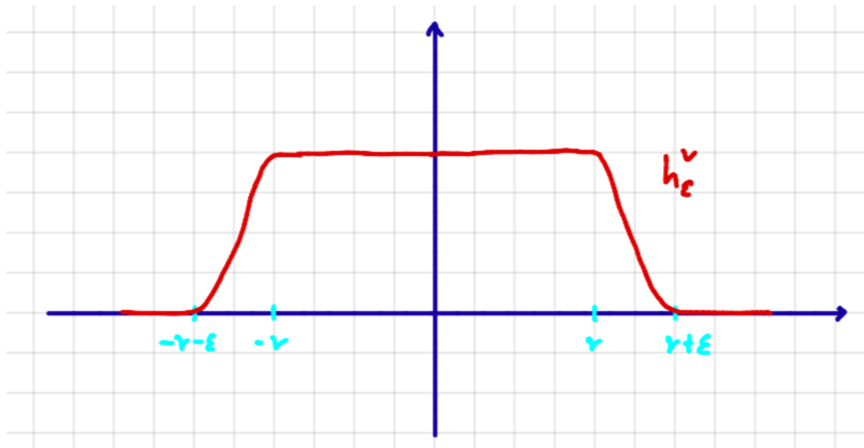
$$\begin{cases} g_\epsilon(x) = 0 & x \leq 0 \\ 0 < g_\epsilon(x) < 1 & 0 < x < \epsilon \\ g_\epsilon(x) = 1 & x \geq \epsilon \end{cases}$$

Finallly, for any fixed point $a \in \mathbb{R}^N$ and for any given $r > 0$, we define

$$h_\epsilon^r\colon \mathbb{R}^N \to \mathbb{R},\ h_\epsilon^r(x) = 1 - g_\epsilon(|x-a| - r).$$

Then h_ϵ^r is smooth, nonincreasing, and

$$\begin{cases} h_\epsilon^r(x) = 1 & |x-a| \leq r \\ 0 < h_\epsilon^r(x) < 1 & r < |x-a| < r+\epsilon \\ h_\epsilon^r(x) = 0 & |x-a| \geq r+\epsilon \end{cases}$$



This gives us a smooth function $\phi\colon \mathbb{R}^N \to \mathbb{R}$ which has value 1 on the compact subset $\overline{B_r(a)}$ and has value 0 outside the closed subset $\overline{B_{r+\epsilon}(a)}$.

Now let $U \subset \mathbb{R}^N$ be open and $K \subset U$ be compact. For this general situation we need to work a bit harder and rearrange the argument as follows:

Let ψ be the function

$$\psi_\epsilon\colon \mathbb{R}^N \to \mathbb{R},\ \psi_\epsilon(x) = \begin{cases} e^{-1/|x|^2} & |x| < 1 \\ 0 & |x| \geq 1. \end{cases}$$

This is a smooth map with $\int_{\mathbb{R}^N} \psi dx = 1$ (using a standard Lebesgue measure dx on $\mathbb{R}^N$).

For a given $\epsilon > 0$, define $\psi_\epsilon\colon R^N \to \mathbb{R}$ by $\psi_\epsilon(x) := \epsilon^{-N}\psi(x/\epsilon)$. This is still a smooth function with $\int_{\mathbb{R}^N} \psi_\epsilon dx = 1$.

Since $\mathbb{R}^N \setminus U$ is closed and K is compact, we can choose a small $\epsilon > 0$ such that, for each point $x \in K$, we have $B_{2\epsilon}(x) \cap U = \emptyset$. Then the $V := \cup_{x \in K} B_\epsilon(x)$ is an open set containing K with **compact closure** $\bar{V} \subset U$ contained in U.

Let χ_V be the characteristic function on V, i.e. the function

$$\chi_V\colon \mathbb{R}^N \to \mathbb{R},\ \begin{cases} \chi_V(x) = 1 & \text{for } x \in V \\ \chi_V(x) = 0 & \text{for } x \notin V. \end{cases}$$

The function χ_V is identically 1 on K and has compact support contained in U. But it is of course not smooth on $\mathbb{R}^N$, not even continuous. Hence we need to modify it, to make it smooth. The function ψ_ϵ, for the fixed ϵ, will serve as a tool to "smoothen" χ_V.

Then the desired smooth function φ is the convolution $\psi_\epsilon * \chi_V$ of χ_V and ψ_ϵ:

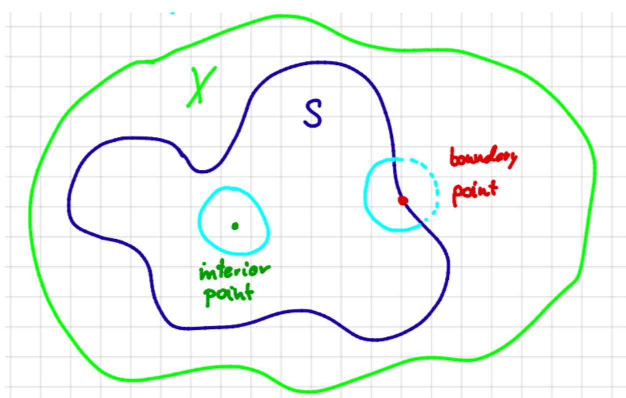
$$\varphi\colon \mathbb{R}^N \to \mathbb{R}, x \mapsto \int_{\mathbb{R}^N} \psi_\epsilon(x-y)\chi_V(y)dy.$$

Note that the integral is well-defined, since the closure of V, which is the support of χ_V, is compact. **QED**

Finally some terminology:

The interior of a set

Let X be a topological space, and S a subset of X. Then the **interior of** S, denoted $\mathrm{int}(S)$, is the union of all open subsets of X contained in S. By definition, the interior of any S is an open subset of X. In fact, it is the largest open subset of X which is contained in S.
If $S \subset \mathbb{R}^N$ then $\mathrm{int}(S)$ is the set of all points $s \in S$ such that there is a small open ball centered at x which is contained in S.
Obviously, if U is an open subset of X then $\mathrm{int}(U) = U$. In particular, if $X \subset \mathbb{R}^N$ is open then $\mathrm{int}(X) = X$. But in general $\mathrm{int}(S)$ is a proper subset of X.



Existence of partitions of unity

We are going to show that partitions of unity exist on manifolds step by step with increasing difficulty. We start with the case of compact subspaces in $\mathbb{R}^N$. Then we are going to transport this result to compact abstract smooth manifolds. Finally, we discuss arbitrary compact smooth k-manifolds. There is no need to restrict to compact manifolds. In fact, partitions of unity exist on every **paracompact** topological space (every open cover has a **locally finite refinement**), a class of spaces much larger than abstract manifolds.

First case: $X \subset \mathbb{R}^N$ compact.

Let $\{U_\alpha\}$ be an open cover of X. Since X is compact, $\{U_\alpha\}$ has a finite subcover $\{U_1, \ldots, U_n\}$. A partition of unity subordinate to the finite subcover is also a partition of unity subordinate to the original cover.

Step 1: We are going to show that we can shrink the covering to an open covering $\{V_1 \ldots, V_n\}$ such that $\bar{V}_i \subset U_i$ for each i.

Consider the closed subset

$$A := X \setminus (U_2 \cup \cdots \cup U_n)$$

of X. Since $\{U_1, \ldots, U_n\}$ cover X, we know $A \subset U_1$. Since A and $X \setminus U_1$ are closed disjoint, we can choose an open subset V_1 containing A such that V_1 is disjoint to an open subset W which contains $X \setminus U_1$. Thus V_1 is contained in the complement $X \setminus W$. Since $X \setminus W$ is a closed subset which contains V_1, we know $\bar{V}_1 \subset X \setminus W$, since the closure of V_1 is the intersection of all closed subsets which contain V_1. Since $X \setminus U_1 \subset W$ by the choice of W, we have $X \setminus W \subset X \setminus (X \setminus U_1) = U - 1$. Thus we have $\bar{V}_1 \subset U_1$. Since V_1 contains the complement of $U_2 \cup \cdots \cup U_n$ in X, the collection $\{V_1, U_2, \ldots, U_n\}$ covers X.

Now we proceed by induction as follows: Given open subsets $V_1, \ldots, V_{k-1}$ such that

$$X = \{V_1, \ldots, V_{k-1}, U_k, U_{k+1}, \ldots, U_n\},$$

let A_k be the subset

$$A_k = X \setminus (V_1 \cup \cdots \cup V_{k-1}) \cup (U_{k+1} \cup \cdots \cup U_n).$$

Then A_k is a closed subset of X which is contained in the open set U_k. Choose an open subset V_k containing A_k such that $\bar{V}_k \subset U_k$. Then $\{V_1, \ldots, V_{k-1}, V_k, U_{k+1}, \ldots, U_n\}$ covers X. At the nth step of the induction we are done.

Step 2: Given the open covering $\{U_1, \ldots, U_n\}$ of X, we use Step 1 to choose an open covering $\{V_1, \ldots, V_n\}$ of X such that $\bar{V}_i \subset U_i$ for each i. Then we repeat this process and choose an open covering $\{W_1, \ldots, W_n\}$ of X such that $\bar{W}_i \subset V_i$ for each i.

For each i, we choose a smooth bump function

$$\varphi_i \colon X \to [0,1] \text{ such that } \varphi_i(\bar{W}_i) = \{1\} \text{ and } \varphi_i(X - V_i) = \{0\}.$$

Since $\varphi_i^{-1}(\mathbb{R} \setminus \{0\}) \subset V_i$, we have

$$\operatorname{supp}(\varphi_i) \subset \bar{V}_i \subset U_i.$$

(Here is the point where see why we need to apply Step 1 twice: If we were working with the V_i's instead of W_i's, then we would have $\mathrm{supp}(\varphi) \subset \bar{U}_i$ instead of $\mathrm{supp}(\varphi) \subset U_i$ as required for a partition subordinate to the cover $\{U_i\}$.)

Since $\{W_1, \ldots, W_n\}$ covers X, we have

$$\varphi(x) := \sum_{i=1}^{n} \varphi_i(x) > 0 \text{ for all } x \in X.$$

Finally, for each i, we define

$$\rho_i(x) := \frac{\varphi_i(x)}{\varphi(x)}.$$

Second case: $X \subset \mathbb{R}^N$ and $X = X_1 \cup X_2 \cup X_3 \cup \cdots$ where each X_i is compact and $X_i \subset \mathrm{int}(X_{i+1})$.

Let $\{U_\alpha\}$ be an open cover of X. For each i, we define

$$U_\alpha^i := U_\alpha \cap (X_{i+1} \setminus \mathrm{int}(X_{i-2})).$$

Then $\{U_\alpha^i\}$ is an open cover of $Y_i := X_i \setminus \mathrm{int}(X_{i-1})$. Since $\mathrm{int}(X_{i-1})$ is an open subset, Y_i is a closed subset of X_i and therefore Y_i is also compact. Then, for each i, the first case implies that there is a partition of unity φ_α^i on Y_i subordinate to the cover $\{U_\alpha^i\}$.

For each $x \in X$, there is an i such that $x \in X_i$ and hence $\varphi_\alpha^j(x) = 0$ for all $j \geq i+2$. Hence, for each $x \in X$, the sum

$$\varphi(x) := \sum_{\alpha, i} \varphi_\alpha^i(x)$$

is a finite sum in some open set containing x.

Now for each α we define

$$\rho_\alpha^i(x) := \frac{\varphi_\alpha^i(x)}{\varphi(x)}$$

This is partition of unity subordinate to the open cover $\{U_\alpha\}$.

Third case: $X \subset \mathbb{R}^N$ is open.

Define subsets

$$X_i := \{x \in X : |x| \leq i \text{ and the distance to } \mathbb{R}^N \setminus X \geq 1/j\}.$$

Then these subsets satisfy:

- each X_i is compact, since it is the intersection $X \cap \overline{B_i(0)} \cap (X \setminus (\cup_{p \in \mathbb{R}^N \setminus X} B_{1/i}(p))$ and therefore closed and bounded in $\mathbb{R}^N$;
- for each i: $X_i \subset \mathrm{int}(X_{i+1})$;
- $X = X_1 \cup X_2 \cup \cdots$.

Hence we can apply the second case.

Fourth case: $X \subset \mathbb{R}^N$ arbitrary.

Let $\{U_\alpha\}$ be an open cover of X. By the definition of the topology on X, for each α, there is a subset $V_\alpha \subset \mathbb{R}^N$ open in $\mathbb{R}^N$ such that $U_\alpha = X \cap V_\alpha$. Let Y be the union of all the V_α in $\mathbb{R}^N$. By the third case, there is a partition of unity on Y subordinate to the open cover $\{V_\alpha\}$. This is also a partition of unity on X subordinate to the open cover $\{U_\alpha\}$.

Last case: X is a compact abstract smooth k-manifold.

Let $\{V_\alpha\}$ be an open over of X. By intersecting with the domains of charts on X, we get a refinement of the cover. Hence we can assume that V_α are the **domains of charts** on X. Since X is **compact**, the domains of finitely many charts on X suffice to cover X. Let us lable them $(V_1,\phi_1),\ldots,(V_n,\phi_n)$. Then each $U_i = \phi_i(V_i)$ is an open subset in $\mathbb{R}^k$.

Now we can **proceed exactly as in the case of a compact subspace in $\mathbb{R}^N$** for the finite cover $\{U_1,\ldots,U_n\}$ of the space $Y := U_1 \cup \cdots \cup U_n \subset \mathbb{R}^k$. This yields a partition of unity $\{\rho_i\}$ subordinate to the cover $\{U_1,\ldots,U_n\}$. Composition of each ρ_i with ϕ_i yields a partition of unity $\{\rho_i \circ \phi_i\}$ on X subordinate to the cover $\{V_1,\ldots,V_n\}$. **QED**

Now we are ready to prove the following embedding result.

Embedding abstract manifolds into Euclidean space

Let X be a compact abstract smooth k-manifold. Then there is an embedding, i.e. an injective proper map, $X \hookrightarrow \mathbb{R}^N$ for some large N.

Proof:

The collection of all V_α for all charts (V_α,ϕ_α) is an open cover of X. Since X is compact, we can cover X by the images of a **finite number of charts** $V_1,\ldots,V_n$.

Let $\{\rho_i\}$ be a partition of unity subordinate to the open cover defined by the V_i's.

For a chart $\phi_i\colon V_i \to U_i \subset \mathbb{R}^k$, we define a new map

$$g_i\colon X \to \mathbb{R}^k,\ g_i(x) = \begin{cases} \rho_i(x)\cdot\phi_i(x) & \text{for } x \in V_i \\ 0 & \text{for } x \in X \setminus \operatorname{supp}(\rho_i). \end{cases}$$

The map g_i is well-defined, since if $x \in V_i \setminus \operatorname{supp}(\rho_i)$, then both definitions agree to be 0. Moreover, g_i is **continuous**, since its restrictions to the two **open** subsets V_i and $X \setminus \operatorname{supp}(\rho_i)$ are continuous (this is why we do not use $X \setminus V_i$ in the definition because that would be a closed subset).

Now we define a map

$$g\colon X \to \mathbb{R}^n \times \mathbb{R}^{nk},\ x \mapsto (\rho_1(x), \dots, \rho_n(x), g_1(x), \dots, g_n(x)).$$

We observe that **g is continuous**, since the g_i's and the ρ_i's are continuous.

Claim: g is an injective proper map.

Since X is compact, g is a **proper** map.

Now we show that g is injective. For assume $g(x) = g(y)$. Then $\rho_i(x) = \rho_i(y)$ for all i by the definition of g. But, by the definition of a partition of unity, for at least one i, we must have $\rho_i(x) = \rho_i(y) \neq 0$.

Thus x and y must lie in the same V_i, since ρ_i is supported on V_i, i.e. $\rho_i(x) \neq 0$ implies $x \in V_i$. Hence, since $g_i(x) = g_i(y)$ and $\rho_i(x) = \rho_i(y) \neq 0$,we must have $\phi_i(x) = \phi_i(y)$. Since ϕ_i is a bijection, this shows $x = y$. Thus **g is injective**.
QED

Actually, g is also an immersion, but we have not defined what that means for an abstract manifold. Since this is just an exercise in translating the defintions, we omit this point and rather move on.

All manifolds can be embedded in Euclidean space

In fact, every abstract k-manifold X can be embedded in Euclidean space. One can just keep on going with the above argument in the non-compact case and use local coordinates to map pieces of X into $\mathbb{R}^k$. Though when using only finitely many copies of $\mathbb{R}^k$ to accomodate infinitely many neighborhoods of X, we loose injectivity. The key tool that restores injectivity are **partitions of unity** which even out the troubles occuring because of overlapping neighborhoods.

For this to work, it is crucial that the topology on X has a **countable** basis. This is a technical point which we did and will not discuss because it would divert us too far from the main story.
We just remark that it is possible to construct topological spaces without a countable basis which are locally homeomorphic to Euclidean space, but which cannot be embedded into Euclidean space.

Another application of the existence of partitions of unity is the following lemma which will turn out to be key tool in the proof of Whitney's Theorem.

Existence of proper functions on manifolds

On any manifold X, there is a proper map $p\colon X \to \mathbb{R}$.

Proof: Let $\{U_\alpha\}$ be the collection of open subsets of X that have **compact closure**, and let ρ_α be a subordinate partition of unity. Then

$$p(x) = \sum_{i=1}^{\infty} i\rho_i(x)$$

is a well-defined smooth function, since, in a neighborhood of every point, it is a finite sum of smooth functions.

In order to show that p is **proper**, we need to show that the preimage of any compact subset of $\mathbb{R}$ is again compact. Every compact subset $K \subset \mathbb{R}$ is contained in a closed interval of the form $[-j,j]$ for some natural number j. Hence if we can show that $p^{-1}([-j,j])$ is compact, then $p^{-1}(K)$ is a closed subset of a compact set and therefore also compact.

For given j, if for any x we had $\rho_1(x) = \cdots = \rho_j(x)$, then

$$\sum_{i=j+1}^{\infty} \rho_i(x) = 1$$

and therefore

$$p(x) \geq (j+1) \sum_{i=j+1}^{\infty} \rho_i(x).$$

This shows

$$p^{-1}([-j,j]) \subset \cup_{i=1}^{j} \{x : \rho_i(x) \neq 0\}.$$

Since $\operatorname{supp}(\rho_i) \subset U_i$ and U_i has compact closure, this shows that $p^{-1}([-j,j])$ is a closed subset in a compact set and therefore it is also compact. **QED**

Whitney's Theorem

Every smooth k-dimensional manifold $X \subset \mathbb{R}^N$ admits an embedding into $\mathbb{R}^{2k+1}$.

Recall that the strongest result is that $N = 2k$ suffices. But that is much harder. And again, this is an upper bound which works for every k-dimensional manifold. For a many manifolds, an even lower dimension suffices, e.g. $S^n \subset \mathbb{R}^{n+1}$.

Proof: The idea is to replace the injective immersion $f\colon X \hookrightarrow \mathbb{R}^N$ with the map $(f,p)\colon X \hookrightarrow \mathbb{R}^{N+1}$ with a proper $p\colon X \to \mathbb{R}$. Then (f,p) is still an injective immersion, and it is proper, since p is proper. It remains to reduce the dimension $N+1$. The details are a bit more involved:

Starting with $X \subset \mathbb{R}^N$ we have seen that we can find an injective immersion $f\colon X \to \mathbb{R}^{2k+1}$. By composing f with the injective immersion map

$$\mathbb{R}^{2k+1} \to B_1(0),\ x \mapsto \frac{x}{1+|x|^2},$$

we can assume that $|f(x)| < 1$ for all $x \in X$.

Let $p\colon X \to \mathbb{R}$ be a proper function which we know to exist by the revious lemma. We define a new injective immersion

$$F\colon X \to \mathbb{R}^{2k+2},\ x \mapsto (f(x),p(x)).$$

Since $2k+2 > 2k+1$, we can apply the argument from last time and find a nonzero vector $a \in \mathbb{R}^{2k+2}$ such that

$$\pi \circ F\colon X \to H$$

is still an injective immersion, where π is the projection onto the orthogonal complement $H = \{b \in \mathbb{R}^{2k+2} : b \perp a\}$ of a in $\mathbb{R}^{2k+2}$. By rescaling we can assume $|a| = 1$.

Since $\pi \circ F$ is an injective immersion for almost every $a \in S^{2k+1}$, we **can assume that a is neither the north nor the south pole on S^{2k+1}**. This will allow us tp show that **$\pi \circ F$ is proper**:

Claim: Given any bound c, there exists another number d such that

$$\{x \in X : |(\pi \circ F)(x)| \leq c\} \subset \{x \in X : |p(x)| \leq d\}.$$

As p is **proper**,

$$\{x \in X : |p(x)| \le d\} = p^{-1}([-d,d])$$

is a **compact** subset of X.

Thus the claim implies that the preimage under $\pi \circ F$ of every closed ball in H is a compact subset of X. Sicne every compact subset K of H is a closed subset of some closed ball in X, this shows that $(\pi \circ F)^{-1}(K)$ is a closed subset of compact subset in X and therefore also compact.

If the claim is false, then there exists a c and a sequence of points $\{x_i\}$ in X for which

$$|(\pi \circ F)(x_i)| \le c, \text{ but } |p(x_i)| \to \infty$$

(becasue there is no d bounding $|p(x_i)|$).

By definition of the projection onto an orthogonal complement, for every $z \in \mathbb{R}^{2k+2}$, $\pi(z)$ is the one point in H for which $z - \pi(z)$ is a multiple of a. In particular,

$$F(x_i) - \pi \circ F(x_i) \text{ is a multiple of } a \text{ for each } i,$$

and hence so is the vector

$$w_i := \frac{1}{p(x_i)}(F(X_i) - \pi \circ F(x_i)).$$

Let us look at what happens when $i \to \infty$:

$$\frac{F(x_i)}{p(x_i)} = \left(\frac{f(x_i)}{p(x_i)},1\right) \to (0,\ldots,0,1)$$

because $|f(x_i)| < 1$ for all i and $p(x_i) \to \infty$. We have

$$\left|\frac{\pi \circ F(x_i)}{p(x_i)}\right| \le \frac{c}{|p(x_i)|}.$$

Thus

$$\frac{\pi \circ F(x_i)}{p(x_i)} \to 0 \Rightarrow w_i \to (0,\ldots,0,1).$$

But each w_i is a multiple of a. Hence the limit of thw w_i must be a mutiple of a. We conclude that a must be either the north or south pole of S^{k+1} which contradicts our assumption on a. This proves the claim and the theorem. **QED**

LECTURE 17

Manifolds with Boundary

In order to be able to analyze a wider class of phenomena we would like to enlarge the class of manifolds. A typical example which we would like to include is the domain of a homotopy $X \times [0,1]$ for a smooth k-dimensional manifold X. The points on $X \times \{0\}$ and $X \times \{1\}$ do not have an open neighborhhod which is diffeomorphic to $\mathbb{R}^k$. In fact, those subsets for the boundary of $X \times [0,1]$. Another example is the **closed** unit ball in $\mathbb{R}^k$. So far such guys do not qualify as a manifold. From now on, We would like to allow such subsets. We will see that most of the theorems we have proved so far are also valid for manifolds with boundary.

The idea for what a manifold with boundary should be is the same as before: it is a space which locally looks like some model space with boundary which we understand well. Hence we need to choose a good model space. But that is not hard to do.

In fact, the standard model of a Euclidean space with boundary is the half-plane

$$\mathbb{H}^k = \{(x_1, \ldots, x_k) \in \mathbb{R}^k : x_k \geq 0\}$$

in $\mathbb{R}^k$. The **boundary of $\mathbb{H}^k$**, denoted $\partial\mathbb{H}^k$, is given by the points

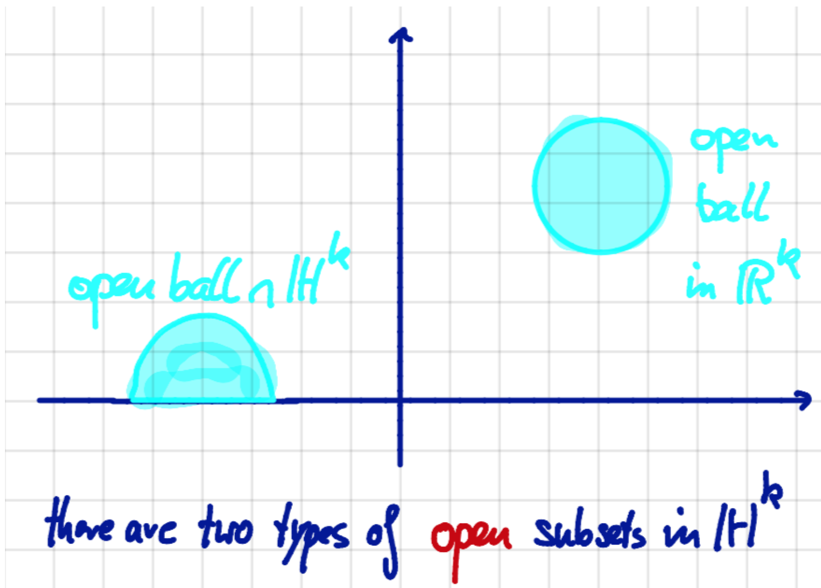
$$\partial\mathbb{H}^k = \{(x_1, \ldots, x_k) \in \mathbb{R}^k : x_k = 0\} = \mathbb{R}^{k-1} \times \{0\} \subset \mathbb{R}^k.$$

Now a manifold with boundary is a space which locally looks like $\mathbb{H}^k$:

Manifolds with boundary

A subset X of $\mathbb{R}^N$ is a **k-dimensional manifold with boundary** if every point x of X there is an open neighborhood $V \subset X$ containing x and an open neighborhood $U \subset \mathbb{H}^k$ together with a diffeomorphism $\phi\colon U \to V$. As before, any such a diffeomorphism is called a **local parametrization of X**.

The **boundary of X**, denoted ∂X, consists of those points that belong to the image of the boundary of $\mathbb{H}^k$ under some local parametrization. Its complement is called the **interior of X**, denoted $\mathrm{Int}(X) = X \setminus \partial X$.
A manifold X with $\partial X = \emptyset$ is just a smooth manifold in our initial terminology. In order to make the distinction clear if necessary, we call them also boundaryless manifolds or manifolds without a boundary.



Warning: The interior of $X \subset \mathbb{R}^N$ as a manifold is in general different from the interior of X as a subspace of $\mathbb{R}^N$. The interior of X as a manifold is the complement of the boundary, whereas the interior of the topologocal space X is the union of all its open subsets. But also every point in ∂X lies in some open neighborhood of X.

Let X be a manifold with boundary. We need to check that our definition of points in the interior and on the boudnary is independent of the choice of a local parametrization.

So let $x \in X$ be a point which is in the image of a local parametrization $\phi\colon U \to V \subset X$ such that $U \subset \mathbb{H}^k$ is an open set of $\mathbb{H}^k$ which is contained in the interior of $\mathbb{H}^k$. Then $\mathbb{R}^k$ is an open subset of $\mathbb{R}^k$. Now assume x is also in the image of another local parametrization $\phi'\colon U' \to V' \subset X$. Then $x \in W := V \cap V' \subset X$, and the composition $\phi' \circ \phi^{-1}\colon \phi^{-1}(W) \to (\phi')^{-1}(W)$ is a diffeomorphism. Hence, after possibly shrinking U', we see that U' is also an open subset in $\mathbb{R}^k$. Thus x is being an interior point is well-defined.

This shows in particular: if X is a manifold with boundary, then the interior of X, $\text{Int}(X)$, is a boundaryless manifold of the same dimension as X.

It remains to show that being a boundary point is also well-defined. We show this by proving the following interest result:

Boundaries are manifolds

If X is a k-dimensional manifold with boundary, then ∂X is a $(k-1)$-dimensional manifold without boundary.

Proof: Let $x \in X$ and let ϕ and ψ be two local parametrizations around x. After possibly shrinking the domains and codomains, we can assume that $\phi\colon U \to V$ and $\psi\colon W \to V$ are both diffeomorphisms from open sets $U \subset \mathbb{H}^k$, $W \subset \mathbb{H}^k$ to the same open subset $V \subset X$.

We would like to show $\phi(\partial U) = \psi(\partial W)$. For then $\partial V = \phi(\partial U)$ is independent of our choice of local parametrization and therefore well-defined. Moreover, since $\partial U = U \cap \partial\mathbb{H}^k$ is an open subset of $\mathbb{R}^{k-1}$, we would get that every point $y \in \partial X$ is contained in a local parametrization $\phi_{|\partial U}\colon U \cap \partial\mathbb{H}^k \to \partial X$. This will show that ∂X is a manifold of dimension $k-1$.

By our assumption on ϕ and ψ, it suffices to show $\psi(\partial W) \subset \phi(\partial U)$. The other inclusion will follow by symmetry. Hence we woud like to show:

Claim: $\phi^{-1}(\psi(\partial W)) \subset \partial U$.

To simplify notation, we define the map $g = \phi^{-1} \circ \psi\colon W \to U$.

Suppose that the claim is false and there is a point $w \in \partial W$ which is mapped to an **interior point** $u = g(w)$ of U by g. Since both ϕ and ψ are diffeomorphisms, **g is a diffeomorphism** of W onto an open subset $g(W)$ of U. The chain rule implies that the derivative $d(g^{-1})_u$ of its inverse is bijective. But, since $u \in \text{Int}(U)$, $g(W)$ contains a neighborhood of u that is **open in $\mathbb{R}^k$**. Thus the **Inverse Function Theorem**, applied to the map g^{-1} defined on this open subset of $\mathbb{R}^k$, implies that the image of g^{-1} contains a neighborhood of w that is **open in $\mathbb{R}^k$**. This contradicts the assumption $w \in \partial W$. **QED**

Tangent spaces and derivatives are still defined in the setting of manifolds with boundary.

Derivatives and tangent spaces vs boundaries

Derivatives of smooth maps can be defined as before. Since smoothness at a point requires a functions to be defined on open neighborhhod around that point, we need to be a bit more careful at boundary points:

Derivatives on $\mathbb{H}^k$:

Suppose that g is a smooth map of an open set U of $\mathbb{H}^k$ to R^l. If u is an interior point of U, then the derivative dg_u is defined as before.

If $u \in \partial U$ is a boundary poin, the smoothness of g means that it may be extended to a smooth map G defined in an open neighborhood of u in $\mathbb{R}^K$. We define dg_u to be the derivative $dG_u \colon \mathbb{R}^k \to \mathbb{R}^l$.

We must show that this definition is independent of the choice of G. So let G' be another local extension of g. We need to show $dG'_u = dG_u$.

The equality of the two derivatives is no problem at points in the interior $\mathrm{int}(U)$ of U, because then we have a small open neighborhood which is still in $\mathrm{int}(U)$. We are going to use this and approximate u be a sequence $\{u_i\}$ of interior points $u_i \in \mathrm{int}(U)$ which converge to u.

Since G and G' agree with g on $\mathrm{int}(U)$, we have

$$dG_{u_i} = dG'_{u_i} \text{ for all } i.$$

Since the derivative of a smooth map at a point depends continuously on the change of point, wthis implies that $dG_{u_i} \to dG_u$ and $dG'_{u_i} \to dG'_u$ when $u_i \to u$ and **both limits agree**. This shows that dg_u is also well-defined at boundary points.

One should note that, at all points, dg_u is still a linear map of all of $\mathbb{R}^k$ to $\mathbb{R}^l$. For we have defined dg_u as the derivative dG_u of an extension G to an **open subset of $\mathbb{R}^k$**.

Tangent spaces:

Let $X \subset \mathbb{R}^N$ be a smooth manifold with boundary, and $x \in X$. Let $\phi \colon U \to X$ be a local parametrization with $U \subset \mathbb{H}^k$ open. Let $u \in U$ be the point with $\phi(u) = x$. (Note that we cannot assume $u = 0$ when x is an interior point.)

Then we have just learned that we can form the derivative

$$d\phi_u \colon \mathbb{R}^k \to \mathbb{R}^N$$

no matter what kind of point x is. Thus, as before, we can define the **tangent space** to X at x, denoted $T_x(X)$, to be the image of $\mathbb{R}^k$ in $\mathbb{R}^N$ under the linear map $d\phi_u$. (One can check that $T_x(X)$ does not depend as a subspace of $\mathbb{R}^N$ on the choice of ϕ just as before using the chain rule.)

Derivatives on tangent spaces:

Now let $f\colon X \to Y$ be a smooth map between manifolds with boundaries with $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$. Given a point $x \in X$. Then after choosing local parametrizations $\phi\colon U \to X$ with $\phi(u) = x$ and $\psi\colon V \to Y$ with $\psi(v) = f(x)$, then we **define**

$$df_x\colon T_x(X) \to T_{f(x)}(Y)$$

as the **linear map** which makes the following diagram commutative

$$\begin{array}{ccc} T_x(X) & \xrightarrow{\;df_x\;} & T_y(Y) \\ {\scriptstyle d\phi_u}\uparrow & & \uparrow{\scriptstyle d\psi_v} \\ \mathbb{R}^k & \xrightarrow[d\theta_u]{} & \mathbb{R}^l. \end{array}$$

where θ is the map $\psi^{-1} \circ f \circ \phi$ (note $v = \theta(u)$).

However, sometimes we do have to be careful when we apply our developed concepts to manifolds with boundaries. For example, the product of two manifolds with boundary may not be a manifold anymore. A simple example is the product $[0,1] \times [0,1]$.

But if only one manifold has a boundary we are ok:

Products and Boundaries

The product of a manifold without boundary X and a manifold with boundary Y is another manifold with boundary. Furthermore,

$$\partial(X \times Y) = X \times \partial Y,$$

and

$$\dim(X \times Y) = \dim X + \dim Y.$$

Proof: If $U \subset \mathbb{R}^k$ and $V \subset \mathbb{H}^l$ are open, then

$$U \times V \subset \mathbb{R}^k \times \mathbb{H}^l = \mathbb{H}^{k+l}$$

is open. Moreover, if $\phi\colon U \to X$ and $\psi\colon V \to Y$ are local parametrizations, so is $\phi \times \psi\colon U \times V \to X \times Y$. **QED**

Regular values and transversality

One of the most important concepts we have studied is transversality of smooth maps to submaifolds. We would like to extend this to manifolds with boundary. This is possible, but requires some care.

We start with the special case of regular values for functions on manifolds without boundary. This is a well-known case, but it turns out that it actually produces manifolds with boundary as follows:

Regular values for real-valued functions

Suppose that S is a manifold without boundary and that $\pi\colon S \to \mathbb{R}$ is a smooth function with regular value 0. Then the subset $\{s \in S : \pi(s) \geq 0\}$ is a manifold with boundary, and the boundary is $\pi^{-1}(0)$.

Proof: The set $\{x \in S : \pi(x) > 0\}$ is open in S, since it is the continuous preimage of the open subset $(0,\infty) \subset \mathbb{R}$. It is therefore a submanifold of the same dimension as S. Hence every point in $\{x \in S : \pi(x) > 0\}$ has an open neighborhood which is diffeomorphic to an open subset of $\mathbb{R}^k$, $k = \dim S$.

So suppose that $\pi(s) = 0$. By assumption, 0 is a regular value which means that s is a regular point of π. Hence π is locally near s equivalent to the canonical submersion. But for the canonical submersion

$$\pi\colon \mathbb{H}^k \to \mathbb{R},\ (x_1, \ldots, x_k) \to x_k$$

the lemma just states the definition of the boundary of $\mathbb{H}^k$:

$$\partial\mathbb{H}^k = \pi^{-1}(0) = \{(x_1, \ldots, x_k) \in \mathbb{R}^k : x_k = 0\}.$$

QED

An immediate consequence of this fact is:

Spheres are boundaries

Let π be the smooth function defined by

$$\pi\colon \mathbb{R}^k \to \mathbb{R},\ (x_1, \ldots, x_k) \mapsto 1 - \sum_i x_i^2.$$

Then 0 is a regular value of π, and the unit ball B^k in $\mathbb{R}^k$ can be described as

$$B^k = \{x \in \mathbb{R}^k : \pi(x) \geq 0\}.$$

The boundary of B^k is the $(k-1)$-sphere $S^{k-1} = \pi^{-1}(0)$.

Recall that transversality is formulated as a criterion on tangent spaces and derivatives. We would like to formulate a similar criterion for maps between manifolds with boundary.

As we learned above, the boundary ∂X of a k-manifold with boundary X is a manifold of dimension $k-1$ without boundary. Let $x \in \partial X$ be a point on the boundary. We have $\dim T_x(\partial X) = k-1$ and $\dim T_x(X) = k$. Moreover, since ∂X is a submanifold of X, we know that

$$T_x(\partial X) \subset T_x(X)$$

is a **vector subspace** of codimension 1 in $T_x(X)$.

For any smooth map $f\colon X \to Y$, we introduce the notation

$$\partial f = f_{|\partial X}$$

for the restriction of f to ∂X. The derivative of ∂f at x is just the restriction of df_x to the subspace $T_x(\partial X)$:

$$d(\partial f)_x = (df_x)_{|T_x(\partial X)}\colon T_x(\partial X) \to T_{f(x)}(Y).$$

Now let $f\colon X \to Y$ be a smooth map from a smooth manifold with boundary X to a boundaryless manifold Y, and let $Z \subset Y$ be a submanifold. We would like to know under which circumstances is $f^{-1}(Z)$ **a submanifold with boundary** of X (i.e. a subset of X which is itself a smooth manifold with boundary) with

$$\partial f^{-1}(Z) = f^{-1}(Z) \cap \partial X. \tag{17}$$

It turns out that it is **not enough** to ask that f is transversal to Z in the previous sense, i.e. $\mathrm{Im}\,(df_x) + T_{f(x)}(Z) = T_{f(x)}(Y)$.

A simple example

Even for the restriction of the canonical submersion

$$\pi\colon \mathbb{H}^2 \to \mathbb{R},\ (x_1,x_2) \mapsto x_2$$

this is not sufficient. For, $d\pi_{(x_1,x_2)}\colon \mathbb{R}^2 \to \mathbb{R}$ is just the projection onto the second factor. Hence it is surjective at every point (x_1,x_2). In particular, 0 is a regular value fo π. Let $Z := \{0\}$. Then

$$\pi^{-1}(Z) = \{(x_1,x_2) \in \mathbb{R}^2 : x_2 = 0\} = \partial\mathbb{H}^2.$$

Since 0 is regular value, we knew that $\pi^{-1}(Z)$ is a submanifold. The problem is that the boundary does not satisfy condition (17). For

$$\partial\pi^{-1}(Z) = \emptyset, \text{ whereas } \pi^{-1}(Z) \cap \partial X = \partial\mathbb{H}^2 \neq \emptyset.$$

In order to make sure that the boundary behaves well, we need to impose an additional transversality condition on ∂f.

We start again with regular values:

Preimages of regular values in manifolds with boundary

Let g be a smooth map of a k-manifold X with boundary onto a boundaryless n-manifold Y, and suppose that $y \in Y$ is a regular value for **both** $g\colon X \to Y$ and $\partial g\colon \partial X \to Y$. Then the preimage $g^{-1}(y)$ is a $(k-n)$-dimensional manifold with boundary

$$\partial(g^{-1}(y)) = g^{-1}(y) \cap \partial X.$$

Proof: To show that $g^{-1}(y)$ is a manifold with boundary is a local question, i.e. it suffices that each point in $g^{-1}(y)$ has an open neighborhood which is a manifold with boundary. So let $x \in X$ be a point with $g(x) = y$. After choosing local coordinates, we can assume that g is a map

$$g\colon \mathbb{H}^k \to \mathbb{R}^n.$$

If x is an interior point in X, then $g^{-1}(y)$ is a manifold without boundary in an open neighborhood around x by the Preimage Theorem for boundaryless manifolds.

So let us look at what happens if $x \in \partial X$. That g is smooth at x means by definition that there is an open subset $U \subset \mathbb{R}^k$ and a smooth map

$$G\colon U \to \mathbb{R}^n \text{ such that } G_{U\cap\mathbb{H}^k} = g_{U\cap\mathbb{H}^k}.$$

After possibly replacing U with a smaller subset, we can assume that **G has no critical points in U**. Then $G^{-1}(y)$ is a smooth manifold by the Preimage Theorem for boundaryless manifolds. We need to show that

$$g^{-1}(y) = G^{-1}(y) \cap \mathbb{H}^K \text{ is a manifold with boundary.}$$

In order to show this, we define a new smooth function π on the manifold $S := G^{-1}(y)$

$$\pi\colon S \to \mathbb{R},\ (x_1, \ldots, x_k) \mapsto x_k$$

as the projection to the last coordinate. Then

$$S \cap \mathbb{H}^k = \{s \in S : \pi(s) \geq 0\}.$$

Claim: 0 is a regular value of π.

If we can show the claim, then our previous lemma shows that $S \cap \mathbb{H}^k$ is a manifold with boundary and the boundary is $\pi^{-1}(0)$.

To show the claim, assume there was an $s \in S$ with both $\pi(s) = 0$, i.e. $s \in S \cap \partial\mathbb{H}^k$, and $d\pi_s = 0$. We want to show that the assumption $d\pi_s = 0$ leads to a contradiction.

To do so, first note that π is a **linear** map, and therefore $d\pi_s = \pi$. Thus,

$$d\pi_s = \pi \colon T_s(S) \to \mathbb{R}$$

being trivial, just means that the last coordinate of every vector in $T_s(X)$ is 0, i.e.

$$d\pi_s = 0 \Rightarrow T_s(S) \subset T_s(\partial\mathbb{H}^k) = \mathbb{R}^{k-1}.$$

Hence we want to show $T_s(S) \not\subset \mathbb{R}^{k-1}$.

The tangent space to $S = G^{-1}(y)$ at s is the kernel of dG_s:

$$T_s(S) = T_s(G^{-1}(y)) = \operatorname{Ker}(dG_s = dg_s \colon \mathbb{R}^k \to \mathbb{R}^n)$$

where $dg_s = dG_s$ by definition of dg_s.

We know that $d(\partial g)_s$ is the restriction of $dg_s \colon \mathbb{R}^k \to \mathbb{R}$ to $\mathbb{R}^{k-1}$:

$$d(\partial g)_s = (dg_s)_{|\mathbb{R}^{k-1}}.$$

Thus, **if $T_s(S) = \operatorname{Ker}(dg_s) \subseteq \mathbb{R}^{k-1}$**, then

$$\operatorname{Ker}(dg_s) = \operatorname{Ker}(d(\partial g)_s). \tag{18}$$

Now, finally, we apply the assumption of regularity of y. Since **y is a regular value of both g and ∂g**, we know that **both dg_s and $d(\partial g)_s$ are surjective**. This implies

$$\dim \operatorname{Ker}(dg_s) = k - n \text{ and } \dim \operatorname{Ker}(d(\partial g)_s) = k - 1 - n.$$

This **contradicts** assertion (18) about the kernels when $\operatorname{Ker}(dg_s) \subset \mathbb{R}^{k-1}$. Thus this assumption must be false, i.e.

$$T_s(S) = \operatorname{Ker}(dg_s) \not\subseteq \mathbb{R}^{k-1}$$

and hence $d\pi_s \neq 0$ and therefore $d\pi_s$ is surjective.

In other words, 0 is a regular value. **QED**

Preimages of manifolds with boundary

Let f be a smooth map of a manifold X with boundary onto a boundaryless manifold Y, and suppose that **both** $f\colon X \to Y$ and $\partial f\colon \partial X \to Y$ are **transversal** with respect to a boundaryless submanifold Z in Y. Then the preimage $f^{-1}(Z)$ is a manifold with boundary

$$\partial(f^{-1}(Z)) = f^{-1}(Z) \cap \partial X,$$

and the codimension of $f^{-1}(Z)$ in X equals the codimension of Z in Y.

Proof: The restriction of f to the boundaryless manifold $\mathrm{Int}(X)$ is transversal to Z. Hence, by the Preimage Theorem for boundaryless manifolds, $f^{-1}(Z) \cap \mathrm{Int}(X)$ **is a boundaryless manifold** of correct codimension. Thus it remains to examine $f^{-1}(Z)$ in a neighborhood of a point $x \in f^{-1}(Z) \cap \partial X$.

Let $l := \operatorname{codim} Z$ in Y. As in the boundaryless case, we can choose a submersion $h\colon W \to \mathbb{R}^l$ defined on an open neighborhood W of $f(x)$ in Y to $\mathbb{R}^l$ such that $Z \cap W = h^{-1}(0)$. Then $h \circ f$ is defined in a neighborhood V of x in X, and $f^{-1}(Z) \cap V = (h \circ f)^{-1}(0)$.

Now let $\phi\colon U \to X$ be a local parametrization around x, where U is an open subset of $\mathbb{H}^k$. Then define

$$g := h \circ f \circ \phi\colon U \to \mathbb{R}^l.$$

Since $\phi\colon V \to \phi(V)$ is a diffeomorphism, the set

$$\begin{aligned} &f^{-1}(Z) \text{ is a manifold with boundary in a neighhorhood of } x \\ \iff\ &(f \circ \phi)^{-1}(Z) = g^{-1}(0) \text{ is a manifold with boundary near } u = \phi^{-1}(x) \in \partial U. \end{aligned}$$

But the transversality assumptions of f and ∂f with respect to Z imply the 0 is a regular value of g. Hence we can apply the previous theorem and we are done. **QED**

Finally, also Sard's Theorem has a version with boundary.

Sard's Theorem with boundary

For any smooth map $f\colon X \to Y$ of a manifold X with boundary to a boundaryless manifold Y, almost every point of Y is a regular value of both f and ∂f.

Proof: For any point $x \in \partial X$ on the boundary of X,

$$d(\partial f)_x = (df_x)_{|T_x(\partial X)} \colon T_x(\partial X) \to T_{f(x)}(Y).$$

Hence if $d(\partial f)_x$ is surjective, then df_x is surjective. Hence if ∂f is regular at x, so is f.

Thus a point $y \in Y$ fails to be a regular value of both f and ∂f only when it is a critical value if both df_x fails to be surjective for all $x \in f^{-1}(y) \cap \mathrm{Int}(X)$ and $d(\partial f)_x$ fails to be surjective for all $x \in f^{-1}(y) \cap \partial X$.

But since $\mathrm{Int}(X)$ and ∂X are both boundaryless manifolds, both sets of critical values have measure zero by Sard's Theorem. Thus the complement of the set of common regular values for f and ∂f is the union of two sets of measure zero, and therefore itself a set of measure zero. **QED**

LECTURE 18

Brouwer Fixed Point Thereom and One-Manifolds

The following theorem gives us a complete list of smooth one-dimensional manifolds. Note that in genera, since every manifold is the disjoint union of its connected components, it suffices to classify connected manifold.

Classification of One-Manifolds

(a) Every compact, connected, one-dimensional smooth manifold without boundary is diffeomorphic to S^1.
(b) Every compact, connected, one-dimensional smooth manifold with boundary is diffeomorphic to $[0,1]$.
(c) Every noncompact, connected, one-dimensional smooth manifold with boundary is diffeomorphic to either $[0,1)$, $(0,1]$ or $(0,1)$.

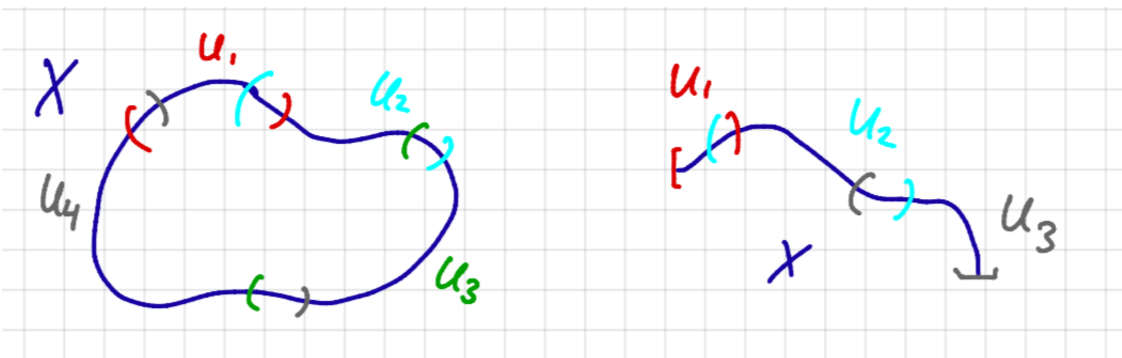
The details of the proof are surprisingly complicated. We content ourselves with a rough idea.

Some heuristics on why the theorem may be true:

(a) Let X be a nonempty, compact, connected 1-manifold. Each point has a neighborhood diffeomorphic to $(-1,1)$. By compactness, finitely many such neighborhoods $U_1, \ldots, U_n$ cover X. If n was equal 1, then $X \cong (-1,1)$. But an open interval is not compact. Thus, there must be at least two neighborhoods. Since X is connected, these two charts must intersect. The union of these two intervals has to be either an open interval (if they intersect on one side of each) or a circle (if they intersect on both sides). But if their union is an open interval, there has to be another chart, by the compactness of X. Since there are only finitely many U_i's, we must eventually arrive at the situation where the neighborhoods intersect on both sides and form a circle. Then one has to use this to construct a diffeomorphism to S^1.

(b) Let X be a compact, connected, one-dimensional smooth manifold with boundary. Since X has at least one boundary point, there must be neighborhood in X containing that boundary point. This neighborhood must be diffeomorphic to $[a,b)$ for some a, b. Since this interval is not compact, there must be another neighborhood in X. This neighborhood either intersects another boundary point which would yield us $X \cong [a,c]$ for some c, or it does not contain a boundary point. In the latter case, the union of the neighborhoods is diffeomorphic to a half-open interval $[a,d)$ which is not compact. Hence there has to be another neighborhood. Since X is compact, this process will end after finitely many steps when we eventually get that X is the union of neighborhoods which is diffeomorphic to a closed interval.

(c) When X is not compact, we repeat the above processes. The difference is that the process may not terminate and we end up with open or half-open intervals.



Much more interesting than the actual theorem are its consequences which are surprisingly rich.

Boundary of One-Manifolds

The boundary of any **compact** one-dimensional manifold with boundary consists of an **even** number of points.

Proof: Every compact one-manifold with boundary X is the disjoint union of **finitely many connected components**. Each component is diffeomorphic to a copy of $[0,1]$. Hence the boundary of each component consists of **two points**. The boundary of X consists of these finitely many **pairs** of points. **QED**

Retractions

Let X be a smooth manifold and $Z \subset X$ be a submanifold. Then a **retraction** is a smooth map $f\colon X \to Z$ such that $f_{|Z}$ is the identity.

There is an important restriction for the existence of such retractions for manifolds with boundary:

No retractions onto boundaries

If X is any **compact** manifold with boundary, then there is no retraction of X onto its boundary.

Proof: Suppose there is such a smooth map $g\colon X \to \partial X$ such that $\partial g\colon \partial X \to \partial X$ is the identity. By Sard's Theorem, we can choose a regular value $z \in \partial X$ of g. Since ∂g is the identity, all values in ∂X a regular for ∂g. Hence z is regular for both g and ∂g. By the Preimage Thoerem for manifolds with boundary, we know that $g^{-1}(z)$ is a submanifold of X with boundary

$$\partial(g^{-1}(z)) = g^{-1}(z) \cap \partial X.$$

Moreover, the codimension of $g^{-1}(z)$ in X equals the codimension of $\{z\}$ in ∂X, namely $\dim X - 1$ as $\{z\}$ has dimension 0. Hence $g^{-1}(z)$ is **one-dimensional**. Since it is a closed subset in the compact manifold X, it is also **compact**.

By definiton of ∂g as the restriction of g to ∂X, we have

$$(\partial g)^{-1}(z) = (g_{|\partial X})^{-1}(z) = g^{-1}(z) \cap \partial X = \partial(g^{-1}(z)).$$

But, since $\partial g = \mathrm{Id}_{\partial X}$,

$$\{z\} = (\partial g)^{-1}(z) = \partial(g^{-1}(z)).$$

This **contradicts** the previous result that the boundary $\partial(g^{-1}(z))$ of the compact one-dimensional manifold $g^{-1}(z)$ consists of an **even** number of points. **QED**

This theorem has a famous consequence:

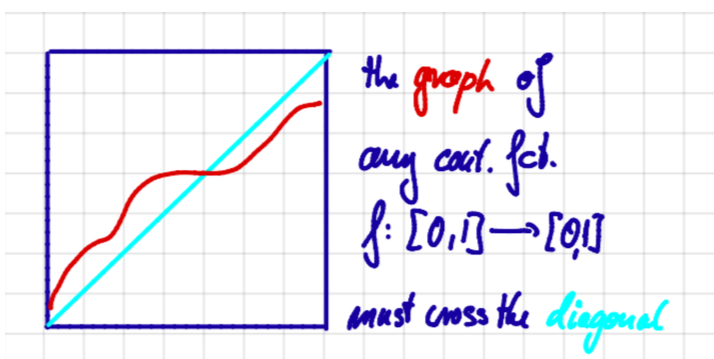
Brouwer Fixed-Point Theorem for smooth maps

Let $f\colon B^n \to B^n$ be a **smooth** map of the closed unit ball $B^n = \{x \in \mathbb{R}^n : |x| \leq 1\} \subset \mathbb{R}^n$ into itself. Then f must have a **fixed point**, i.e. there is an $x \in B^n$ with $f(x) = x$.

Before we prove the theorem, let us have a look at dimension one, where the result is very familiar:

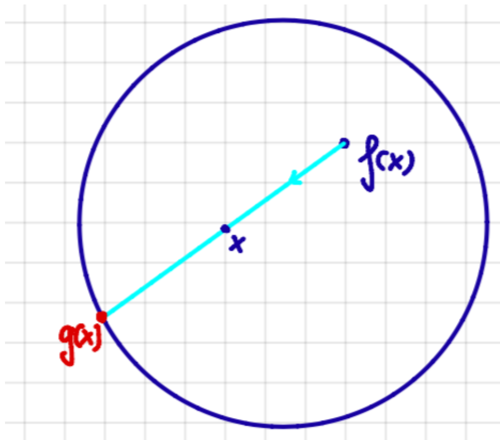
Brouwer FPT is familiar in dimension one

Note that we have seen this theorem for $n = 1$ in Calculus 1. Let $f\colon [0,1] \to [0,1]$ be a continuous map. Then it must have a fixed point. For, if not, then $g(x) = f(x) - x$ is a continuous map defined on [0,1]. We have $g(0) \geq 0$ and $g(1) \leq 0$, since $f(0) \geq 0$ and $f(1) \leq 1$. If $g(0) = 0$ or $g(1) = 1$, we are done. But if $g(0) > 0$ and $g(1) < 1$, then the Intermediate Value Theorem implies that there is an $x_0 \in (0,1)$ with $g(x_0) = 0$, i.e. $f(x_0) = x_0$.



Proof of Brouwer FPT: Suppose that there exists an f without fixed points. We will show that such an f would allow us to construct a retraction $g\colon B^n \to \partial B^n$. But, since B^n is **compact**, we have just proved that such a retraction cannot exist.

So suppose $f(x) \neq x$ for all $x \in B^n$. Then, for every $x \in B^n$, the two points x and $f(x)$ determine a line. Let $g(x)$ be the point where the line segment starting at $f(x)$ and passing through x hits the boundary ∂B^n. This defines a map $g\colon B^n \to \partial B^n$.



If $x \in \partial B^n$, then $g(x) = x$ by construction of g. Hence $g\colon B^n \to \partial B^n$ is the identity on ∂B^n. Thus, in order to show that g is a retraction, it remains to show that g is smooth.

To show this, we describe $g(x)$ explicitly. As a point on the line from $f(x)$ to x, $g(x)$ can be written in the form

$$g(x) = x + tv, \text{ where } v := \frac{x - f(x)}{|x - f(x)|}$$

for some real number t. Note that, since we assume $x \neq f(x)$, the vector v is always defined. In fact, it is the unit vector pointing from $f(x)$ to x. Moreover, since f is smooth, v depends smoothly on x.

We need to calculate t and show that **t depends smoothly on x**. Since $g(x)$ is a point on boundary of B^n, we know $|g(x)| = 1$, and t is determined by the equation

$$1 = |g(x)|^2 = (x + tv) \cdot (x + tv) = x \cdot x + 2tx \cdot v + t^2 v \cdot v$$

or, equivalently,

$$0 = (v \cdot v)t^2 + (2x \cdot v)t + x \cdot x - 1. \tag{19}$$

By definition of v, we know $v \cdot v = |v|^2 = 1$. Since v points from $f(x)$ to x, we know that t must be positive. Now we just need to find the positive solution of the quadratc equation (19) for t and get

$$\begin{aligned} t &= \frac{-2x \cdot v + \sqrt{4(x \cdot v)^2 - 4(x \cdot x - 1)}}{2} \\ &= -x \cdot v + \sqrt{(x \cdot v)^2 - x \cdot x + 1} \end{aligned}$$

where $(x \cdot v)^2 - x \cdot x + 1$ is positive, since $x \cdot x = |x|^2 \leq 1$ and $(x \cdot v)^2 > 0$. Since the scalar products and square roots involved depend smoothly on x, we see that t depends smoothly on x. Hence g is smooth. **QED**

Note that, **for** $n = 1$, in the above proof we would construct a map $g\colon [0,1] \to \{0,1\}$ which would send 0 to 0 and 1 to 1. Such a map cannot be smooth, not even continuous by the Intermediate Value Theorem.

Brouwer Fixed-Point Theorem for continuous maps

Any **continuous** map $F\colon B^n \to B^n$ must have a **fixed point**.

Proof: The idea is to reduce this theorem to the statement on smooth maps by **approximating** F by a smooth mapping. This is possible by **Weierstrass' Approximation Theorem**, an important result from Calculus, which applies as B^n is **compact** and says:

Given $\epsilon > 0$, there is a **polynomial function** $Q\colon B^n \to \mathbb{R}^n$ with

$$|Q(x) - F(x)| < \epsilon \text{ for all } x \in B^n.$$

(Recall that a *polynomial function* is a function that arises by **finitely many** additions and multiplications of the coordinate functions. Such functions are obviously **smooth**.)

However, it is possible that Q sends points in B^n to points outside of B^n. In order to remedy this defect, we replace Q with

$$P(x) := \frac{Q(x)}{1+\epsilon}.$$

Since $|F(x)| \leq 1$, this new polynomial P satisfies:

$$(1+\epsilon)|P(x)| = |Q(x)| \leq |Q(x) - F(x)| + |F(x)| < \epsilon + 1$$

where we appy the trianlge inequality. Hence $|P(x)| \leq 1$ and **P is a map $B^n \to B^n$**. Moreover,

$$\begin{aligned}(1+\epsilon)|P(x) - F(x)| &= |Q(x) - (1+\epsilon)F(x)| = |Q(x) - F(x) + \epsilon F(x)| \\ &\leq |Q(x) - F(x)| + \epsilon|F(x)| < 2\epsilon\end{aligned}$$

where we use that $|F(x)| \leq 1$. Since $1 + \epsilon > 1$, this shows

$$|P(x) - F(x)| < 2\epsilon. \tag{20}$$

Now **suppose that** $F(x) \neq x$ for all $x \in B^n$. Then the continuous function

$$B^n \to B^n,\, x \mapsto |F(x) - x|$$

must have a **minimum** μ, since B^n is **compact**. Since $F(x) \neq x$ for all x, we must have $\mu > 0$.

Now,for $\epsilon = \mu/2$, we choose polynomials Q and then P as above. Since $|F(x) - x| \geq \mu$ for all $x \in B^n$, the triangle inequality yields

$$\begin{aligned} \mu \leq |F(x) - x| &= |F(x) - P(x) + P(x) - x| \\ &\leq |F(x) - P(x)| + |P(x) - x|. \end{aligned}$$

But by (20), we know

$$|F(x) - P(x)| < \mu \text{ for all } x \in B^n.$$

Thus $|P(x) - x| > 0$, and therefore $P(x) \neq x$ for all $x \in B^n$.

Hence $P\colon B^n \to B^n$ is a **smooth map** from B^n to itself **without a fixed point**. This contradicts the statement on smooth maps and completes the proof. **QED**

The theorem is not true for the open ball:

Counterexamples on open balls

Let $B_1^k(0) = \{x \in \mathbb{R}^k : |x| < 1\}$ be the **open** ball in $\mathbb{R}^k$. We define the map

$$\varphi\colon B_1^k(0) \to \mathbb{R}^k,\, x \mapsto \frac{x}{\sqrt{1 - |x|^2}}.$$

This is a **smooth** map with **smooth inverse**

$$\varphi^{-1}\colon \mathbb{R}^k \to B_1^k(0),\, y \mapsto \frac{y}{\sqrt{1 + |y|^2}}$$

Thus φ is a **diffeomorphism** $B_1^k(0) \to \mathbb{R}^k$.
The translation $T\colon \mathbb{R}^k \to \mathbb{R}^k,\ x \mapsto x+1$ does not have a fixed point. Hence the composite map

$$\varphi^{-1} \circ T \circ \varphi\colon B_1^k(0) \to B_1^k(0)$$

does **not** have a fixed point. For if it had a fixed point x, then

$$\varphi^{-1}(T(\varphi(x))) = x \Rightarrow\ T(\varphi(x)) = \varphi(x)$$

and T had a fixed point, which is not the case.

Brouwer's Fixed-Point Theorem has many important consequences. Here is one of them:

Brouwer Invariance of Domain

Let U be an **open** subset of $\mathbb{R}^n$, and let $f\colon U \to \mathbb{R}^n$ be a **continuous injective** map. Then $f(U)$ is also **open**.

Instead of studying the proof of this theorem, let us note a consequence of this result:

Topological Invariance of Dimension

If $n > m$, and U is a nonempty **open** subset of $\mathbb{R}^n$, then there is **no continuous injective** map from U to $\mathbb{R}^m$. In particular, $\mathbb{R}^n$ and $\mathbb{R}^m$ are **not homeomorphic** whenever $n \neq m$.

Even though it sounds like an obvious fact, this is a rather deep theorem. Note that there exist weird things like a continuous surjection from $\mathbb{R}^m$ to $\mathbb{R}^n$ for $n > m$ due to variants of the Peano curve construction. Hence often we have to be careful with our topological intuition.

Proof of Topological Invariance of Dimension: If there was such a continuous injective map from U to $\mathbb{R}^m$, then we could compose it with the embedding $\mathbb{R}^m \hookrightarrow (\mathbb{R}^m \times \{0\}) \subset \mathbb{R}^n$. Hence the composite would yield a **continuous injective** map from U into $\mathbb{R}^n$. By the theorem, the image would be both open in $\mathbb{R}^n$ and lie in the subspace $\mathbb{R}^m \times \{0\}$. But no **open** subset of $\mathbb{R}^n$ can be contained in $\mathbb{R}^m \times \{0\}$, since we must be able to fit at least a tiny **open ball** of $\mathbb{R}^n$ into that subset and there is no room for such a ball in the direction of the remaining $n - m$ coordinates.

Finally, a homeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^m$ would be such a continuous injective map. **QED**

Note that invariance of domain and dimension for **smooth** injective maps is just a consequence of the Inverse Function Theorem. But for maps which are just continuous and injective, it is much harder to achieve.

LECTURE 19

Transversality is generic

Today we are going to review what we have learned about transversality and show that it is actually a generic property. We start with the following extension of Sard's Theorem:

Transversality Theorem

Suppose that $F\colon X \times S \to Y$ is a smooth map of manifolds, where only X has a boundary, and let Z be any boundaryless submanifold of Y. If both F and ∂F are transversal to Z, then for almost every $s \in S$, both f_s and ∂f_s are transversal to Z (where f_s denotes the map $x \mapsto f_s(x) = F(x,s)$, and similarly $\partial f_s(x) = \partial F(x,s)$).

Note that, roughly speaking, the difference between requiring that F is transversal to Z versus f_s is transversal to Z is that for F the image of $T_{(x,s)}(X \times S)$ under $dF_{(x,s)}$ has to be big enough, whereas for f_s we look at the potentially smaller image of $T_{(x,s)}(X \times S)$ under $d(f_s)_x$. Similarly for ∂F and ∂f_s.

Proof: By the Preimage Theorem, the preimage $W := F^{-1}(Z)$ is a submanifold of $X \times S$ with boundary $\partial W = W \cap \partial(X \times S)$. Let $\pi\colon X \times S \to S$ be the natural projection map.

We will show that whenever $s \in S$ is a **regular value** for the restriction $\pi\colon W \to S$ then $f_s \pitchfork Z$, and whenever s is a regular value for $\partial\pi\colon \partial W \to S$, then $\partial f_s \pitchfork Z$. By Sard's theorem (which also holds for manifolds with boundary), almost every $s \in S$ is a regular value for both maps, so the theorem follows.

In order to show that $f_s \pitchfork Z$, suppose that $f_s(x) = z \in Z$. Because $F(x,s) = z$ and $F \pitchfork Z$, we know that

$$dF_{(x,s)}(T_{(x,s)}(X \times S)) + T_z(Z) = T_z(Y).$$

Hence, given any vector $a \in T_z(Y)$, there exists a vector $b \in T_{(x,s)}(X \times S)$ such that

$$dF_{(x,s)}(b) - a \in T_z(Z).$$

What **we need** is to find a vector $v \in T_x(X)$ such that

$$df_s(v) - a \in T_z(Z),$$

as that would show that $df_s(T_x(X)) + T_z(Z) = T_z(Y)$.

Since

$$T_{(x,s)}(X \times S) = T_x(X) \times T_s(S),$$

we can write b as a pair (w,e) for vectors $w \in T_x(X)$ and $e \in T_s(S)$.

If e was zero, we would be done, for since the restriction of F to $X \times \{s\}$ is f_s, it follows that

$$dF_{(x,s)}(w,0) = df_s(w).$$

Although e need not be zero, we may use the projection π to kill it off.

It is easy to check that

$$d\pi_{(x,s)} \colon T_x(X) \times T_s(S) \to T_s(S)$$

is just projection onto the second factor (this holds for every projection map from a product of manifolds).

Now we use the assumption that s is a regular value of π. For this implies that

$$d\pi_{(x,s)} \colon T_{(x,s)}(W) \to T_s(S)$$

is surjective. In particular, the fiber over $e \in T_s(S)$ is nonempty, and there is some vector of the form (u,e) in $T_{(x,s)}(W)$.

But $F \colon W \to Z$, so $dF_{(x,s)}(u,e)$ is an element in $T_z(Z)$. Consequently, the vector $v := w - u \in T_x(X)$ is our solution. For

$$df_s(v) - a = dF_{(x,s)}((w,e) - (u,e)) - a = (dF_{(x,s)}(w,e) - a) - dF_s(u,e),$$

and both of the latter vectors belong to $T_z(Z)$.

Precisely the same argument shows that $\partial f_s \pitchfork Z$ when s is a regular value of $\partial\pi$. **QED**

Transversality is generic - first case

Transversality for smooth maps $X \to \mathbb{R}^M$ is generic in the following sense:

Let $f\colon X \to \mathbb{R}^M$ be any smooth map. Let S be an open ball in $\mathbb{R}^M$, and define

$$F\colon X \times S \to \mathbb{R}^M,\ F(x,s) = f(x) + s.$$

The derivative of F at (x,s) is

$$dF_{(x,s)} = (df_x, \mathrm{Id}_{\mathbb{R}^M})\colon T_x(X) \times \mathbb{R}^M \to \mathbb{R}^M.$$

Thus $dF_{(x,s)}$ is obviously surjective at any (x,s). Hence F is a **submersion**. This implies that F is **transversal to every submanifold $Z \subset \mathbb{R}^M$**. Now we can apply the **Transversality Theorem** we have just proven: Since F and ∂F are transversal to Z, for **almost every** $s \in S$, the map $f_s(x) = f(x) + s$ is transversal to Z. Thus, for any submanifold $Z \subset \mathbb{R}^M$, there is an s, with **arbitrarily small norm** in $\mathbb{R}^M$, such that f may be deformed into a map f_s transversal to Z by the **translation by s**.

This shows us that transversality is generic for maps $X \to \mathbb{R}^M$. We would like to generalize this result to an **arbitrary** boundaryless smooth manifold $Y \subset \mathbb{R}^M$ and smooth map $f\colon X \to Y$.

Given a submanifold $Z \subset Y$, we have just learned how to vary $f\colon X \to Y \subset \mathbb{R}^M$ as a family of maps $X \to \mathbb{R}^M$ such that $f_s \pitchfork Z$ for arbitrarily small s, where we consider Z as a submanifold in $\mathbb{R}^M$.

It remains to understand how we can project these maps down onto Y such that a small perturbation f_s of f remains transversal to the given submanifold $Z \subset Y$. To do so, we must understand a little of the **geometry of Y with respect to its environment**. As usual, the compact case is clearest.

ϵ-Neighborhood Theorem

For a **compact** boundaryless manifold Y in $\mathbb{R}^M$ and a positive number ϵ, let Y^ϵ be the open set of points in $\mathbb{R}^M$ with distance less than ϵ from Y. If ϵ is sufficiently small, then each point $w \in Y^\epsilon$ possesses a unique closest point in Y, denoted $\pi(w)$. Moreover, the map $\pi\colon Y^\epsilon \to Y$ is a submersion. When Y is **not compact**, there still exists a submersion $\pi\colon Y^\epsilon \to Y$ that is the identity on Y, but now ϵ must be allowed to be a positive **smooth function** $\epsilon\colon Y \to \mathbb{R}^{>0}$ on Y, and Y^ϵ is defined as

$$Y^\epsilon = \{w \in \mathbb{R}^M : |w - y| < \epsilon(y) \text{ for some } y \in Y\} \subset \mathbb{R}^M.$$

The manifold Y^ϵ is called a **tubular neighborhood** of Y in $\mathbb{R}^M$.

Note that the important point of the theorem is not so much the existence of the Y^ϵ, but rather that they come equipped with the submersion π. As we will see in a bit, this is related to a key tool, the normal bundle.

Before we prove this theorem, we study a first consequence:

Creating families of submersions

Let $f\colon X \to Y$ be a smooth map where Y is a boundaryless manifold. Let S be the open ball in $\mathbb{R}^M$. Then there is a smooth map $F\colon X \times S \to Y$ such that $F(x,0) = f(x)$, and for any **fixed** $x \in X$, the map

$$S \to Y,\ s \mapsto F(x,s) \text{ is a submersion.}$$

In particular, both F and ∂F are submersions.

Proof: Let $Y \subset \mathbb{R}^M$ and S be the unit ball in $\mathbb{R}^M$. We define

$$F\colon X \times S \to Y,\ F(x,s) = \pi(f(x) + \epsilon(f(x))s). \tag{21}$$

Since $\pi\colon Y^\epsilon \to Y$ restricts to the identity on Y, we have

$$F(x,0) = \pi(f(x) + 0) = f(x).$$

For **fixed** x, the map

$$\varphi\colon S \to Y^\epsilon,\ s \mapsto f(x) + \epsilon(f(x))s$$

is the translation of a linear map. Thus $d\varphi_s$ is just given by multiplying a vector in $T_s(S) = \mathbb{R}^M$ by the real number $\epsilon(f(x)) > 0$ (to get a vector in $T_{\varphi(s)}(Y^\epsilon) \subset \mathbb{R}^M$). This derivative is just $\epsilon(f(x))$ times the identity of $\mathbb{R}^M$, and therefore surjective. Thus φ is a **submersion**.

As the composition of two submersions is a submersion, we get that

$$S \to Y,\ s \mapsto F(x,s) \text{ is a submersion.}$$

Hence the restriction $F_{\{x\}\times S}\colon \{x\} \times S \to Y$ of F to the submanifold $\{x\} \times S$ is submersion for every $x \in X$. Since every point of $X \times S$ lies in one of these submanifolds, **F must be a submersion** as well, since its derivative $dF_{(x,s)}$ is already surjective onto $T_{F(x,s)}$ when restricted to $T_{(x,s)}(\{x\} \times S) \subset T_{(x,s)}(X \times S)$.

The same argument applied to ∂F and ∂X, shows that ∂F is a submersion. **QED**

Now we can prove that transversality is generic:

Transversality Homotopy Theorem

For any smooth map $f\colon X \to Y$ and any boundaryless submanifold Z of the boundaryless manifold Y, there exists a smooth map $g\colon X \to Y$ homotopic to f such that $g \pitchfork Z$ and $\partial g \pitchfork Z$.

Proof: For the family of mappings F of the previous conesequence of the ϵ-Neighborhood Theorem, the Transversality Theorem implies that $f_s \pitchfork Z$ and $\partial f_s \pitchfork Z$ for almost all $s \in S$. But each f_s is homotopic to f, the homotopy being

$$X \times I \to Y, \ (x,t) \mapsto F(x,ts).$$

QED

Now we are going to prove the ϵ-Neighborhood Theorem. To do this we introduce an important geometric tool similar to the tangent bundle.

The Normal Bundle

For each $y \in Y$, define $N_y(Y)$, the normal space of Y at y, to be the orthogonal complement of $T_y(Y)$ in $\mathbb{R}^M$. The normal bundle $N(Y)$ is then defined to be the set

$$N(Y) = \{(y,v) \in Y \times \mathbb{R}^M : v \in N_y(Y)\}.$$

Note that unlike $T(Y)$, $N(Y)$ is not intrinsic to the manifold Y but depends on the specific relationship between Y and the surrounding $\mathbb{R}^M$. There is a natural projection map $\sigma\colon N(Y) \to Y$ defined by $\sigma(y,v) = y$.

The normal bundle $N(Y)$ is actually a manifold itself. In order to show this, we must recall an elementary fact from linear algebra:

Suppose that $A\colon \mathbb{R}^M \to \mathbb{R}^k$ is a linear map. Its **transpose** is a linear map $A^t\colon \mathbb{R}^k \to \mathbb{R}^M$ characterized by the dot product equation

$$Av \cdot w = v \cdot A^t w \text{ for all } v \in \mathbb{R}^M, w \in \mathbb{R}^k.$$

Claim: If A is surjective, then A^t maps $\mathbb{R}^k$ isomorphically onto the orthogonal complement of the kernel of A.

First we note that A^t is injective. For if $A^t w = 0$, then $Av \cdot w = v \cdot A^t w = 0$, so that $w \perp A(\mathbb{R}^M)$. Since A is surjective, w must be zero.

Now, if $v \in \operatorname{Ker}(A)$, i.e. $Av = 0$, then $0 = Av \cdot w = v \cdot A^t w$. Thus $A^t(\mathbb{R}^k) \perp \operatorname{Ker}(A)$. Hence A^t maps $\mathbb{R}^k$ injectively into the orthogonal complement of Ker (A). As Ker (A) has dimension $M-k$, its complement has dimension k, so A^t is surjective, too.

Normal bundles are manifolds

If $Y \subset \mathbb{R}^M$, then $N(Y)$ is a manifold of dimension M and the projection $\sigma\colon N(Y) \to Y$ is a submersion.

Proof: We need to find loal parametrizations for $N(Y)$.

Therefor, we use that we have learned that we can write every manifold locally as the zeros of a smooth function. Hence around every point in Y, there is an open neighborhood $U \subset Y$ and an open subset $\tilde{U} \subset \mathbb{R}^M$ with $U = Y \cap \tilde{U}$ such that we can write U as the zeros of a submersion

$$\varphi\colon \tilde{U} \to \mathbb{R}^k \ (k = \operatorname{codim} Y) \text{ with } U = Y \cap \tilde{U} = \varphi^{-1}(0).$$

The set $N(U)$ equals $N(Y) \cap (U \times \mathbb{R}^M)$, thus is **open in** $N(Y)$.

For each $y \in U$, $d\varphi_y\colon \mathbb{R}^M \to \mathbb{R}^k$ is **surjective** and has **kernel** $T_y(Y)$ by the Preimage Theorem.

Therefore its **transpose** maps R^k isomorphically onto the orthogonal complement of Ker $(d\varphi_y) = T_y(Y)$ which is $N_y(Y)$ by definition:

$$(d\varphi_y)^t\colon \mathbb{R}^k \xrightarrow{\cong} (T_y(Y))^\perp = N_y(Y).$$

Hence the map

$$\psi\colon U \times \mathbb{R}^k \to N(U),\ (y,v) \mapsto (y, d\varphi_y^t(v))$$

is bijective. It is also an embedding of $U \times \mathbb{R}^k$ into $U \times \mathbb{R}^M$, since it is the identity on the first factor and an injective linear map on the second factor. Hence ψ is a diffeomorphism, and $N(U)$ is a **manifold** with local parametrization ψ.

The **dimenion** of $N(U)$ is

$$\dim N(U) = \dim U + k = \dim Y + \operatorname{codim} Y = M.$$

Since every point of $N(Y)$ has such a neighborhood, $N(Y)$ is a **manifold**.

Note that $\sigma \circ \psi\colon U \times \mathbb{R}^k \to U$ is just the projection onto the first factor, which is a submersion. Thus $d(\sigma \circ \psi)_{(u,w)}$, is surjective at every point (u,w). Hence $d\sigma_u$ is surjective at every u, and σ is a **submersion**. **QED**

Before the get to proof the actual theorem, we start with a lemma that will give us the existence of the ϵ-neighborhood Y^ϵ.

ϵ-Neighborhood Lemma

Let $Y \subset \mathbb{R}^M$ be a boundaryless manifold. Then any neighborhood $\tilde{U}$ of Y in $\mathbb{R}^M$, i.e. any open subset $\tilde{U}$ of $\mathbb{R}^M$ with $Y \subset \tilde{U}$, contains

$$Y^\epsilon = \{w \in \mathbb{R}^M : |w - y| < \epsilon(y) \text{ for some } y \in Y\}$$

where $\epsilon\colon Y \to \mathbb{R}^{>0}$ is a suitable smooth function. Moreover, if Y is **compact**, ϵ can be chosen **constant**.

Proof: For each point $\alpha \in Y$, we can find a small radius ϵ_α such that the open ball $B_{2\epsilon_\alpha}(\alpha) \subset \tilde{U}$ is contained in $\tilde{U}$. We set

$$U_\alpha := Y \cap B_{\epsilon_\alpha}(\alpha).$$

Claim:

$$U_\alpha^{\epsilon(\alpha)} = \{w \in \mathbb{R}^M : |w - y'| < \epsilon(\alpha) \text{ for some } y' \in U_\alpha\} \subset \tilde{U}.$$

For, $w \in U_\alpha^{\epsilon_\alpha}$ means there is an $y' \in U_\alpha$ with $|w - y'| < \epsilon_\alpha$. But $y' \in U_\alpha$ means $|y' - \alpha| < \epsilon_\alpha$. Thus the **triangle inequality** implies

$$|w - \alpha| \le |w - y'| + |y' - \alpha| < 2\epsilon_\alpha.$$

Thus $w \in B_{2\epsilon_\alpha}(\alpha) \subset \tilde{U}$ by the choice of ϵ_α.

The collection of all U_α forms an open cover $\{U_\alpha\}$ of $Y \subset \mathbb{R}^M$. By the Theorem in the Existence of Partitions of Unity for subsets in $\mathbb{R}^M$, we can choose a **partition of unity** $\{\rho_i\}$ subordinate to the cover $\{U_\alpha\}$.

Now we define the function

$$\epsilon\colon Y \to \mathbb{R}^{>0},\ y \mapsto \sum_i \rho_i(y)\epsilon_i$$

Note that ϵ is a smooth function, since all the ρ_i's are smooth.

Claim: $Y^\epsilon \subset \tilde{U}$.

Let $w \in Y^\epsilon$. Then there is a $y \in Y$ such that $|w - y| < \epsilon(y)$. For this y, only finitely many of the numbers $\rho_i(y)$ are nonzero, say $\rho_{i_1}(y), \dots, \rho_{i_n}(y)$. This implies $y \in U_{i_1} \cap \dots \cap U_{i_n}$.

Let ϵ_{i_m} be the maximum of the finitely many numbers $\epsilon_{i_1}, \dots, \epsilon_{i_n}$. Then, since $\sum_i \rho_i(y) = 1$, we have $\epsilon(y) \le \epsilon_{i_m}$. Hence

$$|w - y| < \epsilon(y) \le \epsilon_{i_m} \text{ implies } w \in U_{i_m}^{\epsilon_{i_m}} \subset \tilde{U}.$$

Thus $Y^\epsilon \subset \tilde{U}$.

If Y is compact, we can reduce $\{U_\alpha\}$ to a finite cover $U_{\alpha_1}, \ldots, U_{\alpha_n}$ and let ϵ be equal the maximum of the ϵ_{α_j}. **QED**

Now we are equipped for the proof of the ϵ-Neighborhood Theorem.

Proof of the ϵ-Neighborhood Theorem:

The idea of the proof is to use a version of the **Inverse Function Theorem** to show that the ϵ-neighborhood Y^ϵ of $Y = Y \times \{0\}$ in $\mathbb{R}^M \times \mathbb{R}^M$ is diffeomorphic to an **open subspace in the normal bundle**. Then we use the **natural submersion** $\sigma\colon N(Y) \to Y$ from the normal bundle to get the submersion $\pi\colon Y^\epsilon \to Y$:

$$\begin{array}{ccc} Y^\epsilon & \xrightarrow{h^{-1}} & N(Y) \\ & \searrow^{\pi} & \downarrow \sigma \\ & & Y \end{array}$$

To make this precise, we define the map

$$h\colon N(Y) \to \mathbb{R}^M,\ (y,v) \mapsto y + v.$$

We claim that h is **regular at every point** of $Y \times \{0\}$ in $N(Y)$.

For, since h is just the restriction of the linear map

$$H\colon \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}^M, (w,z)) \mapsto w + z,$$

the derivative of h at (y,v) is just

$$dh_{(y,v)} = H\colon \mathbb{R}^M \times \mathbb{R}^M \to \mathbb{R}^M.$$

Hence at any point (y,v) we have

$$dh_{(y,v)}(w,0) = w \text{ and } dh_{(y,v)}(0,z) = z.$$

The tangent space to $N(Y)$ at $(y,0)$ is just

$$T_{(y,0)}(N(Y)) = T_y(Y) \times \{0\} \oplus \{0\} \times N_y(Y),$$

since $T_y(Y)$ and $N_y(Y)$ are orthogonal complements in $\mathbb{R}^M$ and $\dim N(Y) = M$.

At the point $(y,0)$, $dh_{(y,0)}$ maps

$$T_{(y,0)}(Y \times \{0\}) \text{ onto } T_y(Y) \text{ in } \mathbb{R}^M,$$

and it maps

$$T_{(y,0)}(\{y\} \times N_y(Y)) = \{0\} \times N_y(Y) \text{ onto } N_y(Y) \text{ in } \mathbb{R}^M,$$

where we use that $N_y(Y)$ is a vector space and hence its own tangent space.

Hence, in total, we get

$$\begin{aligned} dh_{(y,0)}(T_{(y,0)}(N(Y))) &= T_{(y,0)}(Y \times \{0\}) + T_{(y,0)}(\{y\} \times N_y(Y)) \\ &= T_y(Y) + N_y(Y) = \mathbb{R}^M. \end{aligned}$$

Thus $dh_{(y,0)}$ is surjective and h is regular at $(y{,}0)$.

Since h maps $Y \times \{0\}$ diffeomorphically onto Y and is regular at each $(y{,}0)$, a generalization of the **Inverse Function Theorem** which we prove in the appendix implies that h must map a **neighborhood of $Y \times \{0\}$ in $N(Y)$** diffeomorphically onto a **neighborhood of Y in $\mathbb{R}^M$**.

Now any neighborhood of Y contains some Y^ϵ by the **ϵ-Neighborhood Lemma**. Thus $h^{-1}\colon Y^\epsilon \to N(Y)$ is defined, and

$$\pi = \sigma \circ h^{-1}\colon Y^\epsilon \to Y$$

is the desired submersion.

It is an exercise to check that we can describe π for compact manifolds as given in the theorem. **QED**

As a consquence of the proof of the theorem we note the following useful result:

Tubular Neighborhoods and Normal Bundles

Let $Y \subset \mathbb{R}^M$ be a boundaryless smooth manifold. Then there is a diffeomorphism of an open neighborhood Y^ϵ of Y in $\mathbb{R}^M$ to an open neighborhood $N^\epsilon(Y)$ of $Y \times \{0\}$ in $N(Y)$.

Proof: In the proof of the ϵ-Neighborhood Theorem, we constructed the smooth map h which restricts to a diffeomorphism of open neighborhoods as claimed. **QED**

In the final part of today's lecture, we look at another application of the ϵ-Nieghborhoood Theorem. In fact, there is a stronger form of the Transversality Homotopy Theorem. In order to be able to formulate it, we need some terminology.

Transversality on subsets

Let $f: X \to Y$ be a smooth map, $Z \subset Y$ a submanifold, and $C \subset X$ be a **subset**. We will say f is transversal to Z on C, if the transversality condition

$$\operatorname{Im}(df_x) + T_{f(x)}(Z) = T_{f(x)}(Y) \tag{22}$$

for all $x \in C \cap f^{-1}(Z)$.
Note that, even if C is a submanifold, this is different than requiring $f_{|C} \pitchfork Z$, since (22) involves $\operatorname{Im}(df_x) = df_x(T_x(X))$, not $\operatorname{Im}(d(f_{|C})_x) = df_x(T_x(C))$, which is smaller in general.

Now we can formulate the next important technical result.

Extension Theorem

Let $f: X \to Y$ be a smooth map, Y boundaryless, and Z a closed submanifold of Y without boundary. Let C be a closed subset of X. Assume that $f \pitchfork Z$ on C and $\partial f \pitchfork Z$ on $C \cap \partial X$. Then there exists a smooth map $g: X \to Y$ **homotopic to f**, such that $g \pitchfork Z$ and $\partial g \pitchfork Z$, and on a neighborhood of C we have $g = f$.

Since ∂X is always closed in X, we obtain the important special case:

Extension of maps on boundaries

Assume $f: X \to Y$ is a smooth map such that the boundary map $\partial f: \partial X \to Y$ is transversal to Z. Then there exists a map $g: X \to Y$ **homotopic to f** such that $\partial g = \partial f$ and $g \pitchfork Z$.
In particular, suppose there is a smooth map $h: \partial X \to Y$ transversal to Z. Then, **if** h extends to **any** map on the whole manifold $X \to Y$, it extends to a map that is **transversal to Z** on all of X.

For the proof of the Extension Theorem we need lemma first:

Lemma

If U is an open subset which contains the closed set C in X, then there exists a smooth function $\gamma: X \to [0,1]$ that is identically equal to one outside U but that is zero on a neighborhood of C.

Proof: Let C' be any closed set contained in U that contains C in its interior, and let $\{\rho_i\}$ be a partition of unity subordinate to the open cover $\{U, X\backslash C'\}$. Here it comes handy that we proved the existence of partition of unity for arbitrary subsets of $\mathbb{R}^N$. Then just take γ to be the sum of those ρ_i that vanish outside of $X \setminus C'$. **QED**

Proof of the Extension Theorem:

First we show that $f \pitchfork Z$ on a neighborhood of C i.e. an open subset containing C. If $x \in C$ but $x \notin f^{-1}(Z)$, then since Z is closed, $X \setminus f^{-1}(Z)$ is a neighborhood of x on which $f \pitchfork Z$ automatically.

If $x \in f^{-1}(Z)$, then there is a neighborhood W of $f(x)$ in Y and a submersion $\varphi\colon W \to \mathbb{R}^k$ such that $f \pitchfork Z$ at a point $x' \in f^{-1}(Z \cap W)$ precisely when $\varphi \circ f$ is regular at x'. But if $\varphi \circ f$ is regular at x, so it is regular in a neighborhood of x. Thus $f \pitchfork Z$ on a neighborhood of every point $x \in C$, and so

$$f \pitchfork Z \text{ on a } \textbf{neighborhood } U \text{ of } C \text{ in } X.$$

Second, let γ be the function in the above lemma for the closed subset C and the open neighborhood U of C in X. We set $\tau := \gamma^2$. Since

$$d\tau_x = 2\gamma(x)d\gamma_x, \text{ hence } \gamma(x) = 0 = \tau(x) \Rightarrow d\tau_x = 0.$$

Now we modify the map $F\colon X \times S \to Y$ which we defined in (21) in proving the Homotopy Theorem, where S is the unit ball in $\mathbb{R}^M$. and set

$$G\colon X \times S \to Y,\ G(x,s) := F(x, \tau(x)s).$$

Claim: $G \pitchfork Z$.

For suppose that $(x,s) \in G^{-1}(Z)$, and let us assume first $\tau(x) \neq 0$. Then the map

$$S \to Y,\ r \mapsto G(x,r),$$

is a submersion, since it is the composition of the

$$\text{diffeomorphism } r \mapsto \tau(x)r \text{ with the submersion } r \mapsto F(x,r).$$

Hence G is regular at (x,s), so certainly $G \pitchfork Z$ at (x,s).

To show the claim when $\tau(x) = 0$, we need to check that the image of the derivative $dG_{(x,s)}$ is big enough. To do this, we introuce the map

$$m\colon X \times S \to X \times S,\ (x,s) \mapsto (x,\tau(x)s).$$

We would like to calculate the derivative of m. Therefor, we apply the product rule to the second coordinate and remember that $\tau\colon X \to [0,1]$, i.e. $\tau(x)$ and $d\tau_x(v)$ are both in $\mathbb{R}$ for any $v \in T_x(X)$. Then we get

$$dm_{(x,s)}(v,w) = (v,\tau(x)\cdot w + d\tau_x(v)\cdot s)$$

where w and s are vectors in $\mathbb{R}^M$.

We observe that $G = F \circ m$. Hence in order to calculate the derivative of G, we can apply the chain rule. Since we are interested in the case where $\tau(x) = 0$ and $d\tau_x = 0$ we get

$$dG_{(x,s)}(v,w) = dF_{(x,s)}(v,0).$$

Moreover, since $F(x,0) = f(x)$ for all x by construction of F, we know $F_{|X\times\{0\}} = f$. This implies

$$dF_{(x,s)}(v,0) = dF_{(x,0)}(v,0) = df_x(v).$$

Hence we get

$$dG_{(x,s)}(v,w) = df_x(v)$$

and therefore

$$\mathrm{Im}\,(dG_{(x,s)}) = \mathrm{Im}\,(df_x(v)) \subset T_{f(x)}(Y). \tag{23}$$

Now $\tau(x) = 0$, implies $x \in U$ by definition of γ and τ. But by the choice of U above, this implies $f \pitchfork Z$ at x. Hence (23) implies $G \pitchfork Z$ at (x,s).

The same argument shows $\partial G \pitchfork Z$.

Now we can apply the Transversality Theorem to $G\colon X \times S \to Y$ and get that we can pick and fix an s (almost every s works) for which the map

$$g(x) := G(x,s) \text{ satisfies } g \pitchfork Z \text{ and } \partial g \pitchfork Z.$$

The map G is then a homotopy

$$f = F_{|X\times\{0\}} = G_{|X\times\{0\}} \sim G_{|X\times\{s\}} = g.$$

Finally, if x belongs to the neighborhood of C on which $\tau = 0$, then we even have $g(x) = G(x,s) = F(x,0) = f(x)$. **QED**

Let us summarize what we have done today:

This lecture in a nutshell

We have proven three key results about transversality which can be roughly summarized as follows:

(a) The **Transversality Theorem** says that when a homotopy F is transversal to Z, then, in this homotopy family, **almost every** $f_s = F(-,s)$ is transversal to Z.

(b) The **Transversality Homotopy Theorem** says that given a map f and a submanifold Z, then **there exists** a map g **transversal to Z** and g is **homotopic** to f.

(c) The **Extension Theorem** says that, **given** a map f which is transversal to Z on **a subset C**, then we can always **replace** f with a homotopic map g which is **transversal to Z everywhere** (not only on C) and $f = g$ on an open set containing C.

(a) is a generalization of Sard's Theorem. For (b) and (c), the key for the proof is the **ϵ-Neighborhood Theorem**.

Appendix 1: The Inverse Function Theorem revisited

In the course of this lecture, we used a generalization of the Inverse Function Theorem that we are now going to prove. It will also allow us to show an interesting result on normal bundles and tubular neighborhoods.

As always We start with the compact case:

Generalization of the IFT - compact case

Let $f\colon X \to Y$ be a smooth map that is one-to-one on a **compact** submanifold Z of X. Suppose that for all $x \in Z$,

$$df_x\colon T_x(X) \to T_{f(x)}(Y)$$

is an isomorphism. Then f maps an open neighborhood of Z in X diffeomorphically onto an open neighborhood of $f(Z)$ in Y. If Z is a single point, this is just the usual IFT.

Proof: We know that f maps Z **diffeomorphically onto its image** $f(Z)$, since $f\colon Z \to f(Z)$ is a bijective local diffeomorphism and therefore a diffeomorphism. We would like to show that we can extend this to an open neighborhood around Z.

Since df_x is an isomorphism for all $x \in Z$, for each $x \in Z$, there exists an open neighborhood U_x in X around x on which $f_{|U_x}$ is a diffeomorphism. The collection $\{U_x\}$ is an open cover of Z. Since Z is **compact**, we can choose a **finite** subcover $\{U_1, \ldots, U_n\}$. We set $U := \cup_i U_i$. Restricted to the open subset U, $f_{|U}$ is a local diffeomorphism.

Hence, by a **previous exercise**, it suffices to show that there is some **open subset** V in X which **contains** Z such that $f_{|V}$ is injective. Then $f_{|U\cap V}$ is an injective local diffeomorphism and therefore a diffeomorphism onto its image. Since $Z \subset U$ and $Z \subset V$, we also have $Z \subset U \cap V$ and the assertion is proven.

We are going to show that V exists by **assuming the contrary**.

That means that there exists at least one point $z \in Z$ such that in any small open neighborhood W_i of z there are points a_i and b_i with

$$a_i \neq b_i, \textbf{ but } f(a_i) = f(b_i).$$

For otherwise, every point in Z had an open neighborhood on which f was injective, and we were done.

By choosing the W_i smaller and smaller around z_0 and by choosing subsequences a_j and b_j, we can assume that both the a_i and b_i converge to z. Since $f(a_i) = f(b_i)$ for all i and f is continuous, we have $f(a_i) \to f(z)$ and $f(b_i) \to f(z)$. But since df_z is an isomorphism, the usual Inverse Function Theorem implies that there is a small open neighborhood W_z in X around z such that $f_{|W_z}$ is a diffeomorphism. Since $a_i \to z$ and $b_i \to z$, for N large enough, we have $a_i, b_i \in W_z$ and hence $f(a_i) = f(b_i) \in f(W_z)$ for all $i \geq N$. But since $f_{|W_z}$ is injective, this implies $a_i = b_i$ for all $i \geq N$. This contradicts the choice of the a_i and b_i. **QED**

As it is often the case, it is the existence of partitions of unity that allows us to move from the compact to the general case. We use the technique to show the following lemma:

Local finiteness lemma

An open cover $\{V_\alpha\}$ of a manifold X is called **locally finite** if each point of X possesses a neighborhood that intersects only finitely many of the sets V_α. Any open cover $\{U_\alpha\}$ admits a locally finite refinement $\{V_\alpha\}$.

Now we are equipped to generalize the Inverse Function Theorem.

Generalization of the IFT - general case

Let $f\colon X \to Y$ be a smooth map that is one-to-one on a submanifold Z of X. Suppose that for all $x \in Z$,

$$df_x\colon T_x(X) \to T_{f(x)}(Y)$$

is an isomorphism. Assume that f maps Z diffeomorphically onto $f(Z)$. Then f maps an open neighborhood of Z in X diffeomorphically onto an open neighborhood of $f(Z)$ in Y.

Proof: Since df_x is an isomorphism for all $x \in Z$, for each $x \in Z$, there exists an open neighborhood V_x in X around x on which $f_{|V_x}$ is a diffeomorphism. Let $U_x = f(V_x)$ be the open image in Y. The collection of all U_x is an open cover of $f(Z)$, since each $f(x) \in f(Z)$ lies in some $U_x = f(V_x)$. By the lemma above, we can choose a locally finite subcover $\{U_i\}$ of $f(Z)$ in Y. For each U_i, there is a local inverse $g_i\colon U_i \to X$ of f.

We define

$$W := \{y \in U_i : g_i(y) = g_j(y) \text{ whenever } y \in U_i \cap U_j\}.$$

On the subset W, we can define an inverse

$$g\colon W \to X,\ g(y) = g_i(y) \text{ for any } i.$$

This is well-defined by construction of W, since $g(y) = g_i(y) = g_j(y)$ whenever $y \in U_i \cap U_j$. Since the g_i's are local inverses of f, we have $f(Z) \subset W$.

It remains to show that W contains an open subset which still contains $f(Z)$. Let $x \in Z$, and hence $f(x) \in f(Z)$. Then $f(x)$ lies at least one U_k. We fix one such U_k with $f(x) \in U_k$. After shrinking U_j if necessary, we can assume yy the local finiteness of the cover $\{U_i\}$, that there are only finitely many of the U_i's which intersect U_k, say $U_1, \dots, U_n$. If $U \subset W$, we are done, since then every point in $f(Z)$ has an open neighborhood which is contained in W.

If U is not contained in W, then, for $i = 1, \dots, n$, we set C_{ik} be the **closure** of the set $\{y \in U_i \cap U_k : g_i(y) \neq g_k(y)\}$. Since the union of a **finite** union of closed subsets is closed, $C_k := C_{1k} \cup \cdots \cup C_{nk}$ is closed. Hence

$$U := U_k \setminus C_k$$

is open in Y.

By definition of W and the C_k, we know $U \subset W$. It remains to make sure that we $f(x)$ is still in U, i.e. that it does not beling to one of the closures C_{ik}.

Note that $f(x)$ satisfies $g_i(f(x)) = x = g_k(f(x))$ for all $i = 1, \ldots, n$. Since df_x is an isomorphism, the usual Inverse Function Theorem implies that there is a small open neighborhood $V_\epsilon \subset U$ around x such that $f_{|V_\epsilon}$ is a diffeomorphism. Hence, for each $i = 1, \ldots, n$, we have

$$g_i(f(x')) = x' = g_k(f(x')) \text{ for all } x' \in V_\epsilon \cap g_i(U_i) \cap g_k(U_k).$$

Hence the finite intersection $f(V_\epsilon) \cap U_k \cap U_1 \cap \cdots \cap U_n$ is an open which is not contained in any of the sets $\{y \in U_i \cap U_k : g_i(y) \neq g_k(y)\}$. Thus $f(x)$ is not contained in C_k. Therefore, $U \subset W$ is an open neighorhood of $f(x)$. **QED**

Appendix 2: The Tubular Neighborhood Theorem

We are going to show a generalization of the Tubular neighborhood theorem for submanifolds. First a definition:

Normal Bundles revisited

Let $Y \subset \mathbb{R}^M$ be a boundaryless manifold, and let Z be a submanifold of Y. We define the **normal bundle to Z in Y** to be the set

$$N(Z;Y) := \{(z,v) : z \in Z, v \in T_z(Y) \text{ and } v \perp T_z(Z)\}.$$

Normal bundles are actually manifolds themselves:

Normal bundles are manifolds

One can show that $N(Z;Y)$ is itself a smooth manifold of dimension equal to $\dim Y$. Moroever, the canonical map

$$\sigma \colon N(Z;Y) \to Z, \ \sigma(z,v) = z,$$

is a **submersion**.

Proof: We showed previously that every manifold can be defined **locally** by **independent functions**. So let $\tilde{U} \subset \mathbb{R}^M$ be an open neighborhood of z and $g_1, \ldots, g_n$ be independent functions $\tilde{U} \to \mathbb{R}$ such that

$$U = Z \cap \tilde{U} = \{u \in \mathbb{R}^M : g_1(u) = \cdots = g_n(u) = 0\}$$
$$\text{and } Y \cap \tilde{U} = \{u \in \mathbb{R}^M : g_{k+1}(u) = \cdots = g_n(u) = 0\}$$

where n is the codimension of Z in $\mathbb{R}^M$ and k is the codimension of Z in Y.

Let $g = (g_1, \ldots, g_n) \colon \tilde{U} \to \mathbb{R}^n$. We observed above that the map

$$\psi \colon U \times \mathbb{R}^n \to N_U(Z;\mathbb{R}^M) := (U \times \mathbb{R}^M) \cap N(Z;\mathbb{R}^M), (u,v) \mapsto (u, dg_u^t(v))$$

is a **local parametrization** of $N(Z;\mathbb{R}^M) = N(Z)$.

By restricting ψ to elements in $U \times \mathbb{R}^k \subset U \times \mathbb{R}^n$, we get a smooth map ϕ defined as the composite

$$\begin{array}{ccc} U \times \mathbb{R}^k & \xrightarrow{\psi} & N_U(Z;\mathbb{R}^M) \\ & \searrow^{\phi} & \downarrow \mathrm{id}\times p \\ & & N_U(Z;Y) \end{array}$$

where $N_U(Z;Y) := (U \times \mathbb{R}^M) \cap N(Z;Y)$ and p is the map induced by the **orthogonal projection** $p_z \colon \mathbb{R}^M \to T_z(Y)$ at each z. Note for a vector $w \in \mathbb{R}^M$ which satisfies $w \perp T_z(Z)$, we have $p(w) \in T_z(Y)$ and $p(w) \perp T_z(Z)$. Let $\tilde{g} = (g_{k+1}, \dots, g_n) \colon \tilde{U} \to \mathbb{R}^{n-k}$. We observe that, by our choice of g and $\tilde{g}$, we know

$$T_z(Z) = (\mathrm{Ker}\, dg_z) \subset \mathrm{Ker}\,(d\tilde{g}_z) = T_z(Y)$$

and the orthogonal projection p_z varies smoothly with z.

At each $z \in U$, the dimension of the kernel of the composite

$$\mathbb{R}^n \xrightarrow{dg_z^t} N_z(Z;\mathbb{R}^M) \xrightarrow{p_z} N_z(Z;Y)$$

is

$$\dim \mathrm{Ker}\,(p_z) = \dim N_z(Z;\mathbb{R}^M) - \dim N_z(Z;Y),$$

since dg_z^t is an isomorphism. We can calculate this dimension by

$$\dim N_z(Z;\mathbb{R}^M) - \dim N_z(Z;Y) = M - \dim Z - (\dim Y - \dim Z) = n - k.$$

Thus, ϕ is a diffeomorphism being the identity on the factor and a linear isomorphism on the second factor at each point which varies smoothly with that point.

Hence $\phi \colon U \times \mathbb{R}^k \to N_U(Z;Y)$ is a local parametrization of $N(Z;Y)$. Since $N_U(Z;Y)$ is open in $N(Z;Y)$ and every point in $N(Z;Y)$ lies in such an $N_U(Z;Y)$, we conclude that $N(Z;Y)$ is a smooth manifold. Its dimension is

$$\dim N(Z;Y) = \dim U + \dim \mathbb{R}^k = \dim Z + \dim Y - \dim Z = \dim Y.$$

We note again that $\sigma \circ \phi \colon U \times \mathbb{R}^k \to U$ is just the **projection onto the first factor**, which is a submersion. Thus $d(\sigma \circ \phi)_{(u,v)}$, is surjective at every point (u,v). Hence $d\sigma_u$ is surjective at every u, and σ is a **submersion**. **QED**

Note that for any $z \in Z$, the preimage $\sigma^{-1}(z) =: N_z(Z;Y)$ is the space of **normal vectors** to Z at z in $T_z(Y)$ that we have met before.

Tubular Neighborhoods and Normal Bundles

Let $Y \subset \mathbb{R}^M$ be a boundaryless manifold, and let Z be a submanifold of Y. Then there is a diffeomorphism of an open neighborhood Z^ϵ of Z in Y to an open neighborhood $N^\epsilon(Z;Y)$ of $Z \times \{0\}$ in $N(Z;Y)$.

Proof:

Recall from the ϵ-Neighborhood Theorem the map

$$\pi\colon Y^\epsilon \to Y.$$

We consider again the map

$$h\colon N(Z;Y) \to \mathbb{R}^M,\ (z,v) \mapsto z + v.$$

By the same argument as before, we can show that $dh_{(z,v)}$ is an **isomorphism at every point of $Z \times \{0\}$** in $N(Z;Y)$.

Hence the inverse image

$$W := h^{-1}(Y^\epsilon) \subset N(Z;Y)$$

is an open neighborhood of $Z \times \{0\}$ in $N(Z;Y)$.

Since $h(z,0) = z$ for all $z \in Z$, the composition

$$W \xrightarrow{h} Y^\epsilon \xrightarrow{\pi} Y$$

is the identity when we restrict it to $Z \times \{0\}$. Hence, since $d\pi_z$ is the identity for all $z \in Z \subset Y^\epsilon$, the assumptions of the **generalized Inverse Function Theorem**, are satisfied. Thus we can conclude that there is an **open neighborhood of $Z \times \{0\}$ in $N(Z;Y)$** which is mapped diffeomorphically onto a **neighborhood of Z in Y** by $\pi \circ h$. **QED**

Crucial Point

Note that the fact that we can find an **open neighborhood** of $Z \times \{0\}$ in $N(Z;Y)$ which is diffeomorphic to an **open** neighborhood of Z in Y is **crucial**. For it is clear that Z is diffeomorphic to $Z \times \{0\}$.
To point out the difference between a submanifold Z which is **not oepn** in Y and an **open neighborhood Z^ϵ** of Z in Y, we remark the difference of dimensions:

$$\dim Z^\epsilon = \dim Y, \text{ whereas } \dim Z < \dim Y.$$

Moreover, an **open** neighborhood of $Z \times \{0\}$ is actually **diffeomorphic** to $N(Z;Y)$ as a whole, since we can extend each fiber linearly. This will turn out to be extremely useful for the Pontryagin-Thom construction later.

Let us look at an example for a normal bundle of an embedded submanifold:

An example of a normal bundle

Consider S^{k-1} as a submanifold of S^k via the usual embedding mapping

$$(x_1, \dots, x_k) \mapsto (x_1, \dots, x_k, 0).$$

The tangent space $T_p(S^{k-1})$ is embedded in $T_p(S^k)$ as the subspace consisting of vectors with last coordinate being 0.
Hence the orthogonal complement of $T_p(S^{k-1})$ in $T_p(S^k)$ is spanned by the vector with coordinates $v_k := (0, \dots, 0, 1)$ (in $T_p(S^k)$). Hence we can define a map

$$S^{k-1} \times \mathbb{R} \to N(S^{k-1}; S^k),\ (p, \lambda) \mapsto (p, \lambda v_k).$$

This map is a diffeomorphism with inverse $(p, \lambda v_k) \mapsto (p, \lambda)$.

Note that a n-dimensional vector bundle which is diffeomorphic to the product of the base space with $\mathbb{R}^n$ is called **trivial**. Hence we just showed that $N(S^{k-1}; S^k)$ is a trivial one-dimensional bundle.

We get a similar result when we consider $S^{k-1} \subset \mathbb{R}^k$ for $k \geq 2$. Then, at any $p \in S^{k-1}$, the unit vector $p/|p|$ spans the normal complement to $T_p(S^{k-1})$ in $\mathbb{R}^k$. Hence there is a diffeomorphism

$$S^{k-1} \times \mathbb{R} \to N(S^{k-1}; \mathbb{R}^k),\ (p, \lambda) \mapsto (p, \lambda p/|p|).$$

Hence $N(S^{k-1}; \mathbb{R}^k)$ is a trivial one-dimensional bundle over S^{k-1}.

However, there are a lot of **nontrivial** vector bundles as well. Important examples are the tangent bundle $T(S^2)$ over S^2 and the **universal line bundle** over $\mathbb{R}\mathrm{P}^n$.

LECTURE 20

Intersection Numerbs and Degree modulo 2

A classical geometric approach to classifying maps is to study their fibres. This appraoch is directly related to other fundamental problems in mathematics. For example, if $f\colon X \to Y$ is a map defined by an equation and given a value $y \in Y$, the set $\{x \in X : f(x) = y\}$ is the set of solutions of the equation. In geometric terms, we could rephrase the question which x solve equation f by asking how f meets or intersects the subspace $\{y\}$ in Y.

Building on the methods we have developed so far, we are going to exploit this geometric approach to derive interesting and powerful invariants. We will start with intersection numbers modulo 2. In order to define a $\mathbb{Z}$-valued invariant we will have to introduce orientations later.

Before we get to work, here is a brief summary of the previous long lecture the results of which will play a key role today:

The previous lecture in a nutshell

We proved three key results about transversality which can be roughly summarized as follows:

(a) The **Transversality Theorem** says that when a homotopy F is transversal to Z, then, in this homotopy family, **almost every** $f_s = F(-,s)$ is **transversal to Z**.

(b) The **Transversality Homotopy Theorem** says that given a map f and a submanifold Z, then **there exists** a map g **transversal to Z** and g is **homotopic** to f.

(c) The **Extension Theorem** says that, **given** a map f which is transversal to Z on **a subset C**, then we can always **replace** f with a homotopic map g which is **transversal to Z everywhere** (not only on C) and $f = g$ on an open set containing C.

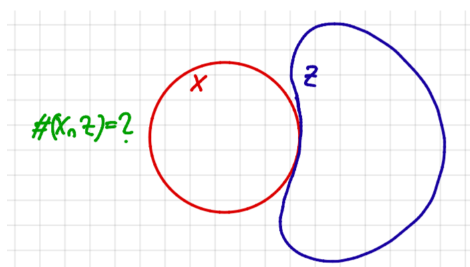
(a) is a generalization of Sard's Theorem. For (b) and (c), the key for the proof was the **ϵ-Neighborhood Theorem**.

We are going to apply these results today. First let us start with a natural situation.

Intersecting manifolds

Two submanifolds X and Z inside Y have **complementary dimension** if $\dim X + \dim Z = \dim Y$. (We assume all manfifolds are boundaryless for the moment.) If $X \pitchfork Z$, the Preimage Theorem tells us that their intersection $X \cap Z$ is manifold with $\operatorname{codim}(X \cap Z)$ in X being equal to $\operatorname{codim} Z$ in Y. Since $\operatorname{codim} Z = \dim X$, **$X \cap Z$ is a zero-dimensional manifold**.
If we further assume that both X and Z are **closed** and that at least one of them, say X, is **compact**, then $X \cap Z$ must be a **finite set of points**. We are going to think of this number of points in $X \cap Z$ as the **intersection number** of X and Z, denoted by $\#(X \cap Z)$.

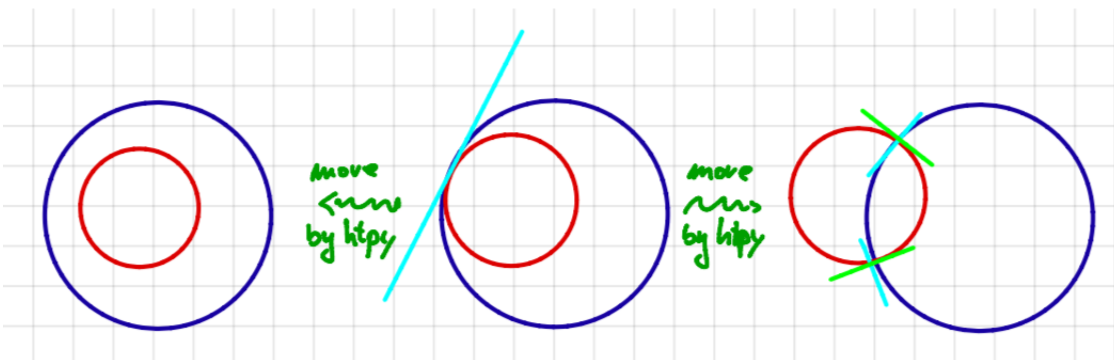
We would like to generalize the notion of intersection numbers. A first obstacle is that if X and Z do **not intersect transversally**, then it makes in general no sense to count the points in $X \cap Z$. Hence, once again, transversality is key.



Luckily, we have learned how to **move or deform** manifolds to make intersections transversal: we can alter them in **homotopic families**. And since **embeddings form a stable class** of maps, i.e. for any homotopy i_t of an embedding i_0, there is an $\epsilon > 0$ such that i_t is still an embedding for all $t < \epsilon$, any small homotopy of i gives us another embedding $X \hookrightarrow Y$ and thus produces an image manifold that is a **diffeomorphic copy** of X adjacent to the original.

But we still have to be **careful**. For the intersection number may depend on how we move or deform the manifold.

For example, take two circles in $\mathbb{R}^2$. Assume they intersect nontransversally, i.e. they touch each other in a point such that both tangent spaces agree and together just span a line. Then we can move the cricles by a simple translations $x \mapsto x + ta$ in direction a such that they either in two points or in no points. In both cases, the intersection is transversal, but the intersection numbers do not agree. But we observe that parity of the intersection numbers is preserved. i.e. up to a multiple of 2 the intersection numbers after moving into a transversal intersection agree.



This observation is the starting point for the following generalization.

Mod 2 Intersection numbers

Let X be a **compact** manifold, and let $f\colon X \to Y$ be a smooth map **transversal** to the **closed** manifold Z in Y. Assume $\dim X + \dim Z = \dim Y$. Then $f^{-1}(Z)$ is a closed submanifold of X of codimension equal to $\dim X$. Hence $f^{-1}(Z)$ is of **dimension zero**, and therefore a **finite** set. We define the **mod 2 intersection number** of the map f with Z, denoted $I_2(f,Z)$, to be **the number of points in $f^{-1}(Z)$ modulo 2**:

$$I_2(f,Z) := \#f^{-1}(Z) \mod 2.$$

For an **arbitrary** smooth map $g\colon X \to Y$, we can **choose** a map $f\colon X \to Y$ that is **homotopic to g** and **transversal** to Z by the **Transversality Homotopy Theorem**. Then we **define** $I_2(g,Z) := I_2(f,Z)$.

Of course, we need to check that the intersection number does not depend on the choice of homotopic map. The key technical result that allows us to show independence is the Extension Theorem. We did not have time to discuss the theorem and its quite technical proof in the lecture. So here is the thereom and one of its applications that will be crucial for us.

The **Extension Theorem** says the following: Let $f\colon X \to Y$ be a smooth map, Y boundaryless, and Z a closed submanifold of Y without boundary. Let C be a closed subset of X. Assume that $f \pitchfork Z$ on C and $\partial f \pitchfork Z$ on $C \cap \partial X$.

Then there exists a smooth map $g\colon X \to Y$ **homotopic to** f, such that $g \pitchfork Z$ and $\partial g \pitchfork Z$, and on a **neighborhood of** C **we have** $g = f$.

We apply this result in the situation we were discussing for intersection numbers, i.e. X, Y and $Z \subset Y$ are boundaryless manifolds. The product $X \times [0,1]$ is then a manifold with boundary. We let C be the boundary of $X \times [0,1]$, i.e. C is the closed subset

$$C := \partial(X \times [0,1]) = X \times \{0\} \cup X \times \{1\}.$$

Now we apply the theorem to the case of a smooth homotopy

$$F\colon X \times [0,1] \to Y.$$

Then ∂F, i.e. F restricted to the boundary of $X \times [0,1]$, is given by the two maps

$$f_0 = F(-,0)\colon X \to Y \text{ and } f_1 = F(-,1)\colon X \to Y.$$

The two conditions $F \pitchfork Z$ on C and $\partial F \pitchfork C$ on $C \cap \partial X$ are thus equivalent, and mean $f_0 \pitchfork Z$ and $f_1 \pitchfork Z$.

Hence, assuming $f_0 \pitchfork Z$ and $f_1 \pitchfork Z$, the Extension Theorem says that there is a smooth map

$$G\colon X \times [0,1] \to Y \text{ with } \mathbf{G \pitchfork Z} \text{ and } \partial G \pitchfork Z,$$

and $G = F$ on a neighborhood of C. The latter means that

$$G \text{ is still a homotopy from } f_0 = G(-,0) \text{ to } f_1 = G(-,1).$$

Mod 2 Intersection Numbers are well-defined

If $f_0\colon X \to Y$ and $f_1\colon X \to Y$ are homotopic and both transversal to Z, then $I_2(f_0,Z) = I_2(f_1,Z)$.

Proof: Let $F\colon X \times I \to Y$ be a homotopy of f_0 and f_1. By the above discussion, we may assume that $F \pitchfork Z$. By the Preimage Theorem with boundary, this implies $F^{-1}(Z)$ is a submanifold of $X \times [0,1]$ such that

$$\operatorname{codim} F^{-1}(Z) \text{ in } X \times [0,1] = \operatorname{codim} Z \text{ in } Y.$$

Hence

$$\begin{aligned} \dim F^{-1}(Z) &= \dim(X \times [0,1]) + \dim Z - \dim Y \\ &= \dim X + 1 + \dim Z - \dim Y \\ &= 1 \end{aligned}$$

since we assume that $\dim X + \dim Z = \dim Y$.

Moreover, the boundary of $F^{-1}(Z)$ is

$$\partial F^{-1}(Z) = F^{-1}(Z) \cap \partial(X \times [0,1]) = f_0^{-1}(Z) \times \{0\} \cup f_1^{-1}(Z) \times \{1\}.$$

Since X is compact, $F^{-1}(Z)$ is **compact**. Hence the **classification of compact one-manifolds** implies that $\partial F^{-1}(Z)$ must have an **even** number of points. Thus

$$I_2(f_0,Z) = \# f_0^{-1}(Z) = \# f_1^{-1}(Z) = I_2(f_1,Z) \quad \text{mod } \mathbf{2}.$$

QED

We can generalize this a bit further.

All homotopic maps have equal Intersection Numbers

If $g_0 \colon X \to Y$ and $g_1 \colon X \to Y$ are arbitrary homotopic maps, then $I_2(g_0,Z) = I_2(g_1,Z)$.

Proof: As before, we can choose maps $f_0 \pitchfork Z$ and $f_1 \pitchfork Z$ such that $g_0 \sim f_0$, $I_2(g_0,Z) = I_2(f_0,Z)$, and $g_1 \sim f_1$, $I_2(g_1,Z) = I_2(f_1,Z)$. Since homotopy is a **transitive** relation (we showed that it is, in fact, an equivalence relation), we have

$$f_0 \sim g_0 \sim g_1 \sim f_1, \text{ and hence } f_0 \sim f_1.$$

By the previous theorem, this implies

$$I_2(g_0,Z) = I_2(f_0,Z) = I_2(f_1,Z) = I_2(g_1,Z).$$

QED

Now that we have a solid notion of intersection numbers modulo 2 for maps and submanifolds, let us return to situation we started with.

mod 2 Intersction Numbers of submanifolds

Assume X is a compact submanifold of Y and Z a closed submanifold of Y. Assume the **dimensions are complementary**, i.e. $\dim X + \dim Z = \dim Y$. Then we can define the **mod 2 intersection number of X with Z**, denoted by $I_2(X,Z)$, by

$$I_2(X,Z) := I_2(i,Z)$$

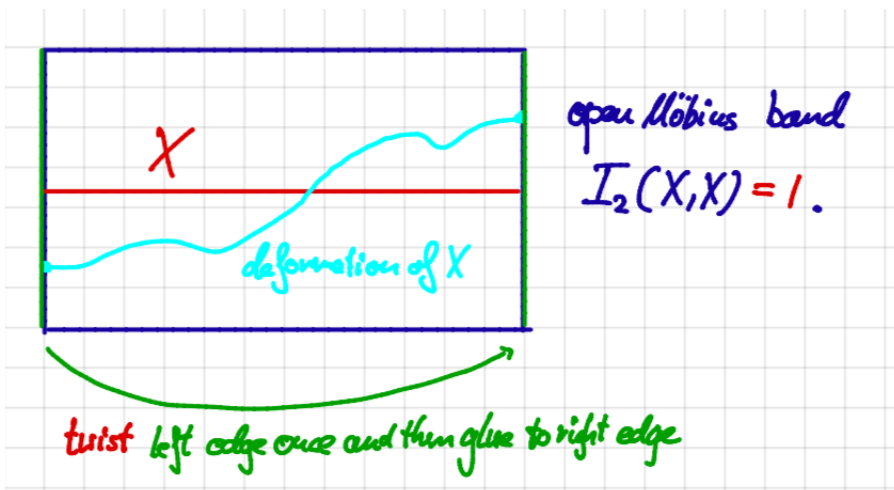
where $i \colon X \hookrightarrow Y$ is the inclusion.
Note that when $X \pitchfork Z$, then $I_2(X,Z) = \#(X \cap Z)$. In general, we have to move or deform X into a **transversal position**.

Some particular situations:

- If $I_2(X,Z) \neq 0$, then no matter how X is moved or deformed, it cannot be pulled entirely away from Z.

  For example, on the torus $Y = S^1 \times S^1$, the two circles $S^1 \times \{1\}$ and $\{1\} \times S^1$ have complimentary dimensions and nonzero mod 2 intersection number.
- If $\dim X = 2 \dim Y$, for then we may consider $I_2(X,X)$ as the **mod 2 self-intersection number** of X.

  An illustrative example is the central curve on the open Möbius band (see Exercise Set 9). Check that $I_2(X,X) = 1$.



- If X happens to be the **boundary** of some W in Y, then $I_2(X,Z) = 0$. For if $Z \pitchfork X$, then, roughly speaking, Z must "pass out" of W as often as it "enters". Hence $\#(X \cap Z)$ is **even**.

The latter case can be made rigorous as follows:

Boundary Theorem

Suppose that X is the **boundary** of some **compact** manifold W and $g\colon X \to Y$ is a smooth map. **If** g can be **extended** to all of W, **then** $I_2(g,Z) = 0$ for any closed submanifold Z in Y of **complementary dimension**, i.e. $\dim X + \dim Z = \dim Y$.

Proof: Let $G\colon W \to Y$ be an extension of g, i.e. $\partial G = g$. From the **Transversality Homotopy Theorem**, we obtain a map $F\colon W \to Y$ **homotopic** to G with $F \pitchfork Z$ and $\partial F \pitchfork Z$. We write $f := \partial F$. Then $f \sim g$ and hence

$$I_2(g,Z) = I_2(f,Z) = \#f^{-1}(Z) \mod 2.$$

Now $F^{-1}(Z)$ is a compact submanifold whose codimension in W is the same as the codimension of Z in Y. Here we use again that X is the boundary of W, for this implies $\dim W = \dim \partial W + 1 = \dim X + 1$, and hence

$$\dim F^{-1}(Z) = \dim X + 1 - \dim Y + \dim Z = 1.$$

Hence $F^{-1}(Z)$ is a **compact one-dimensional manifold with boundary**, so

$$\#\partial(F^{-1}(Z)) = \#(\partial F)^{-1}(Z) = \#f^{-1}(Z) \text{ is } \textbf{even}.$$

QED

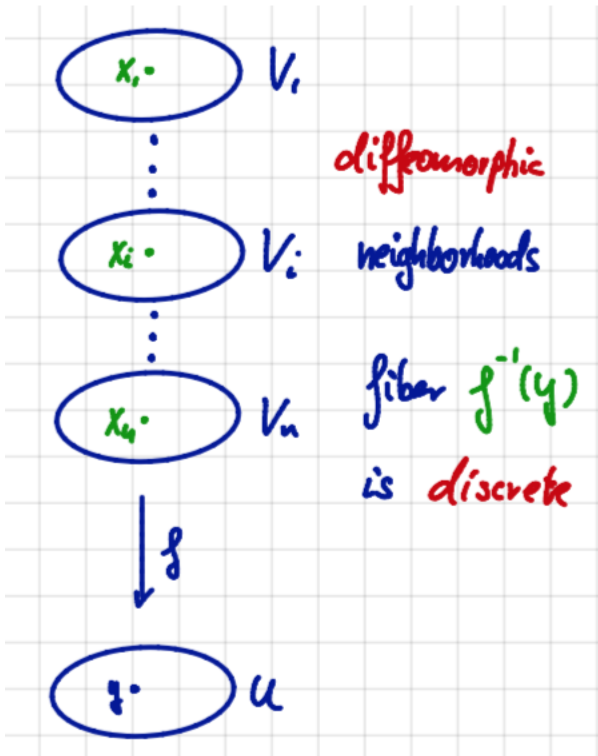
Intersection theory gives us an interesting **homotopy invariant** attached to maps between manifolds of the **same dimension**. The definition depends on the following fact.

The Degree mod 2

If $f\colon X \to Y$ is a smooth map of a **compact** manifold X into a **connected** manifold Y and $\dim X = \dim Y$, then $I_2(f,\{y\})$ is the **same for all points** $y \in Y$. This common value is called the **mod 2 degree of f**, denoted $\deg_2(f)$.

Note: The degree mod 2 is defined only when the range manifold Y is **connected**, the domain X is **compact**, and $\dim X = \dim Y$. Whenever we write $\deg_2$, we assume that these assumptions are satisfied.

Proof: Given any $y \in Y$, we can assume that f is **transversal to** $\{y\}$. For otherwise we can replace it with a homotopic map which is transversal by the **Transversality Homotopy Theorem**. Now by the **Stack of Records Theorem**, we can find a neighborhood U of y such that the preimage $f^{-1}(U)$ is a disjoint union $V_1 \cup \cdots \cup V_n$, where each V_i is an open set in X mapped by f diffeomorphically onto U:



Hence, for all points $z \in U$, we have

$$I_2(f,\{z\}) = \#f^{-1}(\{z\}) = n \quad \text{mod } 2.$$

Consequently, the function

$$Y \to \mathbb{Z}/2,\ y \mapsto I_2(f,\{y\})$$

is **locally constant**. Since Y is **connected**, it must be **globally constant**. **QED**

Since $\deg_2$ is defined as an intersection number, we immediately obtain the following theorems.

$\deg_2$ is a homotopy invariant

Homotopic maps have the same mod 2 degree, i.e.

$$f_0 \sim f_1 \Rightarrow \deg_2(f_0) = \deg_2(f_1).$$

Proof: If $f_0 \sim f_1$, then for every $y \in Y$:

$$\deg_2(f_0) = I_2(f_0,\{y\}) = I_2(f_1,\{y\}) = \deg_2(f_1).$$

QED

Extensions of maps on boundaries have $\deg_2$ equal zero

If $X = \partial W$ for some compact manifold W, and if $f \colon X \to Y$ can be **extended** to all of W, then $\deg_2(f) = 0$.

Note that when W is compact, then the closed subset $X = \partial W$ is also compact. Hence $\deg_2(f)$ is defined.

Proof: This is the Boundary Theorem applied to the zero-dimensional submanifold $\{y\}$ for any $y \in Y$. **QED**

This has an interesting immediate consequence:

Obstruction for extending maps

Let W be a compact manifold, and $f \colon \partial W \to Y$ a smooth map. **If** $\deg_2(f) \neq 0$, then f **cannot be extended** to a smooth map $W \to Y$ on all of W.

Now that we have the invariant $\deg_2$, there are upsides and downsides equipped to $\deg_2$:

The **good news** is that $\deg_2(f)$ is **easy to calculate**: just pick any regular value y for f and count preimage points

$$\deg_2(f) = \# f^{-1}(y) \mod 2.$$

The **bad news** is that its **power is limited**. For example, the map

$$\mathbb{C} \to \mathbb{C},\ z \mapsto z^n,$$

which wraps the circle S^1 smoothly around S^1 n times, has mod 2 degree zero if n is even, and one if n is odd. Hence $\deg_2$ **cannot distinguish** between many different maps, for example $\deg_2$ of the constant map $S^1 \to S^1$ is equal to $\deg_2$ of the map $S^1 \to S^1$ sending $z \mapsto z^2$.

We will remedy this defect soon, when we define intersection numbers and degree functions which have values in $\mathbb{Z}$. This will lead us to the notion of orientation. the idea is that, for example in the case of intersection with a boundary, we need to distinguish between points where a map “goes in” and points where it “goes out”.

Nevertheless, there are some nice and powerful applications of $\deg_2$.

Application: Existence of zeros for complex valued functions.

Suppose that $p\colon \mathbb{C} \to \mathbb{C}$ is a smooth (as a map $\mathbb{R}^2 \to \mathbb{R}^2$), complex function and $W \subset \mathbb{C}$ is a smooth compact region in the plane, i.e. a **two-dimensional compact manifold with boundary**.

Question: Is there a $z \in W$ with $p(z) = 0$?

Assume that p has **no zeros on the boundary** ∂W. Then

$$\frac{p}{|p|}\colon \partial W \to S^1$$

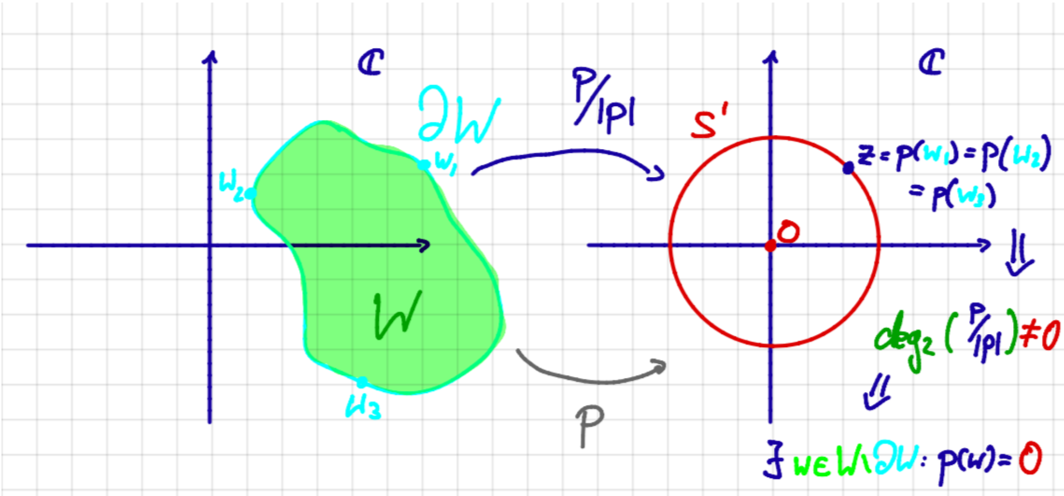
is defined and smooth as a map of **compact one-manifolds**.

Now **if p has no zeros inside W**, then $\frac{p}{|p|}$ is defined on all of W, i.e. $\frac{p}{|p|}\colon \partial W \to \mathbb{C}$ can be **extended** to a smooth map $W \to \mathbb{C}$. If this is the case, we just learned that we must have $\deg_2(\frac{p}{|p|}) = 0$. In other words:

Existence of zeros via $\deg_2$

If the mod 2 degree of $\frac{p}{|p|}\colon \partial W \to S^1$ is **nonzero**, then the function p has a **zero inside** W.

Note that calculating $\deg_2(\frac{p}{|p|})$ simply consists of picking a point $z \in S^1$, we could think of it as a direction, and just counting the number of times we find a $w \in \partial W$ with $p(w) = z$, i.e. how often $p(w)$ points in the chosen direction. The theorem tells us that **this simple procedure can tell us whether p has a zero inside W**. (If you have learned about Complex Analysis, then this should remind you of the Residue Theorem and Cauchy's formula.)



Application: Fundamental Theorem of Algebra in **odd** degrees.

The condition on the degree arises from the **defect of** $\deg_2$ that it cannot distinguish different even numbers. Since we have already seen Milnor's proof of the Fundamental Theorem of Algebra, this is in principle an old story for us. But since we have already done the hard work, so let us have a look at it anyway.

Let

$$p(z) = z^m + a_1 z^{m-1} + \cdots + a_m$$

be a monic complex polynomial. We can define a homotopy from $p_0(z) = z^m$ to $p_1(z) = p(z)$ by

$$p_t(z) = tp(z) + (1-t)z^m = z^m + t(a_1 z^{m-1} + \cdots + a_m).$$

For large z, consider

$$\frac{p(z)}{z^m} = 1 + (\frac{a_1}{z} + \cdots + \frac{a_m}{z^m}).$$

As $z \to \infty$, the term $\frac{a_1}{z} + \cdots + \frac{a_m}{z^m} \to 0$. Hence, if W is a closed ball around the origin in $\mathbb{C}$ with sufficiently large radius, none of the p_t has a zero on ∂W.

Thus the homotopy

$$\frac{p_t}{|p_t|} : \partial W \to S^1$$

is defined for all $t \in [0,1]$. Thus

$$\deg_2\left(\frac{p}{|p|}\right) = \deg_2\left(\frac{p_0}{|p_0|}\right).$$

Since $p_0(z) = z^m$ and $\#\{z \in \partial W : z^m = 1\}$ for closed ball $W \subset \mathbb{C}$ around 0, we have

$$\deg_2\left(\frac{p_0}{|p_0|}\right) = m \mod 2.$$

Hence **if** m is **odd**, then $\deg_2\left(\frac{p}{|p|}\right) \neq 0$, and there must be $w \in W$ with $p(w) = 0$ by the previous result.

More examples: Intersections in projective space

Remember real projective n-space $\mathbb{R}\mathrm{P}^n$ which consists of the set of equivalence classes $[x_0 : \ldots : x_n]$ of $n+1$-tuples of real numbers whith the equivalence relation

$$(x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n) \text{ for } \lambda \in \mathbb{R} \setminus \{0\}.$$

Recall that we showed that $\mathbb{R}\mathrm{P}^2$ is a two-dimensional smooth manifold.

Consider the two embeddings of the unit circle into $\mathbb{R}\mathrm{P}^2$:

$$\iota_1 \colon S^1 \to \mathbb{R}\mathrm{P}^2,\ (\cos(2\pi t), \sin(2\pi t)) \mapsto [\cos(2\pi t) : \sin(2\pi t)) : 0]$$

and

$$\iota_2 \colon S^1 \to \mathbb{R}\mathrm{P}^2,\ (\cos(2\pi t), \sin(2\pi t)) \mapsto [0 : \sin(2\pi t)) : \cos(2\pi t)].$$

As an exercise, check that ι_1 and ι_2 actually are embeddings.

The images of ι_1 and ι_2 meet in the point

$$[0 : 1 : 0] \in \iota_1(S^1) \cap \iota_2(S^1).$$

Note that in $\mathbb{R}\mathrm{P}^2$, there is **exactly one** intersection point. For (0,1,0) and (0, − 1,0) represent the same point in $\mathbb{R}\mathrm{P}^2$.

Moreover, we can check that the intersection in $[0 : 1 : 0]$ is transversal. Thus the mod 2 intersection number satisfies

$$I_2(\iota_1(S^1), \iota_2(S^1)) = 1.$$

This implies that it is **impossible** to move $\iota_1(S^1)$ and $\iota_2(S^1)$ within $\mathbb{R}\mathrm{P}^2$ such that **they do not meet**.

However, in the Euclidean plane $\mathbb{R}^2$ it is very well possible to move two circles such that they do not meet. The mod 2 intersection number of two circles in $\mathbb{R}^2$ is 0 for a transversal intersection either consists of exactly two points or is empty.

This is an example of a phenomenon which motivates the introduction of projective spaces:

A plane V in $\mathbb{R}^3$ can be described as the orthogonal complement of given vector $v \neq 0$ in $\mathbb{R}^3$:

$$V = \{(x_0,x_1,x_2) \in \mathbb{R}^3 : x_0v_0 + x_1v_1 + x_2v_2 = 0\}.$$

Since multiplying the equation $x_0v_0 + x_1v_1 + x_2v_2 = 0$ with a nonzero real number does not change the set of solutions, we can consider the equivalence classes $\mathbb{R}\mathrm{P}^2$ of the points of V. This gives us a line L in $\mathbb{R}\mathrm{P}^2$:

$$L = \{[x_0 : x_1 : x_2] \in \mathbb{R}\mathrm{P}^2 : x_0v_0 + x_1v_1 + x_2v_2 = 0\}.$$

In fact, every line in $\mathbb{R}$P2 is represented by a plane through the origin in $\mathbb{R}^3$ and is hence determined by a nonzero vector $v$in $\mathbb{R}^3$.

Now given two distinct lines L_1 and L_2 in $\mathbb{R}\mathrm{P}^2$ determined by two distinct vectors $v,w \neq 0$ in $\mathbb{R}^3$, i.e.,

$$L_1 = \{[x_0 : x_1 : x_2] \in \mathbb{R}\mathrm{P}^2 : x_0v_0 + x_1v_1 + x_2v_2 = 0\}$$
$$L_2 = \{[x_0 : x_1 : x_2] \in \mathbb{R}\mathrm{P}^2 : x_0w_0 + x_1w_1 + x_2w_2 = 0\}.$$

The orthogonal complements of v and w, respectively, are two planes through the origin. Hence they meet in a line through the origin in $\mathbb{R}^3$ which is the set of solutions of the two linear equations defining $L1$ and L_2 above. This is a one-dimensional vector subspace of $\mathbb{R}^3$ (the kernel of a 2×3-matrix). By definition of $\mathbb{R}\mathrm{P}^2$, this line corresponds to a point in $\mathbb{R}\mathrm{P}^2$. This is the intersection point of L_1 and L_2 in $\mathbb{R}\mathrm{P}^2$.

If this line happens to be the z-axis, i.e., when L_1 and L_2 are represented by the planes given by the xz-plane and the yz-plane, then the intersection point is $[0 : 0 : 1] \in \mathbb{R}\mathrm{P}^2$. We can think of it as the **point at infinity in $\mathbb{R}\mathrm{P}^2$**.

However, in the Euclidean plane $\mathbb{R}^2$ it may very well happen that two lines are parallel and hence do not intersect. The idea for $\mathbb{R}\mathrm{P}^2$ is to add a point at infinity which is the **intersection point for all parallel lines**.

LECTURE 21

Winding Numbers and the Borsuk-Ulam Theorem

Today we are going to exploit intersection numbers and degree modulo 2 a bit further and prove a famous theorem. As a starter, we introduce a useful new invariant.

Let X be a **compact**, **connected** smooth manifold, and let

$$f\colon X \to \mathbb{R}^n$$

be a smooth map. We assume $\dim X = n - 1$.

Let z be a point of $\mathbb{R}^n$ **not** lying in the image $f(X)$. We would like to understand how $f(x)$ **winds around** z. To do this, we look at the unit vector

$$u(x) = \frac{f(x) - z}{|f(x) - z|}.$$

It points in the direction from z to $f(x)$ and has length one.

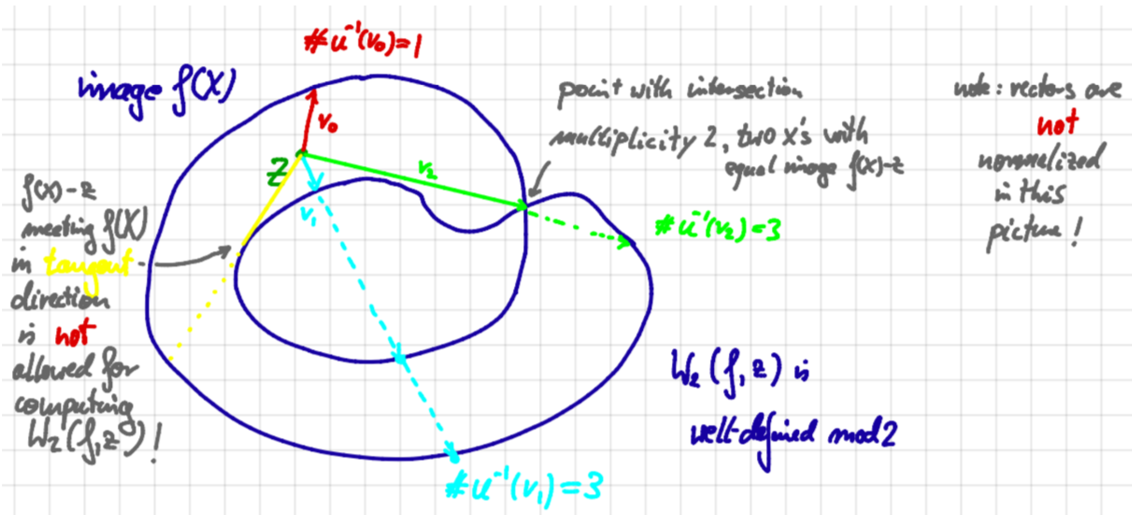
With z **fixed** and x varying, we can consider u as a map

$$u\colon X \to S^{n-1}.$$

We would like to know how often this vector points in a given direction, i.e. how often $u(x)$ has a given value. We learned from the previous lecture, that the degree of u is an invariant that encodes this information. For, we know that, modulo 2, $\#u^{-1}(y)$ is constant for **regular values** y of u, i.e. where $y - z$ hits $f(X)$ transversally, and is equal $\deg_2(u)$ by definition of the latter. (We will see in the proof of our main theorem today, that y being a regular value of u means that the line through z and y must be tranversal to $f(X)$.)

We give this number a name and call it the **winding number of f around z**. We denote it by

$$W_2(f,z) := \deg_2(u).$$



The goal for today is to prove the following famous result:

Borsuk-Ulam Theorem

Let $f\colon S^k \to \mathbb{R}^{k+1} \setminus \{0\}$ be a smooth map, and suppose that f is odd, i.e. satisfies the symmetry condition

$$f(-x) = -f(x) \text{ for all } x \in S^k. \tag{24}$$

Then $W_2(f,0) = 1$.
In other words, any map that is odd, i.e. symmetric around the origin, must wind around the origin an **odd number of times**.

A we will see below, there is a nice interpretation of this result for the meteorologists among us: At any given time, there are **two antipodal points** on the **Earth** that have the **same temperature and pressure**. (Assuming temperature and pressure vary smoothly on the Earth.)

Before we approach the proof, we observe:

Equivalent formulation of BUT

The Borsuk-Ulam theorem is equivalent to the following assertion:
If $f\colon S^k \to S^k$ is a map which sends antipodal points to antipodal points, i.e. $f(-x) = -f(x)$, then $\deg_2(f) = 1$.

Proof: Assume BUT is true: given a smooth map $f\colon S^k \to S^k$ with $f(-x) = -f(x)$, we can consider it as a map $f\colon S^k \to S^k \subset \mathbb{R}^{k+1}$. Then we have $1 = W_2(f,0) = \deg_2(f/|f|) = \deg_2(f)$.

Assume the assertion is true: given a smooth map $f\colon S^k \to \mathbb{R}^{k+1} \setminus \{0\}$ with $f(-x) = -f(x)$, then $f/|f|$ is a well-defined smooth map $f/|f|\colon S^k \to S^k$. Hence $1 = \deg_2(f/|f|) = W_2(f,0)$ by definition of winding number. **QED**

As a slogan, we can remember the Borsuk-Ulam Theorem for a smooth map $f\colon S^k \to S^k$ as follows:

BUT in a nutshell

If f is odd, its degree is odd.

In order to prove the theorem, we first need to investigate the relationship of winding numbers and boundaries:

Winding numbers and boundaries

Suppose that X is the **boundary** ∂D of a compact manifold D of dimension n with boundary, and let $F\colon D \to \mathbb{R}^n$ be a smooth map extending $f\colon X \to \mathbb{R}^n$, i.e. $\partial F = f$. Suppose that z is a **regular value** of F that does **not** belong to the image of f.
Then $F^{-1}(z)$ is a **finite set**, and

$$W_2(f,z) = \#F^{-1}(z) \mod 2.$$

In other words, f winds X around z as often as F hits z, at least modulo 2.

Proof:

First case: $F^{-1}(z) = \emptyset$, i.e. $\#F^{-1}(z) = 0$.

In this case, the map

$$u\colon X = \partial D \to S^{n-1},\ x \mapsto \frac{f(x) - z}{|f(x) - z|}$$

can be extended to a map

$$D \to S^{n-1},\ x \mapsto \frac{F(x) - z}{|F(x) - z|}$$

since $F(x) - z$ is never 0. Hence by the **Boundary Theorem**,

$$W_2(f,z) = \deg_2(u) = 0 \mod 2.$$

Second case: $F^{-1}(z) \neq \emptyset$.

Since D is **compact** and of dimension n, $F^{-1}(z)$ is a zero-dimensional closed submanifold of D, and hence compact and hence a **finite set**. Suppose

$$F^{-1}(z) = \{y_1, \ldots, y_m\}.$$

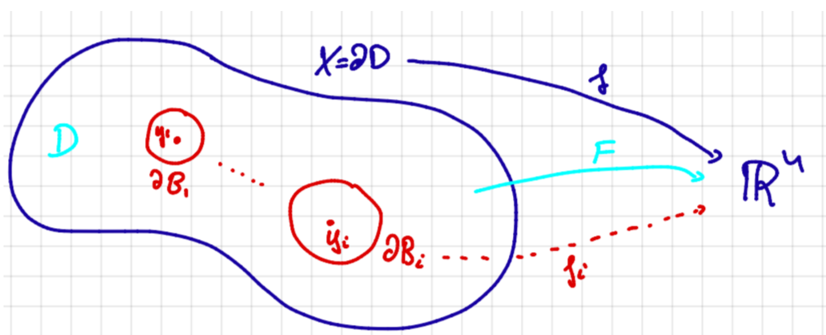
Then we can choose local parametrizations around each y_i in D and let B_i be the image of a closed ball in $\mathbb{R}^n$ around y_i. Since z is a **regular value**, the **Stack of Records Theorem** shows that $F^{-1}(z)$ is discrete and disjoint to $X = \partial D$. Thus we can choose the radii of these balls small enough such that

$$B_i \cap B_j = \emptyset \text{ and } B_i \cap X = \emptyset \text{ for all } i \neq j, \text{ and } i = 1, \ldots, m.$$

We define

$$f_i := F_{|\partial B_i} \colon \partial B_i \to \mathbb{R}^n.$$

to be the restriction of F to ∂B_i.



Now we observe that the subset

$$\tilde{D} := D \setminus (\cup_i \mathrm{Int}(B_i))$$

is a closed submanifold of D with boundary

$$\partial\tilde{D} = \partial D \,\dot{\cup}\, \partial B_1 \dot{\cup} \cdots \dot{\cup}\, \partial B_m$$

the disjoint union of the boundaries of D and the B_i's.

By the choice of the B_i's, we have $F^{-1}(z) \cap \tilde{D} = \emptyset$. Hence

$$F^{-1}(z) \cap \tilde{D} = (F_{|\tilde{D}})^{-1}(z) = \emptyset.$$

Hence the **winding number of $\partial F_{|\tilde{D}}$ at z is zero**.

Since degrees and hence winding numbers are **additive with respect to connected components** this yields

$$0 = W_2(\partial F_{|\tilde{D}}, z) = W_2(f,z) + W(f_1,z) + \cdots + W_2(f_m,z) \mod 2.$$

Since we are working **modulo** 2, this implies

$$W_2(f,z) = W(f_1,z) + \cdots + W_2(f_m,z) \mod 2.$$

Now it **remains to show** $W_2(f_i,z) = 1$ for each $i = 1, \ldots, m$. For then

$$\#F^{-1}(z) = m = \sum_i W_2(f_i,z) = W(f,z) \mod 2.$$

Since z is a **regular value**, dF_{y_i} is an isomorphism (remember $\dim D = n$). Thus, by the **Inverse Function Theorem**, we can choose the radius of B_i small enough such that $F_{|B_i}$ is a **diffeomorphism onto its image** (which contains z). By continuity, this implies also that $f_i = \partial F_{|B_i}$ is one-to-one onto the boundary of $F(B_i)$.

By possibly rescaling and translating, we are **reduced to showing**:

Let B be the closed unit ball in $\mathbb{R}^n$ and $F \colon B \to B$ be a diffeomorphism. Let $f = \partial F \colon S^{n-1} \to S^{n-1}$. Then

$$\#F^{-1}(0) = W(f,0) = 1 \mod 2.$$

But this is obvious, since $W(f,0) = \deg_2(f) = \#f^{-1}(v) = 1$ for any $v \in S^{n-1}$. **QED**

Now we are ready to attack the proof of BUT.

Proof of the Borsuk-Ulam Theorem: The proof is by induction.

The case $k = 1$:

By the previous remark, to show that theorem is equivalent to showing that a map $f \colon S^1 \to S^1$ with $f(-x) = -f(x)$ has $\deg_2(f) = 1$.

The **idea** is that, given any smooth map $f \colon S^1 \to S^1$, we can **lift f locally** using the Stack of Records Theorem and then **patch the pieces together** to get a smooth map

$$g \colon \mathbb{R} \to \mathbb{R} \text{ such that } p(g(t)) = f(p(t))$$

where p is the (covering) map

$$p \colon \mathbb{R} \to S^1,\ t \mapsto e^{2\pi i t}.$$

To make g compatible with f in the above sense, we must have

$$p(g(t+1)) = f(p(t+1)) = f(p(t)) = p(g(t)) \Rightarrow p(g(t+1) - g(t) = 1.$$

Since $p(t) = 1$ if and only if $t \in \mathbb{Z}$, we must have $g(t+1) - g(t) \in \mathbb{Z}$. Since the function $t \mapsto g(t+1) - g(t)$ takes only values in the discrete space $\mathbb{Z}$, it is **locally constant**. Since $\mathbb{R}$ is **connected**, it must be **constant**. Hence q is a **fixed integer** depending only on f. In other words, for all $t \in \mathbb{R}$, we have

$$g(t+1) = g(t) + q \text{ for some } \textbf{fixed } q \in \mathbb{Z}.$$

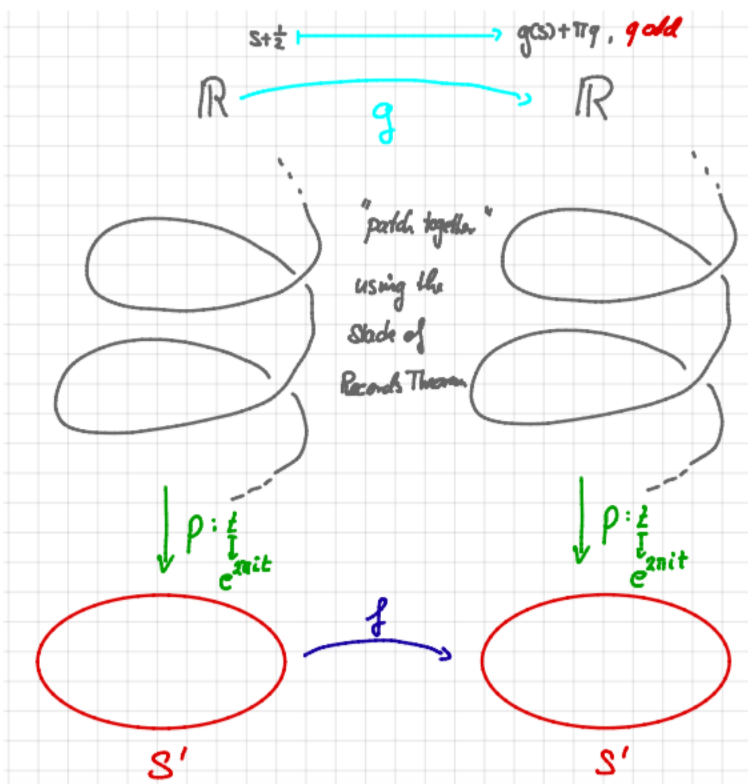
Then we have $\deg_2(f) = q$, since q tells us **how often** f hits the same point when t moves from 0 to 1, or vary t around S^1 once.

When f **is odd**, then

$$\begin{aligned} p(g(t+1/2)) &= f(p(t+1/2)) = f(-p(t)) = -f(p(t)) \\ &= -p(g(t)) = p(g(t) + q/2) \text{ for some fixed } \textbf{odd } q \in \mathbb{Z}. \end{aligned}$$

(For $p(s_1) = -p(s_2) \iff e^{2\pi i s_1} = -e^{2\pi i s_2} = e^{2\pi i s_2} e^{q\pi i}$ for some odd $q \in \mathbb{Z}$, and hence $p(s_1) = -p(s_2) \iff s_1 = s_2 + q/2$ for this odd q.)

Hence $\deg_2(f) = q = 1 \mod 2$.



Aside: There is a deeper general reason why this works. For $\mathbb{R}$ is a **(universal) covering space** of S^1, and continuous paths can always be lifted to a cavering space. You will learn more about this phenomenon later.

Induction step: Assume the theorem is true for $k-1$ and $k \geq 2$. Let $f\colon S^k \to \mathbb{R}^{k+1} \setminus \{0\}$ satisfy the symmetry condition (24). We consider $^{k-1}$ to be the equator of S^k, embedded by

$$(x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_k, 0).$$

The idea is to compute $W_2(f,0)$ by **counting how often f intersects a line L** in $\mathbb{R}^{k+1}$. By choosing L disjoint from the image of the equator, we can use the inductive hypothesis to show that the equator winds around L an odd number of times. Finally, it is easy to calculate the intersection of f with L once we know the behavior of f on the equator.

Let $g\colon S^{k-1} \to \mathbb{R}^{k+1} \setminus \{0\}$ be the restriction of f to the equator. By Sard's Theorem, we can choose a value $y \in S^k$ which is **regular for both** smooth maps

$$\frac{g}{|g|}\colon S^{k-1} \to S^k, \text{ and } \frac{f}{|f|}\colon S^k \to S^k.$$

The **symmetry condition** implies that **y is regular for both** these maps if and only if **$-y$ is regular for both** maps , since the derivatives at preimages of y and $-y$ just differ by multpying with (-1).

Since $\dim S^{k-1} < \dim S^k$, the only way y can be a **regular value of $\frac{g}{|g|}$** is when y is **not** in the image. Hence neither y nor $-y$ are in the image of $\frac{g}{|g|}$.

Thus, for the **line** $L := \mathbb{R} \cdot y = \operatorname{span}(y)$, we have

$$y \text{ is a regular value of } g \iff \operatorname{Im}(g) \cap L = \emptyset.$$

That y is **regular for $\frac{f}{|f|}$** means by definition

$$\operatorname{Im}\left(d\left(\frac{f}{|f|}\right)_x\right) = T_y(S^k).$$

The tangent space to S^k at y is the orthogonal complement of the line pointing in direction of y. The map $x \mapsto \frac{f(x)}{|f(x)|}$ is the composite of f and $x \mapsto x/|x|$ (which is smooth in dimensions $k \geq 2$).

The derivative of the latter map satisfies

$$\operatorname{Im}(d(x/|x|)_x) = (\operatorname{span}(x))^{\perp} \subset \mathbb{R}^{k+1}, \text{ i.e. } \operatorname{Ker}(d(x/|x|)_x) = Span(x).$$

For $f/|f|$, this means

$$\operatorname{Ker}\left(d\left(\frac{f}{|f|}\right)_x\right) = \operatorname{span}(f(x)) \cap \operatorname{Im}(df_x).$$

Thus

$$\begin{aligned}\operatorname{Im}\left(d\left(\frac{f}{|f|}\right)_x\right) = T_y(S^k) &\iff \operatorname{Ker}\left(d\left(\frac{f}{|f|}\right)_x\right) = \{0\} \\ &\iff \operatorname{span}(f(x)) \cap \operatorname{Im}(df_x) = \{0\} \\ &\iff \operatorname{span}(f(x)) \not\subset \operatorname{Im}(df_x) \\ &\iff L + \operatorname{Im}(df_x) = \mathbb{R}^{k+1} \\ &\iff f \pitchfork L.\end{aligned}$$

Summarzing the argument, we have obtained

$$y \text{ is a regular value of } \frac{f}{|f|} \iff f \pitchfork L. \tag{25}$$

Now we are going to exploit these two observations for calculating $W_2(f,0)$. By definition, we have

$$W_2(f,0) = \deg_2\left(\frac{f-0}{|f-0|}\right) = \deg_2\left(\frac{f}{|f|}\right) = \#\left(\frac{f}{|f|}\right)^{-1}(y) \mod 2.$$

By **symmetry**, we have

$$\#\left(\frac{f}{|f|}\right)^{-1}(y) = \#\left(\frac{f}{|f|}\right)^{-1}(-y).$$

From (25) we know

$$\begin{aligned}f^{-1}(L) &= \{x \in S^k : f(x) \in L\} \\ &= \{x \in S^k : \frac{f(x)}{|f(x)|} = \pm y\} \\ &= \left(\frac{f}{|f|}\right)^{-1}(y) \cup \left(\frac{f}{|f|}\right)^{-1}(-y).\end{aligned}$$

Thus

$$\#\left(\frac{f}{|f|}\right)^{-1}(y) = \frac{1}{2}\#f^{-1}(L).$$

Hence we need to **calculate the number** $\frac{1}{2}\#f^{-1}(L)$, at least modulo 2.

By **symmetry**, we can do this on the **upper hemisphere** S^k_+ of S^k, i.e. the points on S^k with $x_{k+1} \geq 0$. Let f_+ be the restriction of f to S^k_+. By the choice of y, L does not meet the equator, and hence no point on the equator is in $f^{-1}(L)$. This implies

$$\frac{1}{2}\#f^{-1}(L) = \#f_+^{-1}(L).$$

The upper hemisphere is a manifold with **boundary**

$$\partial S^k_+ = \{x = (x_1, \ldots, x_{k+1}) : \sum_i x_i^2 = 1 \text{ and } x_{k+1} = 0\} = S^{k-1}$$

being the equator.

Now we would like to apply the previous theorem to the f_+ and $g = \partial f_+$ and use the induction hypothesis. But the target of f_+ has dimension $k+1$, whereas for both the theorem and the induction hypothesis we need as target a Euclidean space of dimension k. So we need to fix this.

The key is that the **orthogonal complement** of L in $\mathbb{R}^{k+1}$, denoted by V, is a vector space of dimension k. By choosing a basis of V, we can identify it with $\mathbb{R}^k$.

To complete the argument, let $\pi\colon \mathbb{R}^{k+1} \to V$ be the orthogonal projection onto V. Since g is symmetric and π is linear,

$$\pi \circ g\colon S^{k-1} \to V \text{ is } \textbf{symmetric} : \pi(g(-x)) = \pi(-g(x)) = -\pi(g(x)).$$

Moreover, we have

$$\pi(g(x)) = 0 \iff g(x) \in L, \text{ hence } \pi(\mathbf{g}(\mathbf{x})) \neq \mathbf{0} \text{ for all } x \in S^{k-1}$$

by the definition of π and the choice of L.

Thus, after choosing a basis for V, we can consider $\pi \circ g$ as a map

$$\pi \circ g\colon S^{k-1} \to \mathbb{R}^k \setminus \{0\}.$$

Now we **apply the induction hypothesis** to $\pi \circ g$ and get $W_2(\pi \circ g, 0) = 1$.

To finish, **recall** $f_+ \pitchfork L$ and hence for

$$\pi \circ f_+\colon S^k \to V, \ (\pi \circ f_+) \pitchfork \{0\}.$$

In other words, 0 **is a regular value** of $\pi \circ f_+$.

Hence we can **apply the previous theorem** to $\pi \circ f_+$ and its boundary map $\partial(\pi \circ f_+) = \pi \circ g$ to get

$$W_2(\pi \circ g,0) = \#(\pi \circ f_+)^{-1}(0).$$

But, by the choice of L, we have

$$\pi(f_+(x)) = 0 \iff f_+(x) \in L, \text{ and hence } (\pi \circ f_+)^{-1}(0) = f_+^{-1}(L).$$

Thus

$$W_2(f,0) = \#f_+^{-1}(L) = W_2(\pi \circ g,0) = 1.$$

QED

Remark: Going back to the definition of $W_2(f,z)$ and the picture at the beginning, we learn from the proof, in particular, that lines **tangential** to $f(X)$ are **not allowed** for calculating $W_2(f,z)$.

Let us look at some of the consequences of this theorem.

Corollary 1 of BUT

If $f\colon S^k \to \mathbb{R}^{k+1} \setminus \{0\}$ is symmetric about the origin, i.e. $f(-x) = -f(x)$, then f intersects every line through 0 at least once.

Proof: Let L be a line in $\mathbb{R}^{k+1}$ through the origin. If f never hits L, then $\#f^{-1}(L) = 0$ and $f \pitchfork L$. By repeating the above proof using this f and L for calculating $W_2(f,0)$, we would get the contradiction

$$W_2(f,0) = \#f^{-1}(L) = 0.$$

QED

Corollary 2 of BUT

Any k smooth odd real-valued functions $f_1, \ldots, f_k$ on S^k must have a common zero.

Proof: If they did not have a common zero, then we can form the smooth odd map

$$f := (f_1, \ldots, f_k, 0)\colon S^k \to \mathbb{R}^{k+1} \setminus \{0\}.$$

Then we can apply Corollary 1 of BUT to f and L being the x_{k+1}-axis. Hence f intersects L at least once. But x with $f(x) \in L$ is a common zero of the $f_1, \ldots, f_k$. Contradiction. **QED**

Corollary 3 of BUT

For any k smooth real-valued functions $g_1, \ldots, g_k$ on S^k, there exists a point $p \in S^k$ such that

$$g_1(p) = g_1(-p), \ldots, g_k(p) = g_k(-p).$$

Proof: We define functions $f_1, \ldots, f_k$ on S^k by

$$f_i(x) := g_i(x) - g_i(-x).$$

Then each f_i is smooth and odd. Hence there is a common zero which is the desired point $p \in S^k$. **QED**

In order to get the meteorologic interpretation, take g_1 measuring temperature and g_2 measuring pressure.

LECTURE 22

Orientations

Our next goal is to improve our definition of intersection numbers and remedy the defect that they only distinguish between even and odd numbers. One of the reasons for this limitation was that a homotopy can turn a nontransversal intersection into either a empty intersection or an intersection in two points. The idea for dealing this phenomenon is to take into account in which "direction" the intersection happens. The solution to implement this idea is to introduce orientations. We will see that, unfortunately, not all manifolds are orientable. But for those manifolds that orientable, we will introduce an improved intersection theory in the next lecture.

Orientations on vector spaces

An **orientation** for a finite dimensional real vector space V is an **equivalence class of ordered bases** where the relation is defined as follows: the ordered basis $(v_1, \dots, v_n)$ has the **same orientation** as the basis $(v_1', \dots, v_n')$ if the matrix A with

$$v_i' = Av_i \text{ for all } i \text{ has } \det(A) > 0.$$

It has the **opposite orientation** if $\det(A) < 0$.

The fact, that this an equivalence relation follows from the multiplicativity of the determinant function.

Thus each finite dimensional vector space **has precisely two orientations**, corresponding to the two equivalence classes of ordered bases.

So an orientation of V is a choice of an equivalence class of ordered bases. To make it easier to talk about the choice of orientation, we attach to the chosen orientation a **positive sign** and a **negative sign** to the other orientation. We say then that an ordered basis is positively oriented (respectively negatively oriented) if its equivalence class belongs to the orientation $+1$ (respectively -1). We often confuse an orientations with their corresponding signs $+1$ or -1.

The vector space $\mathbb{R}^n$ has a **standard orientation** corresponding to the ordered basis $(e_1, \dots, e_n)$. We always assign $+1$ to the standard orientation of $\mathbb{R}^n$.

Warning: The **ordering** of the basis elements **is essential**. Interchanging the positions of two basis vectors changes the sign of the orientation! Check this by calculating the determinant of the corresponding permutation matrix.

In the case of the zero dimensional vector space it is convenient to define an "orientation" as the symbol $+1$ or -1.

If $\varphi\colon V \to W$ is an **isomorphism** of vector spaces, then φ **either preserves or reverses** the orientation. For, given two ordered bases β and β' of V belonging to the the same equivalence class, the ordered bases $\varphi(\beta)$ and $\varphi(\beta')$ either still belong to the same equivalence class of ordered bases of W or not. Whether φ preserves or reverses the orientation is determined by its determinant. If $\det(\varphi)$ is positive, then φ preserves orientations, and if $\det(\varphi)$ is negative, then φ reserves orientations.

Orientations on manifolds

Orienting manifolds

An **orientation of a smooth manifold** X is a **smooth choice of orientations** for all the **tangent spaces** $T_x(X)$. That means: around each point $x \in X$ there must exist a local parametrization $\phi\colon U \to X$ such that the isomorphism $d\phi_u\colon \mathbb{R}^k \to T_{\phi(u)}(X)$ **preserves orientations at each point** u of $U \subseteq \mathbb{H}^k$. The orientation on R^k is alsways assumed to be the standard one.
For zero-dimensional manifolds, orientations are very simple. To each point $x \in X$ we simply assign an orientation number $+1$ or -1.

A manifold X is called **orientable** if such a smooth choice of orientations of tangent spaces exists.

Warning: Not all manifolds possess orientations, the most famous example being the Möbius strip.

Consequence: Orientability helps classifying manifolds: there is the class of orientable manifolds, and the class of non-orientable manifolds.

A manifold is called **oriented** if it is orientable and a choice of orientation has been made. Hence an **oriented manifold** really is a **pair** consisting of a manifold together with a chosen orientation.
A smooth map $f\colon X \to Y$ between oriented manifolds is called **orientation preserving** if its derivative preserves orientations at every point.

We just learned that a manifold may or may not be orientable. To assign $+1$ or -1 to the orientation of $T_x(X)$ for every point is a locally constant function. If X is orientable this assignment is continuous. If X is in addition connected, then this assignment must be constant. Hence on every connected component of an orientable manifold, the orientation is constant $+1$ or -1.

Here is a rigorous proof of this fact:

Orientable manifolds have exactly two orientations

A connected, orientable manifold with boundary admits exactly two orientations.

Proof: Assume we are given two orientations on X. (There are at least two, since given one, we can reverse signs everywhere and get another orientation.)

We show that the **set of points at which two orientations agree** and the **set where they disagree** are **both open**. Consequently, two orientations of a **connected** manifold are either identical or opposite.

Since X is **orientable**, we can choose local parametrizations $\phi\colon U \to X$ and $\phi'\colon U' \to X$ around $x \in X$ with $\phi(0) = x = \phi'(0)$ such that $d\phi_u$ preserves the first orientation and $d\phi'_{u'}$ preserves the second, for all $u \in U$ and $u' \in U'$. After possibly shrinking we can assume $\phi(U) = \phi'(U')$ (replace U and U' with $\phi^{-1}(\phi(U) \cap \phi'(U'))$ and ${\phi'}^{-1}(\phi(U) \cap \phi'(U'))$, respectively).

If the two orientations of $T_x(X)$ **agree**, then the map

$$d(\phi^{-1} \circ \phi')_0 \colon \mathbb{R}^k \to \mathbb{R}^k$$

is an orientation preserving isomorphism. Thus the determinant of $d(\phi^{-1} \circ \phi')_0$ is positive. Hence the function

$$\varphi\colon U' \to \mathbb{R},\ u' \mapsto \det(d(\phi^{-1} \circ \phi')_{u'})$$

satisfies $\varphi(0) > 0$.

Since the derivative depends continuously on u' and the determinant function is continuous, φ is **continuous**. Hence, since $\varphi(0) > 0$, there is an open neighborhood V' around 0 in U' on which $\varphi > 0$. But this implies that the orientations of $T_x(X)$ induced by ϕ and ϕ', respectively, agree for all x in the open subset $\phi'(V')$. Since every point on X has such an open neighborhood, the set of points where the orientations agree is **open**.

If the orientations on $T_x(X)$ indced by ϕ and ϕ', respectively, disagree, the same argument shows that the set of points where the orientations disagree is open. **QED**

Reversed orientation

Hence if X is an oriented manifold X, then we can talk about the manifold with the **reversed orientation**. This is again an oriented manifold which we **denote by $-X$**.

It is now a long and techincal endeavour to check how orientations behave under the main constructions and relate to the concepts we have developed so far. We will go through them one by one:

Products:

If X and Y are oriented and one of them is boundaryless, then $X \times Y$ is a manifold with boundary and inherits an orientation in the following way:

At a point $(x,y) \in X \times Y$, let $\alpha = (v_1, \ldots, v_k)$ be an ordered basis of $T_x(X)$, and $\beta = (w_1, \ldots, w_m)$ be an ordered basis of $T_y(Y)$. We denote by $(\alpha \times 0, 0 \times \beta)$ the ordered basis $((v_1,0), \ldots, (v_k,0), (0,w_1), \ldots, (0,w_m))$ of $T_x(X) \times T_y(Y) = T_{(x,y)}(X \times Y)$.

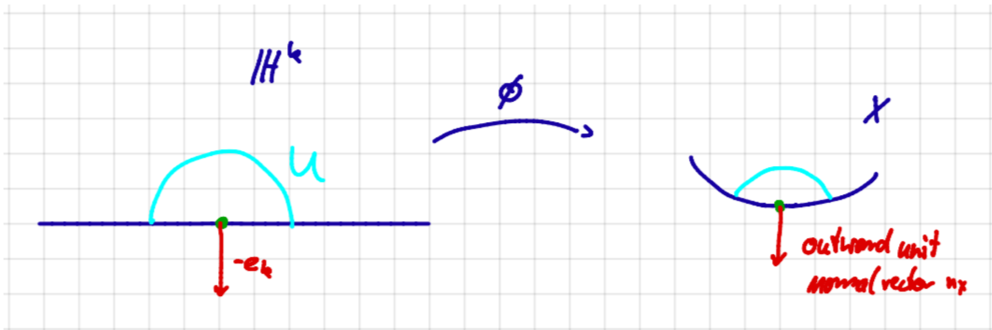
Now it comes handy that we related orientations of ordered bases to signs. For we can define the orientation of $T_x(X) \times T_y(Y)$ simply by determining a sign by setting

$$\mathrm{sign}(\alpha \times 0, 0 \times \beta) = \mathrm{sign}(\alpha) \cdot \mathrm{sign}(\beta).$$

Induced orientation on the boundary

Let X be an oriented smooth manifold with boudary. Then ∂X inherits an orientation as follows:

At every point $x \in \partial X$, $T_x(\partial X)$ is a subspace of codimension one in $T_x(X)$. Its orthogonal complement in $T_x(X)$, is a line which contains exactly two unit vectors: one is pointing **inward** into $T_x(X)$, the other one is pointing **outward** away from $T_x(X)$.



This can be made precise by choosing a local parametrization $\phi\colon U \to X$ around x with $U \subset \mathbb{H}^k$ open and $\phi(0) = x$. The derivative $d\phi_0\colon R^k \to T_x(X)$ is by definition of $T_x(X)$ an isomorphism.

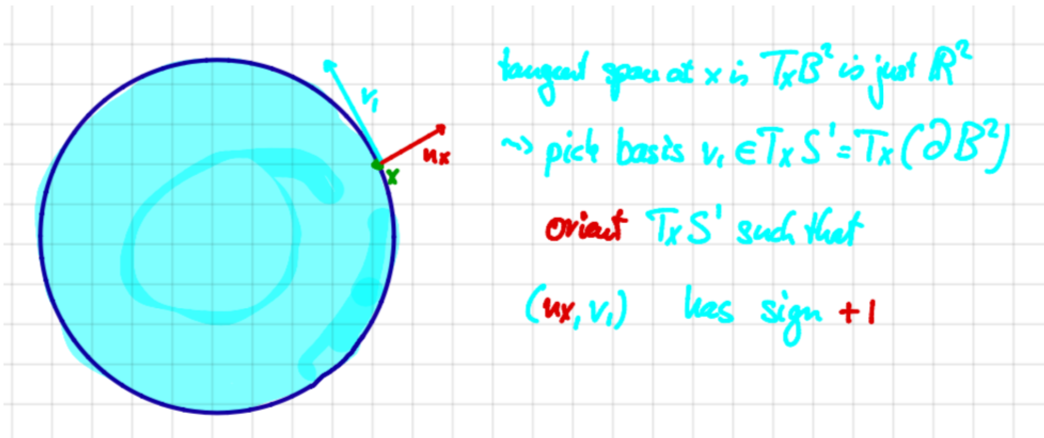
In $\mathbb{R}^k$, there are two unit vectors: $e_k = (0,\dots,0,1)$ one pointing into $\mathbb{H}^k$, and $-e_k = (0,\dots,0,-1)$ pointing out of $\mathbb{H}^k$ Usng the **Gram-Schmidt process** we can orthonormalize the image of e_k under $d\phi_0$ with respect to $T_x(\partial X)$ and get the inward pointing unit normal vector. The orthonormalization with respect to $T_x(\partial X)$ of $d\phi_0(-e_k)$ is the outward pointing unit normal vector. (Note that the inner product on $T_x(X)$ is induced by the standard inner product on $\mathbb{R}^N$, where $X \subset \mathbb{R}^N$ and hence $T_x(X) \subset \mathbb{R}^N$.)

We denote the **outward pointing unit normal vector by** n_x. We checked on Exercise Set 9 that the construction of n_x does not depend on the choice of ϕ and that the assigment $x \mapsto n_x$ is a **smooth** map on ∂X.

Now we are ready to **orient** $T_x(\partial X)$ by declaring the sign of any ordered basis $(v_1,\dots,v_{k-1})$ to be the **sign of the ordered basis** $(n_x,v_1,\dots,v_{k-1})$ **for** $T_x(X)$:

$$\operatorname{sign}(v_1,\dots,v_{k-1}) := \operatorname{sign}(n_x,v_1,\dots,v_{k-1}).$$

Since both the assignment $x \mapsto n_x$ and the choice of sign for ordered bases on $T_x(X)$ are smooth, this defines an orientation of ∂X which is called the **boundary orientation**.



Orientations of One-manifolds

Let us apply what we just learned to the case of a one-manifold with boundary. The boundary ∂X is zero dimensional. The orientation of the zero-dimensional vector space $T_x(\partial X)$ is equal to the sign of the basis of $T_x(X)$ consisting of the outward-pointing unit vector n_x.

As an example, let us look at the compact interval $X = [0,1]$ with its standard orientation inherited from being a subset in $\mathbb{R}$. Note that local parametrizations of $[0,1]$ are given by

$$\phi\colon [0,1) \to [0,1], x \mapsto x$$

around $0 \in [0,1]$ and

$$\psi\colon [0,1) \to [0,1], x \mapsto 1 - x$$

around $1 \in [0,1]$.

Hence, at $x = 1$, the outward-pointing normal vector is $1 \in \mathbb{R} = T_x(X)$. The basis consisting of this vector is positively oriented. At $x = 0$ the outward-pointing normal vector is the negatively oriented $-1 \in \mathbb{R} = T_0(X)$. Thus the orientation of $T_1(\partial X)$ is $+1$, and the orientation of $T_0(\partial X)$ is -1.

Reversing the orientation on $[0,1]$ simply reverses the orientations at each boundary point. Thus the sum of both orientation numbers at the boundary points of $[0,1]$ is always zero.

Since any compact one-manifold with boundary is diffeomorphic is the disjoint union of copies of $[0,1]$, we conlcude:

Boundary orientations of one-manifolds

The **sum of the orientation numbers** at the **boundary points** of any compact oriented one-dimensional manifold with boundary is **zero**.
In particular, the **boundary points of a smooth path** γ on an oriented manifold X, i.e. a smooth map $\gamma \colon [0,1] \to X$, must have **opposite orientation signs**.
This will turn out to be the **crucial point** which will allow us to define **homotopy invariant intersection numbers** with values in $\mathbb{Z}$ in the next lecture.

Oriented Homotopies

As an application of product and boundary orientations, we would like to orient the product $[0,1] \times X$ for a boundaryless smooth oriented manifold X which is the domain of all homotopies on X. This will be crucial for the homotopy invariance of intersection numbers in the next section.

We just learned that a products and boundaries inherit orientations. For each $t \in [0,1]$, the slice $X_t := \{t\} \times X$ is diffeomorphic to X, and the orientation on X_t should be such that the diffeomorphism

$$X \to X_t, \ x \mapsto (t,x) \text{ preserves orientations.}$$

For the future applications, we are particularly interested in the orientation of the boundary

$$\partial([0,1] \times X) = \{0\} \times X \cup \{1\} \times X.$$

So let us try to understand the induced orientation on the boundary.

We start with X_1: We see from the local parametrization ψ above that along X_1 the outward-pointing normal vector is

$$n_{(1,x)} = (1,0) = (1,0,\dots,0) \in T_1([0,1]) \times T_x(X),$$

If $\beta = (v_1,\dots,v_k)$ is an ordered basis of $T_x(X)$, then $0 \times \beta = ((0,v_1),\dots,(0,v_k))$ is an ordered basis of $T_x(X_1)$. By definition of the boundary orientation, $n_{(1,0)},(0 \times \beta)$ is positively oriented if and only if β is positively oriented, in terms of signs:

$$\operatorname{sign}(n_{(1,0)},(0 \times \beta)) = \operatorname{sign}(\beta).$$

If we calculate the orientation induced from the product structure, then we get

$$\operatorname{sign}((1,0),(0 \times \beta)) = \operatorname{sign}(1)\operatorname{sign}(\beta) = \operatorname{sign}(\beta).$$

We learn from these two equations, that the **boundary orientation of X_1** is just the **orientation of X** as a copy in the product $[0,1] \times X$.

This sounds obvious, **but pay attention**:

We see from the local parametrization ϕ that along X_0 the outward-pointing normal vector is

$$n_{(0,x)} = (-1,0) = (-1,0,\ldots,0) \in T_0([0,1]) \times T_x(X).$$

Hence the orientation on $T_0([0,1])$ is opposite to the standard orientation of $\mathbb{R}$. Hence the formula for product orientations yields

$$\operatorname{sign}((-1,0),0 \times \beta)) = \operatorname{sign}(-1)\operatorname{sign}(\beta) = -\operatorname{sign}(\beta).$$

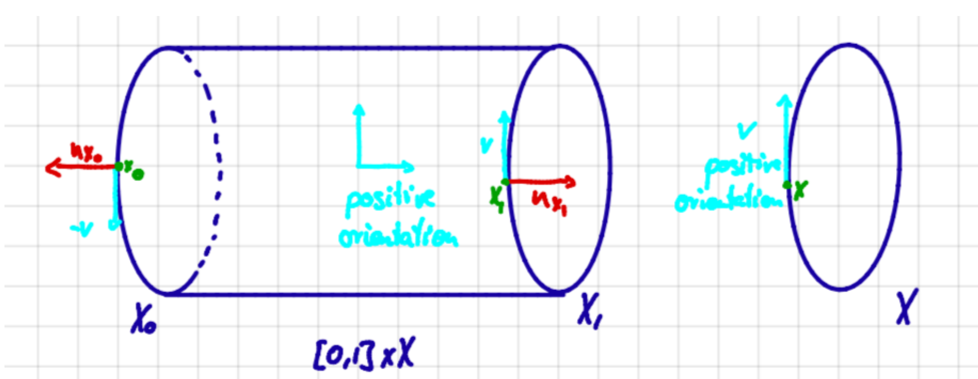
Thus the **boundary orientation on X_0** is the **reverse** of its **orientation as a copy of X** in the product $[0,1] \times X$.

Thus the orientation on the boundary is

$$\partial([0,1] \times X) = X_1 \cup (-X_0).$$

We will also express this fact by using the **notation**

$$\partial([0,1] \times X) = X_1 - X_0.$$



Orientations on direct sums of vector spaces

Our next goal is to **orient preimages**. In order to do so, we will have to look at direct sums (not just products) of vector spaces, and we need to orient those guys.

So suppose that $V = V_1 \oplus V_2$ is a direct sum of vector spaces. Then orientations on any two of these vector spaces automatically induces a direct sum orientation on the third, as follows. Note that this not only means, orientations on V_1 and V_2 determine an orientation on V, but also orientations on V and, say, V_2 determine an orientation on V_1.

Choose ordered bases β_1 of V_1 and β_2 of V_2. Let $\beta = (\beta_1,\beta_2)$ be the combined ordered basis of V (in this order!). For orientations or signs to be compatible with the structure as a direct sum, we require the formula

$$\operatorname{sign}(\beta) = \operatorname{sign}(\beta_1) \cdot \operatorname{sign}(\beta_2).$$

It follows immediately from the way matrices on direct sums are put together that this formula determines an orientation on the third space if two orientations are given. But note again that the order of the summands V_1 and V_2 is crucial.

Orientations of transversal preimages

Let $f\colon X \to Y$ be a smooth map with $f \pitchfork Z$ and $\partial f \pitchfork Z$, where X, Y, and Z are all **oriented** and Y and Z are boundaryless. We would like to define a **preimage orientation** on the manifold with boundary $S = f^{-1}(Z)$.

If $f(x) = z \in Z$, then

$$T_x(S) = (df_x)^{-1}(T_z(Z)) \subset T_x(X).$$

Let $N_x(S;X)$ be the **orthogonal complement** to $T_x(S)$ in $T_x(X)$. By definition, we have a direct sum decomposition

$$N_x(S;X) \oplus T_x(S) = T_x(X).$$

Hence, by our observation on orientations on direct sums, we need only **choose an orientation on** $N_x(S;X)$ to obtain a **direct sum orientation on** $T_x(S)$.

Since $f \pitchfork Z$, we have

$$\begin{aligned} T_z(Y) &= df_x(T_x(X)) + T_z(Z) \\ &= df_x(N_x(S;X) \oplus T_x(S)) + T_z(Z) \\ &= df_x(N_x(S;X)) \oplus T_z(Z) \text{ since } df_x(T_x(S)) = T_z(Z). \end{aligned}$$

Thus the orientations on Z and Y induce a direct image orientation on $df_x(N_x(S;X))$. It remains to show that this also induces an orientation on

$N_x(S;X)$. But

$$\{0\} \subset T_z(Z) \Rightarrow \operatorname{Ker}(df_x) \subset (df_x)^{-1}(T_z(Z)) = T_x(S),$$

and hence the restriction of df_x to $N_x(S;X)$ is in fact an **isomorphism onto its image**. Therefore the induced orientation on $df_x(N_x(S;X))$ defines an orientation on $N_x(S;X)$ via the isomorphism df_x.

Since the orientations on X, Y and Z vary smoothly and df_x also depends smoothly on x, the induced orientation on $T_x(S)$ varies smoothly with x.

Note that we did not really use that $N_x(S;X)$ is orthogonal to $T_x(S)$. All we needed was a direct sum decomposition $H \oplus T_x(S) = T_x(X)$ with a space H with an orientation induced by the orientation of X. We will exploit this fact in the proof below.

Orientations on boundaries of preimages

Let $f\colon X \to Y$ be a smooth map with $f \pitchfork Z$ and $\partial f \pitchfork Z$, where X, Y, and Z are all otientid and Y and Z are boundaryless.

Then the manifold $\partial f^{-1}(Z)$ acquires two orientations:

- one as the boundary of the manifold $f^{-1}(Z)$, and
- one as the preimage of Z under the map $\partial f\colon \partial X \to Y$,

It turns out that there is a formlua that relates these two orientations:

Orientations on boundaries of preimages

$$\partial(f^{-1}(Z)) = (-1)^{\operatorname{codim} Z}(\partial f)^{-1}(Z).$$

This means the orientations of $\partial f^{-1}(Z)$, induced by being a boundary or by being a preimage, are the same if $\operatorname{codim} Z$ is even, and opposite if $\operatorname{codim} Z$ is odd.

Proof:

Denote $f^{-1}(Z)$ again by S.

Let H be a subspace of $T_x(\partial X)$ complementary to $T_x(\partial S)$, i.e.

$$H \oplus T_x(\partial S) = T_x(\partial X).$$

Note that H is also complementary to $T_x(S)$ in $T_x(X)$, i.e.

$$H \oplus T_x(S) = T_x(X).$$

For we have

$$H \cap T_x(S) = \{0\} \text{ and } T_x(S) \cap T_x(\partial X) = T_x(\partial S),$$

and

$$\dim H = \dim T_x(\partial X) - \dim T_x(\partial S) = \dim T_x(X) - \dim T_x(S).$$

Hence we may use H to define the direct sum orientation of both S and ∂S at x.

Since $H \subset T_x(\partial X) \subset T_x(X)$, the maps df_x and $d(\partial f)_x$ agree on H, i.e.

$$df_x(H) = d(\partial f)_x(H).$$

As in the case of $N_x(S; X)$, since

$$\{0\} \subset T_z(Z) \Rightarrow \operatorname{Ker}(df_x) \subset f^{-1}(T_z(Z)) = T_x(S),$$

the intersection $\operatorname{Ker}(df_x) \cap H$ is $\{0\}$. Hence the restrictions of df_x and $d(\partial f)_x$ to H are isomorphisms onto their common image.

Thus $f \pitchfork Z$ and $\partial f \pitchfork Z$ imply that we have two direct sum decompositions $df_x(H) \oplus T_z(Z) = T_z(Y) = d(\partial f)_x(H) \oplus T_z(Z)$, and the two orients of H via these direct sums agree.

To conclude, we obtained that H has a well-defined orientation. Hence we can use this unique orientation on H to orient

$$S \text{ via } H \oplus T_x(S) = T_x(X) \text{ and } \partial S \text{ via } H \oplus T_x(\partial S) = T_x(\partial X).$$

It remains to check how this orientation of $T_x(\partial S)$ relates to the orientation of the boundary induced from the orientation of $T_x(S)$.

Let n_x be the outward unit vector to ∂S in $T_x(S)$, and let $\mathbb{R} \cdot n_x$ represent the one-dimensional subspace spanned by n_x. We orient this space by assigning the sign $+1$ to the basis (n_x).

Even though n_x need not be perpendicular to all of $T_x(\partial X)$, it suffices to know that n_x lies in the halfspace pointing away from $T_x(X)$ to know that the orientations of $\mathbb{R} \cdot n_x$, $T_x(\partial X)$ and $T_x(X)$ are related by the direct sum

$$T_x(X) = \mathbb{R} \cdot n_x \oplus T_x(\partial X).$$

Now we use that H is complementary to both $T_x(S)$ in $T_x(X)$ and $T_x(\partial S)$ in $T_x(\partial X)$ and plugg this into the above direct sum to get

$$H \oplus T_x(S) = \mathbb{R} \cdot n_x \oplus H \oplus T_x(\partial S).$$

This equation is already almost what we need, since we would like to compare the orientations $T_x(S)$ and $\mathbb{R} \cdot n_x \oplus T_x(\partial S)$. For doing so, we need to move $\mathbb{R} \cdot n_x$ passed H. If $\dim H = m$, H has m basis vectors $(w_1, \ldots, w_m)$. Remembering the rule for orienting direct sums, this means we have to apply exactly m transpositions to the ordered set

$$(n_x, w_1, \ldots, w_m) \text{ to get to } (w_1, \ldots, w_m, n_x).$$

This results in m shifts of signs. Hence we get

$$H \oplus T_x(S) = (-1)^{\operatorname{codim} Z} H \oplus \mathbb{R} \cdot n_x \oplus T_x(\partial S).$$

Since H appears on both sides as the first summand, we get disregard it for the computation and get that if ∂S is oriented as a preimage under ∂f, then its orientation relates to the one of $T_x(S)$ by

$$T_x(S) = (-1)^{\operatorname{codim} Z} \mathbb{R} \cdot n_x \oplus T_x(\partial S).$$

Now, if ∂S is oriented as a boundary, then we have

$$T_x(S) = \mathbb{R} \cdot n_x \oplus T_x(\partial S).$$

Thus

$$\operatorname{sign}(\partial S) \text{ as a boundary} = (-1)^{\operatorname{codim} Z} \cdot \operatorname{sign}(\partial S) \text{ as a preimage.}$$

QED

The following theorem shows that an important class of manifolds is orientable. Recall that a manifold X is called **simply-connected** if it is conneced and every smooth map $S^1 \to X$ is homotopic to a constant map.

Simply-connected implies orientable

Every simply-connected manifold is orientable.

Proof:

We start by picking any point $x \in X$, and choose an orientation for the tangent space $T_x(X)$. Since $T_x(X)$ is a vector space, this is always possible.

Now let $y \in X$ be any other point in X. Since X is simply-connected, it is in particular also connected. By a previous exercise, since X is a smooth manifold,

X is therefore even **path-connected**. Hence there is a smooth map $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. For every point in $z \in \gamma([0,1])$ we choose a local parametrization $\phi_z\colon V_z \to U_z$ around z. By shrinking V_z if necessary, we can assume that each V_z is an open ball in $\mathbb{R}^k$.

The sets $U_z \cap \gamma([0,1])$ is open in $\gamma([0,1])$, and the collection of $\{U_z \cap \gamma([0,1])\}$ for all $z \in \gamma([0,1])$ is an open covering of $\gamma([0,1])$. Since $[0,1]$ is compact and γ continuous, the image $\gamma([0,1])$ is compact. Hence **finitely many** of the U_z suffice to cover $\gamma([0,1])$. We label these open sets $U_1, \ldots, U_m$ and **order them** such that $U_i \cap U_{i+1} \neq \emptyset$ and $x \in U_1$, $y \in U_m$.

For U_1, we choose the orientation which is compatible with the chosen orientation of $T_x(X)$. That means: let $\phi_1\colon U_1 \to X$ be the associated local parametrization with $\phi_1(0) = x$. If $d(\phi_1)_0\colon \mathbb{R}^k \to T_x(X)$ is orientation preserving, we orient the vector space $T_a(U_1)$ such that $d(\phi_1)_{\phi_1^{-1}(z)}\colon \mathbb{R}^k \to T_a(X)$ is orientation preserving for all $a \in U_1$.

If $d(\phi_1)_0 \to \mathbb{R}^k \to T_x(X)$ reverses orientation, we first replace ϕ_1 with $\tilde{\phi}_1\colon V_1 \to X$, $v \mapsto \phi_1(-v)$. This new map $\tilde{\phi}_1$ is also a local parametrization of X with domain V_1, since V_1 is an open ball in $\mathbb{R}^k$ and ϕ_1 is therefore symmetric with respect to the origin.

Hence after replacing ϕ_1 with $\tilde{\phi}_1$, we can assume that $d(\phi_1)_0$ is orientation preserving, and we orient all $T_a(U_1)$ as above. Note that switching from $\phi_1(v)$ to $\phi_1(-v)$ corresponds to switching the orientation on $\mathbb{R}^k$.

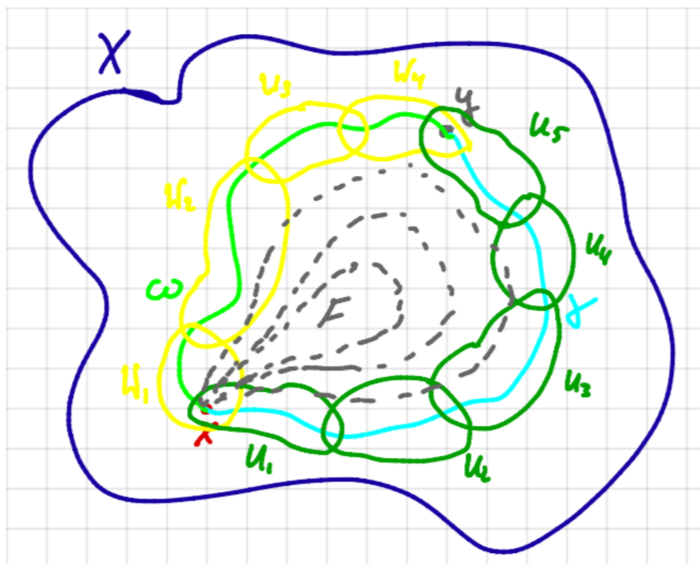
For U_2, we choose the orientation which is compatible with the orientation of the $T_a(X)$ for all poins $a \in U_1 \cap U_2$. That means: if $d(\phi_2)_{\phi_2^{-1}(a)}$ is orientation preserving on $T_a(X)$ for $a \in U_1 \cap U_2$, we orient $T_a(X)$ such that $d(\phi_2)_{\phi_2^{-1}(a)}\colon \mathbb{R}^k \to T_a(X)$ is orientation preserving for all $a \in U_2$. If it is not orientation preserving, then we replace $\phi_2(v)$ by $\phi_2(-v)$.

Continuing this way, we obtain an orientation for U_m and therefore $T_y(X)$ after finitely many steps.

It remains to show that the induced orientation on $T_y(X)$ **does not depend on the choice** of γ and the U_i's.

So let $\omega\colon [0,1] \to X$ be another smooth path with $\omega(0) = x$ and $\omega(1) = y$. As for γ, we choose open sets $W_1, \ldots, W_l$ covering all points in $\omega([0,1])$ with $x \in W_1$ and $y \in W_l$ and $W_i \cap W_{i+1} \neq \emptyset$. Then we orient $T_y(X)$ following the same procedure using the W_i's.

Arriving at y, we do not know a prioir whether the orientation of $T_y(X)$ induced by γ and the orientation of $T_y(X)$ induced by ω agree. But now we can use that X is **simply-connected**.



For, walking first along γ and then back on ω defines, after readjusting the speed and smoothing things out, a **loop** $\alpha\colon [0,1] \to X$ with $\alpha(0) = x = \alpha(1)$, i.e. a smooth map $\alpha\colon S^1 \to X$. Walking along α, we obtain an **isomorphism**

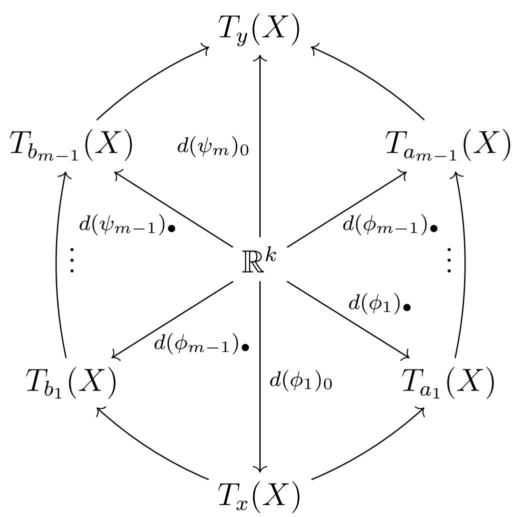
$$T(\alpha)\colon T_x(X) = T_{\alpha(0)}(X) \xrightarrow{\cong} T_{\alpha(1)}(X) = T_x(X)$$

by composing

$$T_x(X) \xrightarrow{d(\phi_1)_\bullet^{-1}} \mathbb{R}^k \xrightarrow{d(\phi_2)_\bullet} T_z(X) \xrightarrow{d(\phi_2)_\bullet^{-1}} \mathbb{R}^k \xrightarrow{d(\phi_2)_\bullet} \cdots \xrightarrow{d(\psi_{m-1})_\bullet^{-1}} \mathbb{R}^k \xrightarrow{d(\psi_m)_\bullet} T_x(X)$$

where the subscript $\bullet$ stands for the varying points at which we take derivatives.

Another way to picture how we get from $T_x(X)$ to $T_y(X)$ via γ and ω, respectively, is the following diagram:



The isomorphism $T(\alpha)$ is either orientation preserving or reversing. If it preserves the orientation, then its determinant is positive, and if it reverses the orientation, then its determinant is negative. And **$T(\alpha)$ is orientation preserving** if and only if the **two orientations** on $T_y(X)$ induced by γ and ω, respectively, **agree**.

Since X is **simply-connected**, α is homotopic to the constant map $c_x\colon S^1 \to \{x\}$.

Let $F\colon S^1 \times [0,1] \to X$ be a homotopy from α to c_x. Since $S^1 \times [0,1]$ is **compact**, its image in X is compact and we can add **finitely many** open subsets to the collection $U_1,\dots,U_m,W_1,\dots,W_l$ to cover $F(S^1 \times [0,1])$ with the codomains of local parametrizations.

For each $t \in [0,1]$, $F(-,t)$ defines a **smooth loop** from x to x. Using the above procedure for orienting tangent spaces along a path, we obtain an isomorphism

$$T(F(-,t))\colon T_x(X) = T_{F(0,t)}(X) \xrightarrow{\cong} T_{F(1,t)}(X) = T_x(X) \text{ for each } t \in [0,1].$$

Taking the **determinant** of $T(F(-,t))$ defines a map

$$[0,1] \to \mathbb{R},\ t \mapsto \det(T(F(-,t)))$$

which is **continuous**, since each point of X is contained an open neighborhood on which the orientation is determined by the derivatives of local parametrizations, and these derivatives vary smoothly with the basepoints.

Since each $T(F(-,t))$ is an isomorphism, its determinant is either strictly positive > 0 or strictly negative < 0. Since $[0,1]$ is **connected** and $t \mapsto \det(T(F(-,t)))$ is **continuous**, we have

$$\text{either } \det(T(F(-,t))) > 0 \text{ or } \det(T(F(-,t))) < 0 \text{ for all } t \in [0,1].$$

But we know that, for $t = 1$, $F(-,1) = c_x$ is the **constant loop at** x. Thsus

$$\det(T(F(-,1))) = \det(\mathrm{Id}_{T_x(X)}) > 0.$$

Hence we must have $\det(T(F(-,t))) > 0$ **for all** $t \in [0,1]$. In other words, $T(F(-,t))$ must be orientation preserving for all t, and in particular, $T(\alpha)$ **is orientation preserving**.

This shows that the orientation of $T_y(X)$ does not depend on the choice of γ. **QED**

Let us summarize the key points we shoud remember from this technical lecture.

Key points we need to take from this lecture

- An orientation of a vector space is a choice of a sign, $+1$ or -1, for an equivalence of orderings of a bases. We can think of it as choosing a positive and negative direction.
- An orientation on a manifold is a smooth choice of orientations of the tangent spaces for each point. Such a choice may or may not exist. Hence manifolds can be orientable or not.
- Orientability helps us classifying manifolds: there is a box with orientable and a box with non-orientable manifolds.
- The boundary of a cylinder has opposite orientations:

$$\partial([0,1] \times X) = X_1 - X_0.$$

- As a conseuqnce: For any compact oriented one-dimensional manifold with boundary, the sum of the orientation numbers at the boundary points is zero. This is the key point for defining homotopy invariant intersection numbers soon.
- There is a formula for the boundary of preimages:

$$\mathrm{sign}(\partial f^{-1}(Z)) \text{ as a boundary} = (-1)^{\mathrm{codim}\, Z} \cdot \mathrm{sign}(\partial f^{-1}(Z)) \text{ as a preimage}.$$

- Simply-connected manifolds are orientable.

LECTURE 23

Intersection Theory

The assumptions for our intersection theory to work will be always:

Assumptions for intersection theory

- We consider a smooth map $f\colon X \to Y$, where X, Y are boundaryless smooth manifolds, $Z \subset Y$ is a boundaryless submanifold.
- The dimensions are complementary, i.e. $\dim X + \dim Z = \dim Y$.
- X will always be assumed to be compact.
- All manifolds are oriented, i.e. they are orientable and we have chosen an orientation.

The idea for the new intersection number is now very simple:

If $f\colon X \to Y$ is transversal to Z, then $f^{-1}(Z)$ consists of a **finite number of points** (since $f^{-1}(Z)$ is zero-dimensional and compact becasue of the assumptions on X, Z and the dimensions; the assumptions are all important). Each point in $f^{-1}(Z)$ has an orientation number ± 1 provided by the **preimage orientation**.

If $x \in f^{-1}(Z)$ is a point in the preimage, the orientation number at x is determined as follows. If $f(x) = z \in Z$, then transversality implies $df_x(T_x(X)) + T_z(Z) = T_z(Y)$. But since the dimensions are complementary, this sum must be **direct**, i.e.,

$$df_x(T_x(X)) \cap T_z(Z) = \{0\}, \text{ and } df_x(T_x(X)) \oplus T_z(Z) = T_z(Y). \tag{26}$$

This direct sum decompostion implies that

$$\dim T_x(X) = \dim df_x(T_x(X)),$$

since $\dim T_x(X) = \dim T_z(Y) - \dim T_z(Z)$. Thus df_x must be an **isomorphism onto its image**. In particular, the orientation of $T_x(X)$ provides an **orientation of $df_x(T_x(X))$**.

Then the **orientation number at x** is $+1$ if the orientation of $T_z(Y)$ as the direct sum in (26) induced by the orientations on $df_x(T_x(X))$ and $T_z(Z)$ agrees with the given orientation of $T_z(Y)$. And it is -1 if the induced orientation disagrees.

Intersection numbers as sums of orientation numbers

If $f \pitchfork Z$, we define the **intersection number** $I(f,Z)$ to be the **sum of the orientation numbers** at the finitely many points $x \in f^{-1}(Z)$.

We claimed that introducing orientations would yield **homotopy invariant** intersection numbers in $\mathbb{Z}$. Now we we have to demonstrate that this claim holds. This will then allow us to define intersection numbers for nontransversal intersections.

Suppose that $X = \partial W$ is the **boundary** of a compact W and that f extends to a smooth map $F\colon W \to Y$, i.e. $f = \partial F = F_{|\partial W}$.

By the **Extension Theorem**, we may assume $F \pitchfork Z$. Thus, by the Preimage Theorem for manifolds with boundary, $F^{-1}(Z)$ is a compact oriented manifold with boundary $\partial F^{-1}(Z) = f^{-1}(Z)$. Since $\operatorname{codim} \partial W = 1$ in W, we have $\operatorname{codim} F^{-1}(Z) = 1$ in Y, and hence

$$\begin{aligned} \dim W - \dim F^{-1}(Z) &= \operatorname{codim} F^{-1}(Z) \text{ in } W \\ &= \operatorname{codim} Z \text{ in } Y = \dim Y - \dim Z = \dim X. \end{aligned}$$

But $\dim W = \dim X + 1$, and **thus** $\dim F^{-1}(Z) = 1$. Hence $F^{-1}(Z)$ is a **compact oriented one-manifold with boundary**. As we learned in the previous lecture, the sum of the orientation numbers at points in the boundary $f^{-1}(Z)$ must be **zero**.

As a consequence we get:

Intersection numbers for maps on boundaries

If $f \pitchfork Z$ and $X = \partial W$ is the **boundary** of a compact W and that f extends to a smooth map $F\colon W \to Y$, then the sum of orientation numbers of points in $f^{-1}(Z)$ is zero, i.e. $I(f,Z) = 0$.

This enables us to prove the key fact:

Homotopy invariance for transversal maps

Let f_0 and f_1 be two homotopic maps $X \to Y$ which are both transversal to Z. Then $I(f_0,Z) = I(f_1,Z)$.

Proof: Let $F\colon X \times [0,1] \to Y$ be a homotopy between them. Then we just learned that $I(\partial F,Z) = 0$. The boundary map ∂F is just f_0 on the copy X_0 at 0 and f_1 on the copy X_1 at 1. Now recall that the orientations of X_0 and X_1 as the boundary of $X \times [0,1]$ are given by

$$\partial(X \times [0,1]) = X_1 - X_0.$$

Hence as oriented manifolds we get

$$\partial F^{-1}(Z) = f_1^{-1}(Z) - f_0^{-1}(Z).$$

By our definition of **interesction numbers as sums of orientation numbers**, this implies

$$0 = I(\partial F,Z) = I(f_1,Z) - I(f_0,Z).$$

QED

As in the mod 2-theory, the previous theorem allows us to define intersection numbers for arbitrary maps.

Intersection numbers for arbitrary maps

Let $g\colon X \to Y$ be any smooth map. By the Transversality Homotopy Theorem, we can **choose** a smooth map $f\colon X \to Y$ which is homotopic to g and transversal to Z. Then we **define** $I(g,Z)$ to be $I(f,z)$, i.e.

$$I(g,Z) := I(f,Z).$$

We just shows that the definition does not depend on the choice of f. Moreover, all homotopic maps have equal intersection numbers:

All homotopic maps have equal Intersection Numbers

If $g_0\colon X \to Y$ and $g_1\colon X \to Y$ are arbitrary **homotopic** maps, then $I(g_0,Z) = I(g_1,Z)$.

Proof: The proof is the same is in the mod 2-case. We can choose maps $f_0 \pitchfork Z$ and $f_1 \pitchfork Z$ such that $g_0 \sim f_0$, $I(g_0,Z) = I(f_0,Z)$, and $g_1 \sim f_1$, $I(g_1,Z) =$

$I(f_1,Z)$. Since homotopy is a **transitive** relation, we have

$$f_0 \sim g_0 \sim g_1 \sim f_1, \text{ and hence } f_0 \sim f_1.$$

By the previous theorem, this implies

$$I(g_0,Z) = I(f_0,Z) = I(f_1,Z) = I(g_1,Z).$$

QED

The Brouwer degree

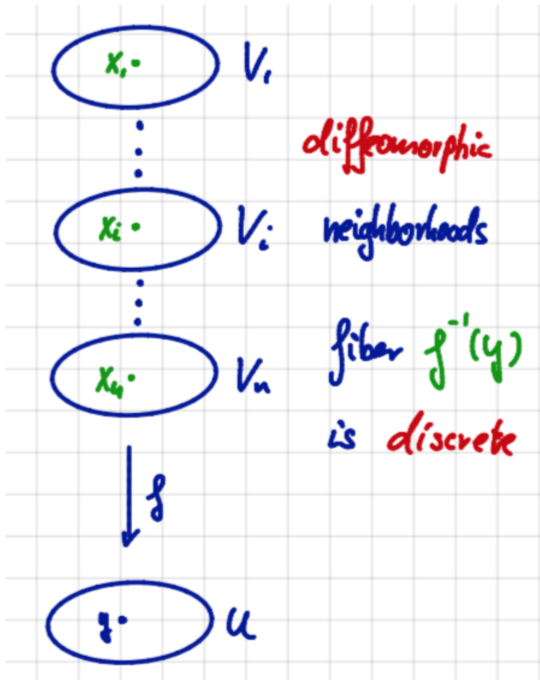
Let us look again at the special case when $\dim X = \dim Y$:

The Brouwer degree

Let $f\colon X \to Y$ be a smooth map with $\dim X = \dim Y$, X compact, and Y connected. We define the **degree of f**, denoted by $\deg(f)$, to be the intersection number $I(f,\{y\})$ at any regular value $y \in Y$ of f. In particular, we claim that the integer $I(f,\{y\})$ does not depend on the choice of the regular value y.
The degree is homotopy invariant, i.e. $f_0 \sim f_1$ implies $\deg(f_0) = \deg(f_1)$.

Proof of the claim of independence: Actually, the proof in the mod 2-case gave us this result already. But only observed the weaker consequence for mod 2-intersction numbers. To be sure, let us go through it again.

Given any $y \in Y$, we can assume that f is **transversal to** $\{y\}$. For otherwise we can replace it with a homotopic map which is transversal by the **Transversality Homotopy Theorem**. Now by the **Stack of Records Theorem**, we can find a neighborhood U of y such that the preimage $f^{-1}(U)$ is a disjoint union $V_1 \cup \cdots \cup V_n$, where each V_i is an open set in X mapped by f diffeomorphically onto U:



Hence, for all points $z \in U$, we have $\#f^{-1}(\{z\}) = n$. But this is not enough for knowing that the intersection numbers agree. For we we have to take orientations into account.

Since $f_{|V_i} \colon V_i \to U$ is a diffeomorphism, we know that

$$df_{x_i} \colon T_{x_i}(X) \to T_y(Y)$$

is an isommorphism. Now both $T_{x_i}(X)$ and $T_y(Y)$ are oriented, and hence df_{x_i} is either orientation preserving or reversing. But by our definition of orientations on manifolds, we have **either**

- $\det(df_{x_i}) > 0$ and hence, for all $z \in U$, $\det(df_{w_i}) > 0$, where w_i is the unique point in V_i with $f(w_i) = z$; in other words, df_{w_i} preserves orientations for all points $w_i \in V_i$;
- **or** $\det(df_{x_i}) < 0$ and hence, for all $z \in U$, $\det(df_{w_i}) < 0$, where w_i is the unique point in V_i with $f(w_i) = z$; in other words, df_{w_i} reverses orientations for all points $w_i \in V_i$.

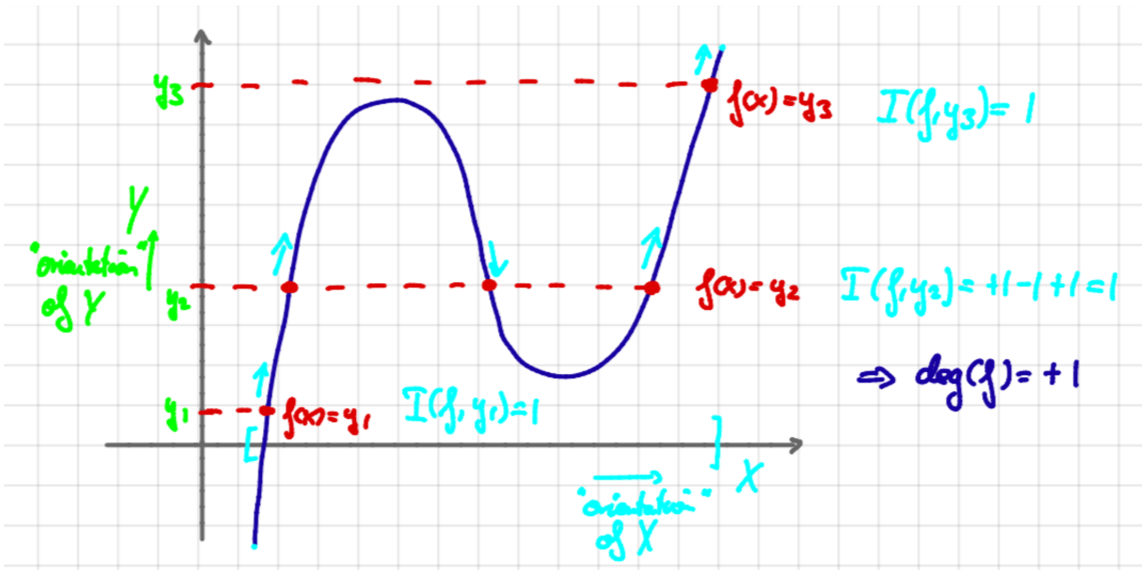
Thus the orientation number is the same for all points in V_i. Hence the sum of orientation numbers of the points in $f^{-1}(z)$ is the same for all points $z \in U$.

Consequently, the function

$$Y \to \mathbb{Z},\ y \mapsto I(f,\{y\})$$

is **locally constant**. Since Y is **connected**, it must be **globally constant**. **QED**

Here is a simple example of how to calculate a degree:



Degree of a diffeomorphism

A special case of the situation $\dim X = \dim Y$ is that of a **diffeomorphism** $f\colon X \to Y$. It follows immedtiately from the definition that f has **degree** $+1$ **or** -1 according to if f preserves or reverses orientation. In particular, we get:
An **orientation reversing diffeomorphism** of a compact boundaryless manifold is **not** smoothly homotopic to the identity.
An example of such an orientation reversing diffeomorphism is provided by the **reflection** $r_i\colon S^n \to S^n$ which we have seen in the Exercises before:

$$r_i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, -x_i, \ldots, x_{n+1}).$$

As in the mod 2-case, the boundary result for intersection numbers imply the following fact on extensions of maps.

Extendable maps on boundaries have degree zero

Suppose that $f\colon X \to Y$ is a smooth map of compact oriented manifolds having the same dimension and that $X = \partial W$ is the boundary of a compact manifold W. If f can be extended to all of W, then $\deg(f) = 0$.

Example: Degree of self-maps of S^1

Recall that the restriction of complex multiplication $z \to z^m$ defines a smooth map $f_m\colon S^1 \to S^1$ for every $m \in \mathbb{Z}$. For $m \neq 0$, let us calculate the derivative $d(f_m)_z\colon T_z(S^1) \to T_{f_m(z)}(S^1)$.

We use the parametrization $\phi t \mapsto (\cos t, \sin t)$. We have the commutative diagram

$$\begin{array}{ccc} S^1 & \xrightarrow{\ f_m\ } & S^1 \\ {\scriptstyle\phi}\big\uparrow & & \big\uparrow{\scriptstyle\phi} \\ \mathbb{R} & \xrightarrow[t\mapsto mt]{} & \mathbb{R}. \end{array}$$

Taking derivatives yields, where we note that $t \mapsto mt$ is a linear map and therefore equal to its derivative:

$$\begin{array}{ccc} T_z(S^1) & \overset{d(f_m)_z}{\cdots\cdots\cdots\rightarrow} & T_{z^m}(S^1) \\ {\scriptstyle d\phi_t}\big\uparrow & & \big\uparrow{\scriptstyle d\phi_{mt}} \\ \mathbb{R} & \xrightarrow[t\mapsto mt]{} & \mathbb{R}. \end{array}$$

In order to determine $d(f_m)_z$, recall

$$d\phi_t\colon \mathbb{R} \to \mathbb{R}^2,\ s \mapsto (-\sin t, \cos t) \cdot s$$

and, hence at $z = \phi(t)$ (we have done this a long time ago):

$$T_z(S^1) = (-\sin t, \cos t) \cdot \mathbb{R}.$$

Putting these information together we obtain we get

$$\begin{aligned} &d(f_m)_z\colon T_z(S^1) \to T_{z^m}(S^1), \\ &(-\sin t, \cos t) \cdot s \mapsto m(-\sin(mt), \cos(mt)) \cdot s. \end{aligned}$$

Hence, when $m > 0$, f_m wraps the circle uniformly around itself m times preserving orientation. The map is everywhere regular and orientation preserving, so its degree is the number of preimages of any point, that is m.

Similarly, when $m < 0$ the map is everywhere regular but orientation reversing. As each point has $|m|$ preimages, the degree is $-|m| = m$.

Finally,when $m = 0$ the map is constant, so its degree is zero.

One homotopy class $S^1 \to S^1$ for every integer

One immediate consequence of this calculation (which could not have been proven with mod 2 theory) is the interesting fact that the circle admits an **infinite number** of homotopically distinct mappings. For since $\deg(z^m) = m$, none of these maps can be homotopic to another one.

Application: The Fundamental Theorm of Algebra - again

Now we can finish the proof of the Fundamental Theorem of Algebra using degrees. Remember that mod 2-degrees were only good enough for polynomials of odd order. Now we can deal with all of them.

So let

$$p(z) = z^m + a_1 z^{m-1} + \cdots + a_m$$

be a monic complex polynomial. For the argument in the case m odd, we used the homotopy from $p_0(z) = z^m$ to $p_1(z) = p(z)$ defined by

$$p_t(z) = tp(z) + (1-t)z^m = z^m + t(a_1 z^{m-1} + \cdots + a_m).$$

We observed that, if W is a closed ball around the origin in $\mathbb{C}$ with sufficiently large radius, none of the p_t has a zero on ∂W.

Thus the homotopy

$$\frac{p_t}{|p_t|} \colon \partial W \to S^1$$

is defined for all $t \in [0,1]$. Thus

$$\deg\left(\frac{p}{|p|}\right) = \deg\left(\frac{p_0}{|p_0|}\right).$$

Since $p_0(z) = z^m$, the degree of $p_0/|p_0|$ is the same as $\deg(z^m) = m$, and hence

$$\deg\left(\frac{p}{|p|}\right) = m.$$

Thus, if $m > 0$, $p/|p|$ does not extend to all of W, since otherwise its degree had to be zero. Hence p must have a zero inside W.

Hopf Degree Theorem in dimension one

We return our attention to self-maps of S^1. We learned that there is a homotopy class of maps $S^1 \to S^1$ for every integer m. Actually, the following theorem, the one-dimensional case of a famous theorem of Hopf, shows that the degree is a **bijective map**

$$\deg\colon [S^1,S^1] \to \mathbb{Z},\ f \mapsto \deg(f),$$

where $[S^1,S^1] = \mathrm{Hom}(S^1,S^1)/\sim$ denotes the set of equivalence classes of maps from S^1 to S^1 modulo the homotopy relation.

The same is true for every $n \geq 1$: For every $m \in \mathbb{Z}$, there is exactly one homotopy class of maps $S^n \to S^n$. We will get back to this important result later. Today we show:

Hopf Degree Theorem in dimension one

Two maps $f_0,f_1\colon S^1 \to S^1$ are homotopic if and only if they have the same degree.

Proof: We already know that if f_0 and f_1 are homotopic, then $\deg(f_0) = \deg(f_1)$.

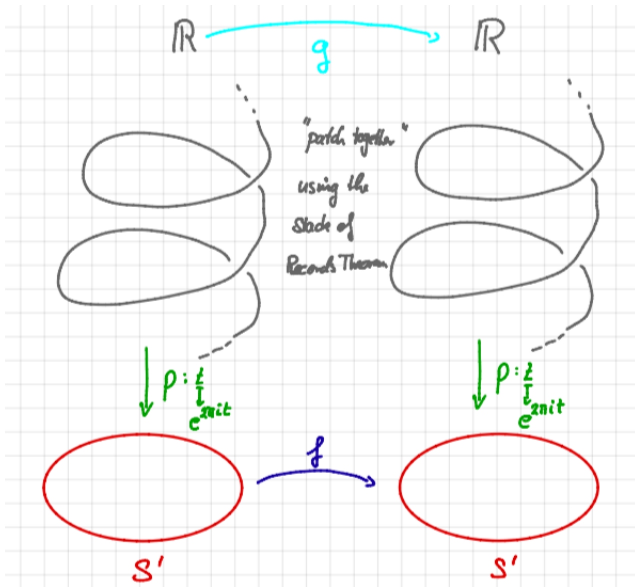
So assume $\deg(f_0) = \deg(f_1)$, and we want to show $f_0 \sim f_1$.

Remember that earlier we used the map p defined by

$$p\colon \mathbb{R} \to S^1,\ t \mapsto e^{2\pi it},$$

and remarked that every smooth map $f\colon S^1 \to S^1$ can be lifted (lift piecewise and then patch together) to a map $g\colon \mathbb{R} \to \mathbb{R}$ with

$$g(t+1) = g(t) + q \text{ for some } q \in \mathbb{Z} \text{ such that } f(p(t)) = p(g(t)).$$



If we can show $q = \deg(f)$, then we **get a homotopy** $f_0 \sim f_1$ **as follows**:

Let g_0 and g_1 be smooth maps $\mathbb{R} \to \mathbb{R}$ with $g_0(t+1) = g_0(t) + q$, $g_1(t+1) = g_1(t) + q$ and $f_0(p(t)) = p(g_0(t))$, $f_1(p(t)) = p(g_1(t))$. Then the map $g_s(t) := sg_1 + (1-s)g_0$ also satifsies $g_s(t+1) = g_s(t) + q$. Note $g_s(t)$ defines a homotopy G from g_0 to g_1 by $G(t,s) = g_s(t)$.

But any homotopy

$$G\colon \mathbb{R} \times [0,1] \to \mathbb{R} \text{ with } G(t+1,s) = G(t,s) + q \text{ for all } t,s$$

induces a well-defined homotopy

$$F\colon S^1 \times [0,1] \to S^1, (z,s) \mapsto p(G(t,s)) \text{ for any } t \in p^{-1}(z).$$

Hence the above $g_s(t)$ induces a homotopy from

$$f_0 = p \circ g_0 \text{ to } p \circ g_1 = f_1.$$

It remains to show:

Claim: $q = \deg(f)$**.**

First, note that if f is **not surjective**, then we can pick a point $y \notin f(S^1)$. This y is automatically a regular value. Since $\#f^{-1}(p) = 0$, we must have $\deg(f) = 0$. In this case, we need to have $q = 0$, i.e. $g(t+1) = g(t)$. For otherwise $p \circ g$ was surjective and hence f would be surjective.

Note that, since the stereographic projection map $S^1 \setminus \{y\} \to \mathbb{R}$ is a diffeomorphism and $\mathbb{R}$ is contractible, this shows directly that $S^1 \setminus \{y\}$ is contractible. Hence f is a map to a contractible space and therefore **homotopic to a constant map**.

Now we assume that f is **surjective**. Let $y \in S^1$ be a regular value of f, and let $z \in f^{-1}(y)$. Since p is surjective, there is a $t \in \mathbb{R}$ with $p(t) = z$. Since y is a regular value, f is a local diffeomorphism around z. Its derivative is related to the one of g by the chain rule

$$df_z \circ dp_t = dp_{g(t)} \circ dg_t.$$

The derivative of $p \colon \mathbb{R} \to S^1$ at any t is

$$dp_t \colon \mathbb{R} \to T_{p(t)}(S^1),\ w \mapsto 2\pi(-\sin(2\pi t), \cos(2\pi t)) \cdot w.$$

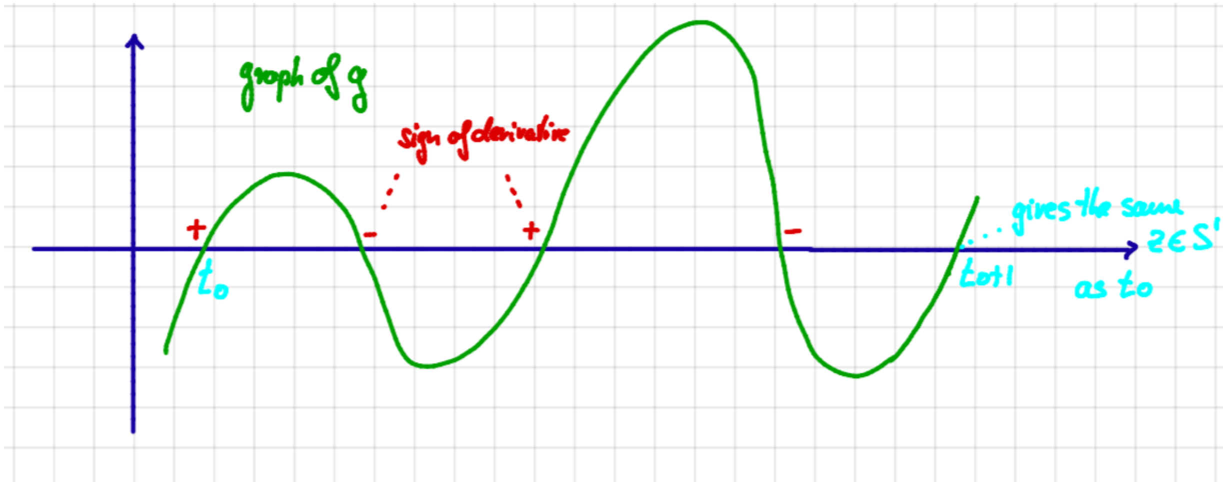
Hence the determinant of dp_t at any t is positive (in fact equal $+2\pi$). Thus the sign of the determinant of df_z equals the sign of $dg_t \in \mathbb{R}$.

As above, let $y \in S^1$ be a regular value of f and $z \in f^{-1}(y)$. Let us fix a $t_0 \in \mathbb{R}$ with $p(t_0) = z$. When we walk from t_0 to $t_0 + 1$ we need to **count** how many preimages of y we collect along the way, **with their orientation (!)**.

We start with the case $q = 0$, i.e. $g(t+1) = g(t)$. It will actually teach us all we need to remember from this proof.

We need to count how often $g(s) = g(t_0)$ with $dg_s = g'(s) > 0$ and how often $g(s) = g(t_0)$ with $dg_s = g'(s) > 0$. Note that since y is **regular**, dg_s is always $\neq 0$ at such those s.

Since g is a smooth function $\mathbb{R} \to \mathbb{R}$, this is now just an exercise from Calculus. Using the periodicity of g, i.e., that $g'(t_0)$ must have the same sign as $g'(t_0+1)$, we see that there are **exactly as many** points s with $g(s) = g(t_0)$ and $dg_s = g'(s) > 0$ as there are points with $g(s) = g(t_0)$ and $dg_s = g'(s) > 0$. Thus $\deg(f) = 0$.



Now assume $q > 0$, and $g(t+1) = g(t) + q$.

Again, we walk from t_0 to $t_0 + 1$ and sum up the orientation numbers of all the preimages of y that we collect along the way. This corresponds to counting how often we have $g(s) = g(t_0) + i$ for some $i = 0,1,\dots,q-1$ and $s \in [t_0,t_0+1]$.

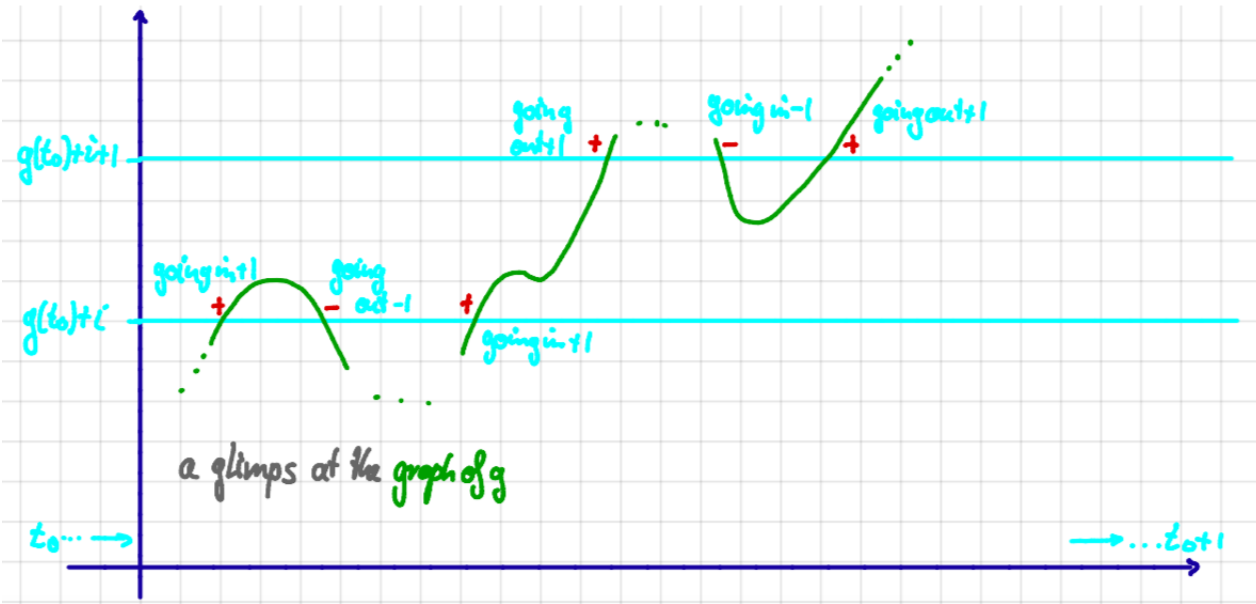
Let us look at one interval $[g(t_0)+i, g(t_0)+i+1]$ at a time. We would like to know how many $s \in [t_0,t_0+1]$ are sent to either $g(t_0)+i$ or $g(t_0)+i+1$ together with the sign of the derivative.

Therefore we look at the preimage

$$g^{-1}([g(t_0)+i, g(t_0)+i+1]).$$

This set is a disjoint union of closed intervals. For each of these intervals the start and endpoints are sent to either $g(t_0)+i$ or $g(t_0)+i+1$.

Let us think of the graph of g passing $g(t_0)+i$with a positive sign of the derivative as **going in with** $+1$ and passing $g(t_0)+i+1$ with a positive sign of the derivative as **going out** $+1$, and the other two alternatives as the ones with -1. Then we see that the graph has to go in with $+1$ for a first time, and has to go out with $+1$ for a last time (since the graph starts at $g(t_0) \le g(t_0)+i$ and ends at $g(t_0)+q \ge g(t_0)+i+1$). In between those two points, the graph is going out with -1 as often as it goes in $+1$ and goes in with -1 as often as it goes out with $+1$.



Thus in total the orientation numbers for $g^{-1}([g(t_0)+i,g(t_0)+i+1])$ add up to $+2$. Repeating this for all $i = 0,1,\dots,q-1$ gives a sum of orientation numbers equal to q, since we have to account for that we counted the inner points twice.

Since the sum of orientation numbers of f equals the one of g, this shows $\deg(f) = q$.

If $q < 0$, the same argument works with signs and directions reversed. **QED**

LECTURE 24

Intersection Numbers and Euler Characteristics

Let us return to one of the initial motivations for the intersetion numbers and see what happens if both X and Z are submanifolds.

Intersection of submanifolds

Let X and Z be submanifolds of Y, with X compact and complementary dimensions $\dim X + \dim Z = \dim Y$, and all are oriented. Then we define the **intersection number of X and Z** in Y to be

$$I(X,Z) := I(i,Z)$$

where $i\colon X \hookrightarrow Y$ is the inclusion map.

Recall that calculating $I(X,Z)$ requires to bring X in transversal position to Z and then take the sum of the orientation numbers at the **finitely many intersection points** in $X \cap Z$.

A point $y \in X \cap Z$ has sign $+1$ if the orientation of $T_y(Y)$ induced by the direct sum decomposition

$$T_y(X) \oplus T_y(Z) = T_y(Y)$$

is the given orientation on $T_y(Y)$, and the sign is -1 if it is the opposite orientation.

Since the **order** of the summands in a direct sum **matters** for the orientation, it is clear that when both X and Z are compact we cannot expect $I(X,Z)$ to be equal $I(Z,X)$ in general.

All we should expect is $I(X,Z) = \pm I(Z,X)$. An example is given by intersecting the two circles on the torus. There we get $I(X,Z) = -I(Z,X)$.

Our next goal is to show that $I(X,Z)$ is homotopy invariant in both variables, and to determine the sign when we flip the factors.

Homotopy Invariance of intersection numbers revisited

Recall that a **deformation** of X in Y is a smooth homotopy from the embedding $i_0\colon X \hookrightarrow Y$ of X in Y to an embedding $i_1\colon X \hookrightarrow Y$ such that each i_t is an embedding.

We know that $I(X,Z)$ is **invariant under deformations of X**, since we calculate it point by point in $X \cap Z$ and a deformation of X is a homotopy of the inclusion. We need to prove that $I(X,Z)$ is **invariant under deformations of Z as well**. In order to show this we generalize our approach.

Let $f\colon X \to Y$ and $g\colon Z \to Y$ be two smooth maps with X and Z compact, all manifolds ae boundaryless and the dimensions satisfy $\dim X + \dim Z = \dim Y$. In particular, that the images of f and g are closed in Y and for g being the inclusion of Z into Y, we are back at the familiar situation.

As always we start with the case of transversal maps and then extend our defintion via homotopy.

In order to do so, we need to say what it means for **two maps** to be transversal:

Transversal maps

We say that $f\colon X \to Y$ and $g\colon Z \to Y$ are **transversal**, denoted $f \pitchfork g$, if

$$df_x(T_x(X)) + dg_z(T_z(Z)) = T_y(Y) \text{ whenever } f(x) = y = g(z).$$

In our situation, the assumption on dimensions implies that if $f \pitchfork g$ then the above **sum is direct**, i.e.

$$df_x(T_x(X)) \oplus dg_z(T_z(Z)) = T_y(Y) \text{ whenever } f(x) = y = g(z).$$

Moreover, the derivatives df_x and dg_z are both injective. Thus these derivatives map $T_x(X)$ and $T_z(Z)$ **isomorphically onto their images**. In particular, the **image spaces inherit an orientation** from X and Z, respectively.

Intersection numbers for maps

We define the **local intersection number** at (x,z) to be $+1$ if the direct sum orientation of $df_x(T_x(X)) \oplus dg_z(T_z(Z))$ equals the given orientation of $T_y(Y)$, and -1 otherwise.
Then $I(f,g)$ is defined as the **sum of the local intersection numbers** of all pairs (x,z) at which $f(x) = g(z)$.

When $g\colon Z \hookrightarrow Y$ is the inclusion map of a submanifold then $f \pitchfork g$ if and only if $f \pitchfork Z$, and if so $I(f,g) = I(f,Z)$. So everything remained consistent so far.

In the definition, we quietly assumed that the sum is **finite**. We should better check this! To do so, we are going to look at intersections from yet another angle. It will seem artificial at first glance, but it is actually a very useful perspective. For it can be generalized to many other situations, e.g. in Algebraic Geometry.

Let Δ denote the **diagonal** of $Y \times Y$, i.e. the set of points (y,y), and let

$$f \times g\colon X \times Z \to Y \times Y,\ (x,z) \mapsto (f(x),g(z))$$

be the product map. Then we have

$$f(x) = g(z) \iff (x,z) \in (f \times g)^{-1}(\Delta).$$

The dimension of $\dim(X \times Z)$ is $\dim X + \dim Z = \dim Y$, and the dimension of Δ is $\dim Y$. Thus $\dim(X \times Z) = \operatorname{codim}(\Delta)$ in $Y \times Y$. Hence if $f \times g \pitchfork \Delta$, then $(f \times g)^{-1}(\Delta)$ is a **compact zero-dimensional manifold**. Hence it is a **finite set**.

Transversality, $f \times g \pitchfork \Delta$, will follow from the following lemma from linear algebra:

Help from Linear Algebra

Let V be a finite dimensionsal vector space, and U and W be vector subspaces of V. Let Δ be the diagonal in $V \times V$. Then

$$U \oplus W = V \iff U \times W \oplus \Delta = V \times V.$$

Assume now that $U \oplus W = V$, and in addition that U and W are oriented, and give V the direct sum orientation. We assign Δ the orientation carried from V by the natural isomorphism $V \to \Delta$ which sends $v \mapsto (v,v)$. Then the **product orientation** on $V \times V$ **agrees** with the **direct sum orientation** induced from $U \times W \oplus \Delta$ if and only if **W is even dimensional**.

We skip the proof of the lemma which can be found in [GP], page 113+114. Instead we are ging to exploit its implications.

Transversality and diagonals

The maps f and g are transversal if and only if $f \times g$ is transversal to Δ, i.e.

$$f \pitchfork g \iff (f \times g) \pitchfork \Delta.$$

If $f \pitchfork g$, then

$$I(f,g) = (-1)^{\dim Z} I(f \times g,\Delta).$$

Proof: We apply the lemma to $U = df_x(T_x(X))$, $W = dg_z(T_z(Z))$, and $V = T_y(Y)$. Then the first part of the lemma yields the equivalence of transversality. The second part implies the formula on the signs, keeping in mind that we know $X \cap Z = (f \times g)^{-1}(\Delta)$. **QED**

The main point of the previous effort is that considering intersections as preimages of the diagonal allows us to extend our definition:

Intersection numbers via diagonals

For maps f and g as above which are not necessarily transversal, we **define** $I(f,g)$ to be

$$I(f,g) := (-1)^{\dim Z} I(f \times g,\Delta).$$

Moreover, the desired properties of $I(f,g)$ follow right away:

Homotopy Invariance

If f_0 and g_0 are **homotopic** to f_1 and g_1, respectively, i.e. $f_0 \sim f_1$ and $g_0 \sim g_1$, then

$$I(f_0,g_0) = I(f_1,g_1).$$

Proof: If F is a homotopy from f_0 to f_1 and G is a homotopy from g_0 to g_1, then $F \times G$ is a homotopy from $f_0 \times g_0$ to $f_1 \times g_1$. Then the homotopy invariance of $I(f \times g,\Delta)$ which we proved before implies the invariance of $I(f,g)$. **QED**

Recovering the previous definition

If Z is a **submanifold** of Y and $i\colon Z \to Y$ is its **inclusion map**, then $I(f,i) = I(f,Z)$ for any map $f\colon X \to Y$ (with the usual assumption that X is compact and complementary dimensions).

Proof: This follows just from the definition of $f \pitchfork Z$. If f is arbitrary, then we use the homotopy invariance of both $I(f,i)$ and $I(f,Z)$. **QED**

When we applied $I(f,Z)$ to the case $\dim X = \dim Y$ and $Z = \{y\}$, we obtained the degree of f. Let us check that this definition still works in the new setup.

Degrees are still well defined

If $\dim X = \dim Y$ and Y is connected, then $I(f,\{y\})$ is the same for every $y \in Y$. Thus $\deg(f) = I(f,\{y\})$ is well defined.

Proof: Since Y is connected and a smooth manifold, it is path-connected. Hence the inclusion maps i_0 and i_1 for any two points $y_0,y_1 \in Y$ are homotopic. Therefore

$$I(f,\{y_0\}) = I(f,i_0) = I(f,i_1) = I(f,\{y_1\}).$$

QED

How signs switch when we flip maps

When we flip the order of the maps, we get

$$I(f,g) = (-1)^{(\dim X)(\dim Z)} I(g,f).$$

Proof: We must compare the direct sum orientations of

$$T_y(Y) = df_x(T_x(X)) \oplus dg_z(T_z(Z)) \text{ and } T_y(Y) = dg_z(T_z(Z)) \oplus df_x(T_x(X)).$$

As we remarked in a previous lecture, switching the order of the summands reuqires to apply $\dim X) \cdot (\dim Z)$ many transpositions of the basis vectors. This gives the sign in the assertion. **QED**

Applying this result to the inclusions of two submanifolds yields the follwoing formula for signs when we switch the order of factors in intersection numbers:

How signs switch when we flip submanifolds

If X and Z are both compact submanifolds, then

$$I(X,Z) = (-1)^{(\dim X)(\dim Z)} I(Z,X).$$

Self-intersections and Euler Characteristic

As a special case, we can look at at the **self-intersection number** $I(X,X)$ when $\dim Y = 2\dim X$.

But the above sign formula implies that **if** $\dim X$ **is odd**, then

$$I(X,X) = (-1)^{(\dim X)^2} I(X,X) = -I(X,X) \text{ and hence } I(X,X) = 0.$$

As a consequence we also get $I_2(X,X) = I(X,X) \mod 2 = 0$.

This observation yields an insight into the nonorientability of some manifolds.

Obstruction for orientability

Let Y be any smooth manifold of **even dimension**. Then we can calculate the mod 2-self-intersection number $I_2(X,X)$ for any compact submanifold $X \subset Y$ of dimension $\dim X = \frac{1}{2}\dim Y$ as in the previous lecture without assuming orientability of Y.
If one of these self-intersection numbers **fails to vanish**, then Y is **not orientable**.

For example, the central circle in the Mobius strip has nonzero mod 2 self-intersection number, so the Möbius strip is nonorientable.

Self-intersection numbers can be used to define a very powerful and famous invariant. You will see different constructions for this invariant later in your mathematical life. Here is the first:

Euler Characterstics

Let Y be a compact, oriented manifold. Its **Euler characteristic**, denoted $\chi(Y)$, is defined to be the self-intersection number of the diagonal Δ in $Y \times Y$:

$$\chi(Y) := I(\Delta,\Delta).$$

Note: Our methods and contruction here makes it look like a differential invaiant. But note that the Euler characteristic is a **topological invariant** in the sense that it only depends on the topology of Y and not the differentiable structure.

As a first calculation of an Euler number, we deduce from the previous observations:

Euler characteristic in odd dimensions vanishes

The Euler characteristic of an odd-dimensional, compact, oriented manifold is zero.

Proof: If $\dim Y$ is odd, then $\dim \Delta = \dim Y$ is odd. Hence

$$\chi(Y) = I(\Delta,\Delta) = (-1)I(\Delta,\Delta) = 0$$

must be zero. **QED**

Lefschetz Fixed-Point Theorem

For a (smooth) map $f \to X \to X$ it is often desirable to know if the equation $f(x) = x$ has a solution, i.e., if f has a fixed point. In particular, we could ask how many fixed point does f have. On a compact oriented manifold X, intersection theory can help us answering that question.

Again it turns out to formulate the question first using diagonals. A point $x \in X$ is a fixed point of f if and only if $(x,f(x))$ is a point in the intersection of the graph $\Gamma(f)$ of f with the diagonal Δ of X in $X \times X$:

$$f(x) = x \iff (x,f(x)) \in \Delta \cap \Gamma(f).$$

Both Δ and $\Gamma(f)$ are submanifolds of X and their dimensions satisfy

$$\dim \Delta + \dim \Gamma(f) = \dim X + \dim X = \dim(X \times X).$$

Moreover, both receive an orientation from X via the natural diffeomorphism $X \to \Delta$ and $X \to \Gamma(f)$.

Thus we may use intersection theory to count their common points (if it is a finite number):

Global Lefschetz numbers

The **global Lefschetz number of f**, denoted by $L(f)$, is defined to be the intersection number

$$L(f) := I(\Delta, \Gamma(f))$$

Note: Again, our methods and contruction here makes it look like a differential invaiant. But the **Lefschetz number** is a **topological invariant** in the sense that it only depends on the topology of X and not the differentiable structure

Of course, f may have an infinite number of fixed points, as the identity map demonstrates. Thus the sense in which $L(f)$ measures the fixed-point set is somewhat subtle. However, we shall see that when the fixed points of f do happen to be finite, then $L(f)$ may be calculated directly in terms of the local behavior of f around its fixed points.

The significance of Lefschetz numbers may be illustrated by the following immediate consequences of the intersection theory approach. The following famous theorem in its many variations plays a crucial role in many branches in mathematics:

Smooth Lefschetz Fixed-Point Theorem

Let $f\colon X \to X$ be a smooth map on a compact orientable manifold. If $L(f) \neq 0$, then f has a **fixed point**.

Proof: If f has no fixed points, then Δ and $\Gamma(f)$ are disjoint, and hence trivially transversal. Consequently,

$$L(f) = I(\Delta, \Gamma(f)) = 0.$$

QED

Since $L(f)$ is an intersection number, we immediately get:

Lefschetz numbers are homotopy invariant

If $f_0 \sim f_1$, then $L(f_0) = L(f_1)$.

The graph of the identity map is just the diagonal itself. thus $L(\mathrm{Id}) = \chi(X)$ is just the Euler characteristic of X:

Lefschetz numbers and Euler characteristics

If f is homotopic to the identity, then $L(f)$ equals the Euler characteristic of X. In particular, if X admits any smooth map $f\colon X \to X$ that is **homotopic to the identity** and has **no fixed points**, then $\chi(X) = 0$.

Trnasversality is crucial for intersection theory. So let us call a smooth map $f\colon X \to X$ a **Lefschetz map** if $\Gamma(f) \pitchfork \Delta$.

Note that a Lefschetz map has only finitely many fixed points, since there are only finitely many points in the coplementary intersection $\Gamma(f) \cap \Delta$. Also not that the converse is false. Since Lefschetz maps are defined by a transversality condition, it should be plausible that most maps are Lefschetz.

Most maps are Lefschetz

Every smooth map $f\colon X \to X$ is homotopic to a Lefschetz map.

Proof: In the lecture on transversality we proved the following fact: Given $X \subset \mathbb{R}^N$ and $f\colon X \to X$, we can find an open ball S in $\mathbb{R}^N$ and a smooth map $F\colon X \times S \to X$ such that $F(x,0) = f(x)$ and $s \mapsto F(x,s)$ is a submersion for each $x \in X$.

Given this F, the map

$$G\colon X \times S \to X \times X,\ (x,s) \mapsto (x,F(x,s))$$

is also a submersion. For suppose that $G(x,s) = (x,y)$. Since G acts like the identity on the first X factor, the image of $dG_{(x,s)}$ contains a vector of the form (u,w) for every $u \in T_x(X)$. Since G restricted to $\{x\} \times S$ is a submersion to $\{x\} \times X$, the image also contains a vector of the form $(0,w)$ for every $w \in T_y(X)$. Therefore G is a submersion.

In particular, $G \pitchfork \Delta$. By the Transversality Theorem, for almost every s the map

$$X \to X \times X,\ \ x \mapsto G(x,s)$$

is transversal to Δ.

Now we observe that the image of this map is just the graph of the map $x \mapsto F(x,s)$. Hence, for any s, the map

$$X \to X, x \mapsto F(x,s)$$

is Lefschetz and homotopic to f. **QED**

Let us try to understand Lefschetz maps better. Suppose that x is a **fixed point** of f. As we showed in the exercises, the **tangent space of** $\Gamma(f)$ in $T_x(X \times X)$ is the graph of the derivative $df_x \colon T_x(X) \to T_x(X)$. Moreover, the **tangent space of the diagonal** Δ is the diagonal Δ_x in $T_x(X) \times T_x(X)$.

This implies

$$\Gamma(f) \pitchfork \Delta \text{ in } (x,x) \iff \Gamma(f) + \Delta_x = T_x(X) \times T_x(X).$$

As $\Gamma(df_x)$ and Δ_x are vector subspaces of $T_x(X) \times T_x(X)$ with **complementary dimension**, we have

$$\Gamma(f) + \Delta_x = T_x(X) \times T_x(X) \iff \Gamma(f) \cap \Delta_x = \{0\}.$$

But $\Gamma(f) \cap \Delta_x = \{0\}$ just means that df_x does **not have a fixed point**. In the language of linear algebra, this means that df_x **has no eigenvector of eigenvalue** $+1$.

Lefschetz fixed points

We call a fixed point x a **Lefschetz fixed point of f** if df_x has no nonzero fixed point, i.e., if the eigenvalues of df_x are all unequal to $+1$.
This shows that f is a **Lefschetz map** if and only if **all its fixed points are Lefschetz**.

Notice that the Lefschetz condition on x is simply the infinitesimal analog of the demand that x be an **isolated fixed point** of f. We have met Lefschetz fixed points on Exercise Set 6.

Local Lefschetz fixed points

If x is a Lefschetz fixed point, we denote the orientation number ± 1 of (x,x) in the intersection $\Delta\Gamma(f)$ by $L_x(f)$. It is called the **local Lelschetz number of f at x**.
For **Lefschetz maps**, we have

$$L(f) = \sum_{f(x)=x} L_x(f)$$

where the sum is taken over the finite number of fixed points of f.

Hence in order to calculate the global Lefschetz number $L(f)$, it suffices to calculate all the local Lefschetz numbers $L_x(f)$.

So let us have a closer look at the $L_x(f)$. First we observe that the condition for x to be a Lefschtz fixed point means that, for the identity map I on $T_x(X)$, $df_x - I$ is still an isomorphism on $T_x(X)$, since the kernel of $df_x - I$ is the space of fixed points of df_x. (We used that also to solve the exercise on Lefschetz fixed points and Lefschetz maps.) Now we observe:

Local Lefschetz numbers and orientations

Let x be a Lefschetz fixed point of f. Then $L_x(f)$ is $+1$ if the isomorphism $df_x - I$ preserves orientations on $T_x(X)$, and it is -1 if $df_x - I$ reverses orientations.
In other words,

$$L_x(f) = \operatorname{sign}(\det(df_x - I)).$$

Again, we skip the proof of this exercise in linear algebra ([GP] pages 121+122) and rther look at an important example.

The Euler characteristic of the two-sphere

As an example, we consider $X = S^2 \subset \mathbb{R}^3$. Let $g\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the rotation by $\pi/2$ about the z-axis. The matrix representing g in the standard basis is

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In particular, g is a linear map and its own derivative dg_x at any point is just g.

Now let $f\colon S^2 \to S^2$ be the restriction of g to S^2. Then f has **exactly two fixed points**, the north pole $N = (0,0,+1)$ and the south pole $S = (0,0,-1)$. At both poles, $df_x\colon T_x(S^2) \to T_x(S^2)$ can be represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Hence $\det(df_x - I) = 2$. In particular, f is a **Lefschetz map**, and the sign of the determininant is $+1$ at both poles. Thus $L(f) = L_N(f) + L_S(f) = 2$.

Any rotation with positive determinant is **homotopic to the identity** map of S^2. For a concrete homotopy from g to the identity map we can take

$$F(-,t) = \begin{pmatrix} t & t-1 & 0 \\ 1-t & t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By our previous discussion, this implies $L(f) = \chi(S^2)$. Hence we have proved the folowing important fact:

Euler characterstic of the two-sphere

The Euler characteristic of S^2 is 2: $\chi(S^2) = 2$.

As a consequence we get:

Self-maps on the two-sphere

Every map $S^2 \to S^2$ that is homotopic to the identity must possess a fixed point. In particular, the **antipodal map** $x \mapsto -x$ is **not** homotopic to the identity.

LECTURE 25

Euler characteristic and surfaces

Euler characteristic and surfaces

How can we prove that the two surfaces

are not homeomorphic? One way is to use the Euler characteristic.

The 'classical' definition

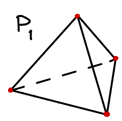
P polyhedron in $\mathbb{R}^n$ with a_0 vertices, a_1 edges, a_2 faces (2-dim 'sides'), ..., a_n n-dim 'sides'.

$$\chi(P) := \sum_{i=0}^{n} (-1)^i a_i .$$

Note: $\chi(P) \in \mathbb{Z}$.

$X \subseteq \mathbb{R}^n$ with $X \cong P$ then $\chi(X) = \chi(P)$, and is independent of P as long as $P \cong X$. (Poincaré - Alexander)

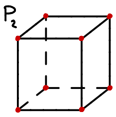
Examples: (1) P_1



$a_0 = 4$ (V)
$a_1 = 6$ (E)
$a_2 = 4$ (F)

$\chi(P_1) = a_0 - a_1 + a_2 = 2.$

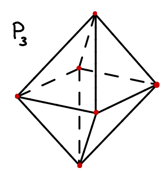
(2) P_2



$a_0 = 8$
$a_1 = 12$
$a_2 = 6$

$\chi(P_2) = a_0 - a_1 + a_2 = 2.$

(3) P_3



$a_0 = 6$
$a_1 = 12$
$a_2 = 8$

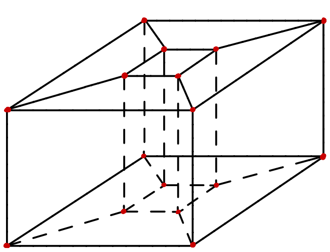
$\chi(P_3) = a_0 - a_1 + a_2 = 2.$

(See also dodecahedron and icosahedron.)

Note: $P_1 \cong P_2 \cong P_3 \cong S^2$, $\chi(S^2) = 2$ (Euler).

The Euler characteristic is a topological invariant: if $X \cong Y$ then $\chi(X) = \chi(Y)$. In other words: if $\chi(X) \neq \chi(Y)$ then $X \ncong Y$.

Example: P

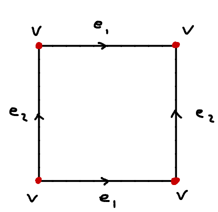


$a_0 = 16$
$a_1 = 32$
$a_2 = 16$

$\chi(P) = a_0 - a_1 + a_2 = 16 - 32 + 16 = 0.$

$P \cong T^2$. $\chi(T^2) = 0$.

As $\chi(T^2) = 0 \neq \chi(S^2)$, T^2 and S^2 can't be homeomorphic, i.e. $T^2 \ncong S^2$.

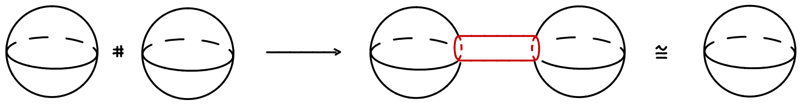


$a_0 = 1$
$a_1 = 2$
$a_2 = 1$

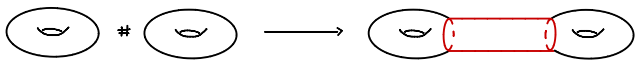
$\Rightarrow \chi = 0.$

X, Y surfaces. The connected sum $X \# Y$ is (roughly) obtained by removing a (small) disk from each of X and Y and connecting the resulting holes with a cylinder.

<u>Examples</u>: 1) $S^2 \# S^2 \cong S^2$ (For an arbitrary surface X, $S^2 \# X \cong X$.)

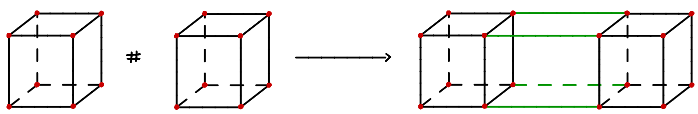


2) $T^2 \# \overset{g}{\cdots} \# T^2 = \Sigma_g$ (genus g)



$\chi(X \# Y) = ?$

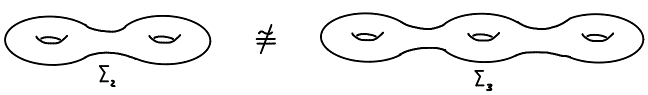
<u>Example</u>: As $S^2 \# S^2 \cong S^2$, $\chi(S^2 \# S^2) = 2$. Let P be a cube. Then we remove 2 faces and add 4, add 4 edges and no vertices



when constructing $P \# P$ as above. Hence, $\chi(P \# P) = \chi(P) + \chi(P) + 0 - 4 + (4-2) = \chi(P) + \chi(P) - 2 = 2$.

<u>Theorem</u>: X, Y surfaces. Then $\chi(X \# Y) = \chi(X) + \chi(Y) - 2$.

Thus $\chi(\Sigma_g) = 2(1-g)$, and hence it follows that Σ_2 and Σ_3 are not homeomorphic as $\chi(\Sigma_2) = 2(1-2) = -2 \neq \chi(\Sigma_3) = 2(1-3) = -4$.



<u>Theorem</u> (Classification of surfaces): Two connected compact surfaces are homeomorphic if and only if they have the same Euler characteristic and the same number of boundary components, and both are orientable or both are non-orientable.

By a deep theorem in differential topology any pair of homeomorphic smooth surfaces are diffeomorphic. (Holds for dim ≤ 3.) The first example of homeomorphic but not diffeomorphic was given by Milnor where he constructed a smooth 7-manifold homeomorphic but not diffeomorphic to the standard S^7.

A proof of the classification of surfaces (as stated above) is given by Hirsch (GTM 33, SpringerLink). Another proof (and statement) is given by Lawson (OUP, GTinM 9).

How do we relate the 'classical' and the 'intersection number' definition of the Euler charateristic (when they both make sense)?

<u>The Poincaré-Hopf theorem</u>

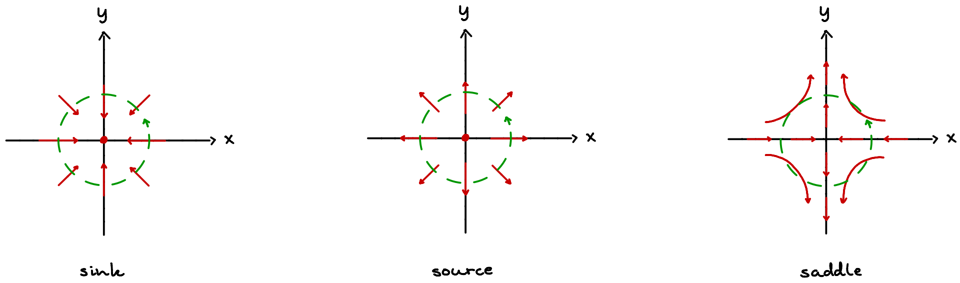
The Poincaré-Hopf theorem provides a way of computing the Euler characteristic by relating it to the indices of vector fields.

A smooth manifold M^n is parallelizable if the tangent bundle TM (Lecture 15) is trivial: $TM \cong M \times \mathbb{R}^n$, $T_pM \longrightarrow \{p\} \times \mathbb{R}^n$.

Using the Poincaré-Hopf theorem we can compute the Euler characteristic for every parallelizable manifold M: $\chi(M) = 0$. Thus, $\chi(M) = 0$ for all Lie groups M, as all Lie groups are parallelizable

Consider the following three (smooth) vector fields in $\mathbb{R}^2$:

(1) $F_1(x,y) = (-x,-y)$ (2) $F_2(x,y) = (x,y)$ (3) $F_3(x,y) = (-x,y)$



The **index** of F_i at $(0,0)$ counts the number of times F_i rotates completely while traversing the (small) circle centered at $(0,0)$ with rotation of F_i counterclockwise gives $+1$ and rotation of F_i clockwise gives -1.

Hence, $\text{ind}_0 F_1 = +1$, $\text{ind}_0 F_2 = +1$ and $\text{ind}_0 F_3 = -1$.

A **vector field** on a manifold M in $\mathbb{R}^N$ is a smooth map $F: M \longrightarrow \mathbb{R}^N$ such that $F(x) \in T_x M$ for every $x \in M$.

F vector field in $\mathbb{R}^k$ with an isolated zero at 0. We define the **index** of F at 0 as

$$\text{ind}_0(F) := \deg(u) \quad , \quad u: S_\varepsilon \longrightarrow S^{k-1}, \quad x \longmapsto F(x)/\|F(x)\|.$$

Note that F_1 corresponds to the antipodal map on S^1, hence $\text{ind}_0(F_1) = \deg(F_1) = (-1)^2 = 1$. F_2 corresponds to the identity map, hence $\text{ind}_0(F_2) = \deg(F_2) = 1$. Finally, F_3 corresponds to the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix} \quad , \quad \det(A) = -1$$

with $\text{ind}_0(F_3) = \deg(F_3) = -1$.

To define the index of vector fields at isolated zeros on arbitrary manifolds, use local parametrization or charts. The index does **not** depend on the choice of local parametrization or chart.

$\varphi: \mathcal{U} \longrightarrow M$ local parametrization, $\varphi(0) = x$, $0 \in \mathcal{U} \subseteq \mathbb{R}^k$. The **pullback** vector field $\varphi^* F$ on $\mathcal{U}$ is defined by

$$\varphi^* F(u) = d\varphi_u^{-1} F(\varphi(u)) \quad , \quad u \in \mathcal{U}. \qquad (d\varphi_u : T_{\varphi(u)} M \xrightarrow{\cong} \mathbb{R}^k)$$

If F has an isolated zero at x, $\varphi^* F$ has an isolated zero at 0. Hence,

$$\text{ind}_x(F) := \text{ind}_0(\varphi^* F).$$

<u>Theorem</u> (Poincaré – Hopf): If F is a smooth vector field on a compact oriented manifold M with only finitely many zeros. Then

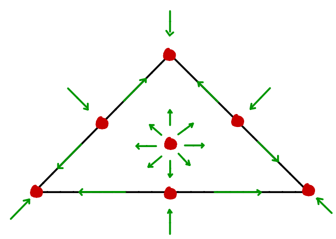
$$\sum_x \text{ind}_x(F) = \chi(M).$$

A proof using (local) Lefschetz numbers is presented in Guillemin and Pollack, pp. 134 – 137.

As a consequence, we have the following:

Theorem: For a smooth oriented compact 2-manifold, the 'classical' and the 'intersection number' definition of the Euler characteristic agree.

Proof (sketch): Triangulate the manifold (can always be done; see Cairns (1935)). Define a vector field F on M with a source on each face, a saddle on each edge and a sink at each vertex:



For each source there is a zero of F of index 1, and similarly each saddle has a zero of index −1 and each sink has a zero of index 1.

By Poincaré–Hopf, $\sum_x \text{ind}_x(F) = \chi(M) = I(\Delta,\Delta)$ [Δ: diagonal in $M \times M$].

But this sum is precisely $a_0 - a_1 + a_2$ with $a_0 = \#$ vertices, $a_1 = \#$ edges and $a_2 = \#$ faces. □

The theorem also holds for higher dimensions.

The Euler characteristic can be defined in many ways. One way that uses homology is as follows: For a space X the ith Betti number of X, $b_i(X)$, is the rank of $H_i(X)$ (rank of an abelian group is somewhat like the dimension of a vector space).

$b_0(X)$ is the number of path components in X. $b_i(X)$ measure a form of higher-dimensional connectivity of X.

The Euler characteristic of X is then given by

$$\chi(X) = \sum_i (-1)^i b_i(X).$$

LECTURE 26

Two dimensional Quantum Field Theories

2-dim TQFTs

TQFTs are rich geometric gadgets, encoding many fundamental manifold invariants. Roughly speaking, they capture the idea of cutting a manifold into pieces (cobordisms), attaching invariants to these pieces, and then gluing these invariants together to obtain an invariant of the original manifold.

A TQFT is a symmetric monoidal functor $Z : n\mathrm{Cob} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$ (linear category). When $n = 2$ these are equivalent to Frobenius algebras:

Theorem: $2\mathrm{TQFT}_{\mathbb{C}} \simeq \mathrm{cFA}_{\mathbb{C}}$

Categorical preliminaries

A category $\mathcal{C}$ consists of

- objects : $A, B, C, \dots$ $(A \in \mathcal{C})$
- morphisms ('arrows'): $A \xrightarrow{f} B$ $(f \in \mathcal{C}(A,B))$

subject to:

1) Given $A \xrightarrow{f} B$, $B \xrightarrow{g} C$ we can compose: $A \xrightarrow{g \circ f} C$
2) Composition is associative: $h \circ (g \circ f) = (h \circ g) \circ f$, $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$
3) For every $A \in \mathcal{C}$ there is a (unique) identity morphism 1_A (id_A): $f \circ 1_A = f = 1_B \circ f$.

Examples: $\mathrm{Vect}_{\mathbb{C}}$: vector spaces over $\mathbb{C}$, linear maps

Top : topological spaces, continuous maps

A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ consists of

- map from the objects of $\mathcal{C}$ to the objects of $\mathcal{D}$
- map $F_{A,B} : \mathcal{C}(A,B) \longrightarrow \mathcal{D}(F(A), F(B))$

subject to:

1) Given $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$ $F_{A,C}(g \circ f) = F_{B,C}(g) \circ F_{A,B}(f)$ (F covariant)
2) $F_{A,A}(1_A) = 1_{F(A)}$ for all $A \in \mathcal{C}$.

$F, G : \mathcal{C} \longrightarrow \mathcal{D}$ functors. A natural transformation $\eta : F \Rightarrow G$ assigns to each $A \in \mathcal{C}$ a morphism $\eta(A) : F(A) \longrightarrow G(A)$ in $\mathcal{D}$ such that for each $A \xrightarrow{f} B$ in $\mathcal{C}$

$$\begin{array}{ccc} F(A) & \xrightarrow{F_{A,B}(f)} & F(B) \\ \eta(A)\downarrow & \circ & \downarrow \eta(B) \\ G(A) & \xrightarrow{G_{A,B}(f)} & G(B). \end{array}$$

η is natural isomorphism if $\eta(A), \eta(B)$ are isomorphisms: $X \underset{\beta}{\overset{\alpha}{\rightleftarrows}} Y$ $\beta \circ \alpha = 1_X$, $\alpha \circ \beta = 1_Y$, $X \cong Y$.

We write $F \cong G$.

$\mathcal{C}$ and $\mathcal{D}$ are equivalent if there exists functors $F : \mathcal{C} \longrightarrow \mathcal{D}$, $G : \mathcal{D} \longrightarrow \mathcal{C}$ such that $1_{\mathcal{C}} \cong G \circ F$, $1_{\mathcal{D}} \cong F \circ G$.

A strict monoidal category $(\mathcal{C}, \otimes, I)$ is a category $\mathcal{C}$ with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is associative and with an object $I \in \mathcal{C}$ which is a left and right unit for $\otimes$. $(\mathcal{C}, \otimes, I)$ is symmetric if for each pair of objects A, B in $\mathcal{C}$ there is a twist (braid) map $\tau_{A,B} : A \otimes B \longrightarrow B \otimes A$ subject to:

- for every pair $A \xrightarrow{f} A'$, $B \xrightarrow{g} B'$ in $\mathcal{C}$

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\tau_{A,B}} & B \otimes A \\ f \otimes g \downarrow & \circ & \downarrow g \otimes f \\ A' \otimes B' & \xrightarrow[\tau_{A',B'}]{} & B' \otimes A' \end{array}$$

- for every triple $A, B, C \in \mathcal{C}$

$$\begin{array}{ccccc} A \otimes B \otimes C & & \xrightarrow{\tau_{A,B \otimes C}} & & B \otimes C \otimes A \\ & \tau_{A,B} \otimes 1_C \searrow & \circ & \nearrow 1_B \otimes \tau_{A,C} & \\ & & B \otimes A \otimes C & & \end{array} \qquad \begin{array}{ccccc} A \otimes B \otimes C & & \xrightarrow{\tau_{A \otimes B, C}} & & C \otimes A \otimes B \\ & 1_A \otimes \tau_{B,C} \searrow & \circ & \nearrow \tau_{A,C} \otimes 1_B & \\ & & A \otimes C \otimes B & & \end{array}$$

- $\tau_{B,A} \circ \tau_{A,B} = 1_{A \otimes B}$ for every pair $A, B \in \mathcal{C}$.

Cobordisms

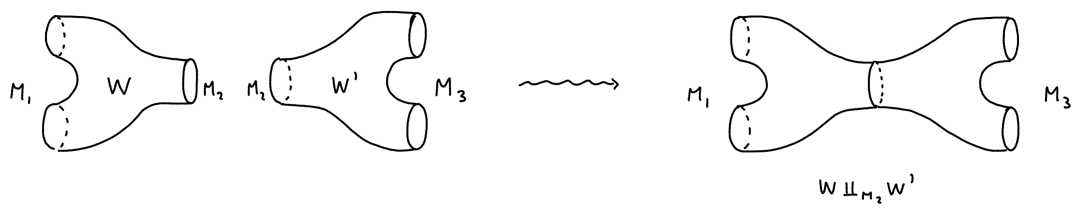
(See exercise set 11, problem 5.) We will only consider 2-dim cobordisms. (Manifolds are always assumed to be compact smooth.)

Let 2Cob be the category with

- objects: closed oriented 1-manifolds
- morphisms: $M, N \in 2Cob$, a morphism from M to N is a cobordism W from M to N, i.e. W is an oriented 2-manifold equipped with an orientation-preserving diffeomorphism $\partial W \xrightarrow{\cong} M \amalg \bar{N}$.

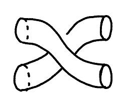
W, W' define the same morphism in 2Cob if there is an orientation-preserving diffeomorphism $W \xrightarrow{\cong} W'$ (extending $\partial W \cong M \amalg \bar{N} \cong \partial W'$). For any $M \in 2Cob$, 1_M is represented by the cobordism $W = M \times I$.

$M_1, M_2, M_3 \in 2Cob$, cobordisms $W : M_1 \longrightarrow M_2$, $W' : M_2 \longrightarrow M_3$. The composition $W' \circ W : M_1 \longrightarrow M_3$ is defined to be the morphism represented by $W \amalg_{M_2} W'$.



Note: To give $W \amalg_{M_2} W'$ a smooth structure, we can make a choice of a smooth collar around M_2 inside of W and W'. Different choices of collars (can) lead to different smooth structures on $W \amalg_{M_2} W'$, but the resulting cobordisms are diffeomorphic (but there is no canonical diffeomorphism). See Milnor's Lectures on the h-cobordism theorem for full details.

$(2Cob, \amalg, \emptyset)$ is a monoidal category.

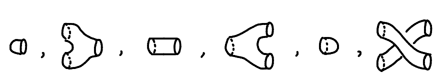
The cobordism induced by the twist diffeomorphism $M \amalg M' \longrightarrow M' \amalg M$ is the twist cobordism: 

$(2Cob, \amalg, \emptyset, T)$ is a symmetric monoidal category.

2Cob can be described explicitly in terms of generators and relations, where we use the classification of surfaces.

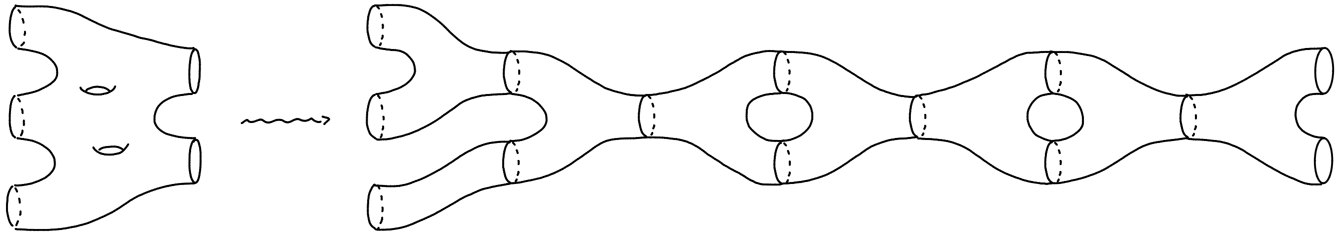
A generating set for a monoidal category is a set S of morphisms such that all morphisms in the category can be obtained from elements in S by composition and $\otimes$.

A skeleton of 2Cob (full subcategory comprising exactly one object from each isomorphism class) is the full subcategory $\{\underline{0}, \underline{1}, \underline{2}, \dots\}$ with $\underline{n} = \amalg S^1$. Let 2Cob denote this skeleton.

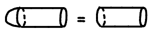
Theorem: 2Cob is generated by the six cobordisms: 

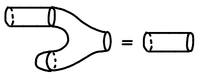
(We will use the classification of surfaces for this theorem.)

The normal form of a connected surface with m in-boundaries, n out-boundaries, genus g is a decomposition of the surface into a number of basic cobordisms.

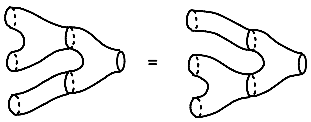


The relations we need are as follows:

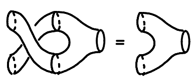
1. Identity:  etc.
2. Unit and counit:

 , etc.

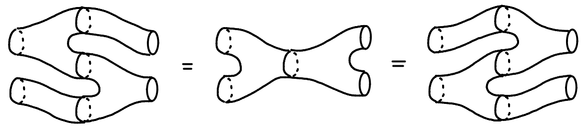
3. Associativity and coassociativity:

 , etc.

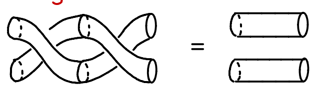
4. Commutativity and cocommutativity:

 , etc.

5. Frobenius:



6. Twisting:

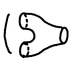


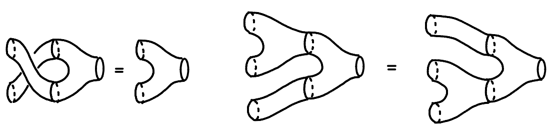
These relations are sufficient but not minimal.

<u>2-dim TQFTs and commutative Frobenius algebras</u>

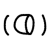
A 2-dim TQFT is a symmetric monoidal functor $Z: 2\mathrm{Cob} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$.

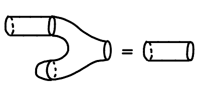
Let $Z(S^1) = Z(\underline{1}) = A$. Then $Z(\underline{n}) = A^{\otimes n}$. Furthermore,

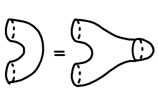
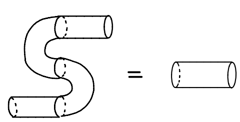
$Z($  $): A \otimes A \xrightarrow{m} A$

m is commutative and associative: 

Moreover,

$Z($$) = \mathbb{C} \xrightarrow{u} A$

$Z($$) = A \xrightarrow{tr} \mathbb{C}$, 

 $: A \otimes A \xrightarrow{m} A \xrightarrow{tr} \mathbb{C}$ is nondegenerate:  (use the Frobenius relation).

A is a commutative Frobenius algebra (i.e. commutative $\mathbb{C}$-algebra together with a linear map $tr: A \longrightarrow \mathbb{C}$ such that $(a,b) \longmapsto tr(ab)$ is nondegenerate.

<u>Example</u>: (1) $A = M_n(\mathbb{C})$, $tr((a_{ij})) = \sum_i a_{ii}$.

(2) $A = \mathbb{C}[t]/(t^n - 1)$, $tr(1) = 1$, $tr(t^i) = 0$ for $i = 1, 2, \ldots, n-1$.

<u>Theorem</u>: $2\mathrm{TQFT} \simeq \mathrm{cFA}_{\mathbb{C}}$.

For a proof see J. Kock's book (CUP, No. 59 of LMSST, 2003).

TQFTs produce topological invariants: every closed surface can be considered as a cobordism from $\emptyset$ to $\emptyset$, so its image under a TQFT is a linear map $\mathbb{C} \longrightarrow \mathbb{C}$ (i.e. a constant) which is a topological invariant of the surface.

TQFTs and physics

TQFTs posses certain features that we expect from quantum gravity.

The closed manifolds represent space. The cobordisms represents space-time. The $Z(M)$'s are the state spaces. An operator associated to a space-time is the time-evolution operator (Feynman path integral).

Topological means that these do not depend on any additional structure on space-time (e.g. Riemannian metric, curvature) but only on the topology.

See Barrett (J. Math. Phys. Vol. 36, 1995) or Freed (Bulletin AMS, 2013).

Also, Milnor's paper (Bulletin, AMS, 2015) is definitely worth reading. (No physics.)

LECTURE 27

The Hopf Degree Theorem

Today we are going to generalize an important result on the homotopy classes of maps to spheres. We proved previously that there is exactly one homotopy class of maps $S^1 \to S^1$ for every integer $n \in \mathbb{Z}$. By our classification of one-manifolds, we can read this also as follows:

For every compact, connected, boundaryless one-manifold X, there is exactly one homotopy class of maps $X \to S^1$ for every integer $n \in \mathbb{Z}$.

Today we are going to prove a generalization of this result to higher dimensions. It is a famous theorem of Hopf:

The Hopf Degree Theorem

Two maps $X \to S^k$ of a compact, connected, **oriented**, boundaryless k-manifold X to S^k are **homotopic** if and only if they have the **same degree**.

Recall that the degree of a map $f\colon X \to S^k$ as in the theorem is defined as

$$\deg(f) = \sum_{x \in f^{-1}(y)} \operatorname{sign}(df_x)$$

where y is a regular value of f and $\operatorname{sign}(df_x)$ is $+1$ if df_x preserves orientations and -1 if df_x reverses orientations. We will refer to this sign rule as **our usual orientation convention**.

Some Remarks on Hopf's Theorem

- We can think of the degree as a map
$$\mathrm{Hom}(X,S^k)_{/\sim} =: [X,S^k] \to \mathbb{Z}.$$
Hopf's Theorem tells us that this map is **injective**, where $\sim$ denotes the homotopy relation. One can show that it is also surjective, i.e., there is exactly one homotopy class of maps $X \to S^k$ for every integer $n \in Z$.
- For $X = S^k$, one usually rephrases this result by saying that the kth homotopy group of S^k is $\mathbb{Z}$, i.e.,
$$\pi_k(S^k) =: [S^k,S^k] := \mathrm{Hom}(X,S^k)_{/\sim} = \mathbb{Z}.$$
- Note that the situation is different for nonorientable manifolds: Two maps of a compact, connected, **nonorientable**, boundaryless k-manifold X to S^k are **homotopic** if and only if they have the **same degree modulo** 2.

Now we start our march towards a proof Hopf's theorem. We will follow the guideline of Guillemin-Pollack as usual. But it is worth noting that there are many different ways to prove this theorem. In particular, there is Pontryagin's proof as presented in Milnor's book which introduces an extremely important and interesting concept, called cobordism. We recommend to have a look at that proof as well.

Strategy for the proof of Hopf's Theorem

Assume given two maps f_0 and f_1 from X to S^k.
• Set $W := X \times [0,1]$, define $f\colon \partial W \to S^k$ by $f := f_0$ on $X \times \{0\}$ and $f := f_1$ on $X \times \{1\}$. Then $\deg(f) = \deg(f_1) - \deg(f_0) = 0$. Moreover, a homotopy between f_0 and f_1 is a **global extension of f to W**.
• Show the **Extension Theorem**: $f\colon \partial W \to S^k$ has a **global extension** $W \to S^k$ if and only if $\deg(f) = 0$, for any compact, connected, oriented $k+1$-manifold W. (We knew already: existence of global extensions $\Rightarrow \deg(f) = 0$.)
• To show the Extension Theorem, use the **Isotopy Lemma** to move W inside some ball $B \subset \mathbb{R}^{k+1}$ with $\mathrm{Int}(W) \subset B$. This reduces to checking an extension statement on balls and spheres.
• Use **winding numbers** to show that a map which is homotopic to a **constant map** on the boundary of a ball B extends to all of B.
• Show the **Special Case**: For $f\colon S^k \to S^k$,

$$\deg(f) = 0 \Rightarrow f \sim \text{constant map}.$$

This follows by **induction on the dimension** k of S^k. We have shown previously that $f,g\colon S^1 \to S^1$ are homotopic if and only if $\deg(f) = \deg(g)$. The induction step is actually a zigzag argument using **winding numbers**. The Isotopy Lemma is frequently used to move points into appropriate open neighborhoods and balls.

In order to make this strategy work, we need to prove a series of technical results. This will occupy the rest of the lecture. Two main technical ingredients are isotopies which allow to move points, and winding numbers which help us calculating degrees.

Isotopies and the Isotopy Lemma

We will need an important special type of homotopy which preserves more information than homotopies in general:

Isotopies

An **isotopy** is a homotopy h_t in which **each map h_t is a diffeomorphism**, and two diffeomorphisms are isotopic if they can be joined by an isotopy. An isotopy is **compactly supported** if the maps h_t are all equal to the identity map outside some fixed compact set.

A particular case of isotopies are linear isotopies.

Linear Isotopy Lemma

Suppose that E is a linear isomorphism of $\mathbb{R}^k$ that preserves orientations. Then there exists a homotopy E_t consisting of linear isomorphisms, such that $E_0 = E$ and E_1 is the identity. If E reverses orientation, then there exists such a homotopy with E_1 equal to the reflection map

$$r_1(x_1, \ldots, x_k) = (-x_1, x_2, \ldots, x_k).$$

Proof: First we remark that it suffices to deal with the case that E preserves orientations. For if E is orientation reversing, then $r_1 \circ E$ preserves orientations. Then if there is a homotopy F between $r_1 \circ E$ and Id, then, after composing all maps with r_1, $r_1 \circ F$ is a homotopy between $E = r_1 \circ r_1 \circ E$ and r_1.

So let E be a linear isomorphism of $\mathbb{R}^k$ that preserves orientations. The proof is by induction on the dimension k. We need to check two initial cases.

First, let $k = 1$. Then $E\colon \mathbb{R} to \mathbb{R}$ is given by multiplication by a real number $\lambda > 0$. Then $E_t = t \cdot 1 + (1-t) \cdot \lambda$ is a homotopy between $E = \lambda$ and $\text{Id} = 1$. Note that each E_t is nonzero and therefore a linear isomorphsm.

Now let $k = 2$ and assume that E has only complex eigenvalues. Then $E_t = tE + (1-t)\text{Id}$ is a linear homotopy between Id and E. Moreover, each E_t is a linear isomorphism. To show this we show that $\det(E_t) \neq 0$ for all $t \in [0,1]$. If $E = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then we get

$$\begin{aligned} \det(E_t) &= (t(a-1)+1)(t(d-1)+1) - t^2bc \\ &= t^2(a-1)(d-1) + t(a+d-2) + 1 - t^2bc \\ &= t^2(ad-bc-a-d+1) + t(a+d-2) + 1. \end{aligned}$$

The discriminant of this quadratic equation in t is

$$\begin{aligned} &(a+d-2)^2 - 4(ad-bc-a-d+1) \\ =&(a+d)^2 - 4(a+d) + 4 - 4(ad-bc) + 4(a+d) - 4 \\ =&(a+d)^2 - 4(ad-bc). \end{aligned}$$

But this is exactly the discriminant of the equation

$$t^2 + t(a+d) - (ad-bc) = 0$$

which is the characteristic polynomial (in t) of E. By assumption, this polynomial has only complex roots, i.e. its discriminant is negative. Hence there is no real t such that $\det(E_t) = 0$.

Now we show the induction step. So assume $k \geq 2$ and the assertion to be true in all dimensions $< k$. Then E has either at least one real eigenvalue or at least one complex eigenvalue. Let $V \subset \mathbb{R}^k$ be the corresponding eigenspace, which is either one- or two-dimensional. Then E maps V into itself. Hence $\mathbb{R}^k$ splits into a direct sum $\mathbb{R}^k = V \oplus W$. By choosing a basis of $\mathbb{R}^k$ consisting of a basis of V and one for W, we can represent E as a matrix of the form

$$E = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}.$$

(Here A is either a 1×1- or a 2×2-matrix given by the eigenvalue.)

Then we can define a linear homotopy E_t by

$$E_t = \begin{pmatrix} A & tB \\ 0 & C \end{pmatrix}.$$

Since E is a linear isomorphism and the determinant is multiplicative, we have

$$0 \neq \det(E) = \det(A)\det(C) = \det(E_t).$$

Thus E_t is also a linear isomorphism for every t.

For $t = 0$, we see that E_0 maps V to V by A and W to W by C. Since $\dim W$ is strictly less than k, we can apply the induction hypothesis to C and W and the initial cases to A and V, respectively. Hence we have a homotopyC_t consisting of linear isomorphisms between C and the identity and a homotopy A_t between A and the identity. Then

$$\begin{pmatrix} A_t & tB \\ 0 & C_t \end{pmatrix}$$

is a homotopy consisting of linear homotopies between E and the identity of $\mathbb{R}^k$.
QED

The following theorem will allow us to move points on connected manifolds via a family of diffeomorphisms. The fact that every map in the homotopy family is a diffeomorphism makes it much easier to keep track of the orientation numbers at preimages.

The Isotopy Lemma

Given any two points y and z in the **connected** manifold Y, there exists a diffeomorphism $h\colon Y \to Y$ such that $h(y) = z$ and h is isotopic to the identity. Moreover, the isotopy may be taken to be compactly supported.

Today we are lazy and skip the proof of this reuslt (it is in [GP] on pages 142, 143). Instead we look at a consequence which we will actually use later.

Corollary to the Isotopy Lemma

Suppose that Y is a **connected** manifold of dimension greater than 1, and let $\{y_1, \dots, y_n\}$ and $\{z_1, \dots, z_n\}$ be two sets of distinct points in Y. Then there exists a diffeomorphism $h\colon Y \to Y$ which is isotopic to the identity with

$$h(y_1) = z_1, \dots, h(y_n) = z_n.$$

Moreover, the isotopy may be taken to be compactly supported.

Proof of the Corollary: The proof works by induction. The Isotopy Lemma is the case $n = 1$. Now we assume the corollary being true for $n - 1$. Then we have a compactly supported isotopy $h'_t\colon Y \setminus \{y_n, z_n\} \to Y \setminus \{y_n, z_n\}$ such that $h'_1(y_i) = z_i$ for all $i < n$ and $h'_0 = \mathrm{Id}$.

Since $\dim Y > 1$, the punctured manifold $Y \setminus \{y_n, z_n\}$ is connected. Since the isotopy h'_t has compact support, there are open neighborhoods around y_n and z_n in Y on which the h'_t are all equal to the identity. Hence we can extend the family h'_t to a family of diffeomorphisms of Y that fix those two points.

Now we apply the induction hypothesis again to the punctured manifold

$$Y \setminus \{y_1, \dots, y_{n-1}, z_1, \dots, z_{n-1}\} \text{ and the points } y_n, z_n.$$

Then we get a compectly supported isotopy h''_t with $h''_1(y_n) = z_n$ and $h''_0 = \mathrm{Id}$. By the same argument as for h'_t, we can extend h''_t to an isotopy on all of Y such that all h''_t satisfy $h''_t(y_i) = z_i$ for all $i < n$. Then

$$h_t := h''_t \circ h'_t$$

is the desired isotopy. **QED**

Winding numbers revisited

As for many results on maps between spheres, the winding number is useful concept. We used it before with values modulo 2. Today, we need an integral version:

Integer winding numbers

Let X be a compact oriented k-dimensional smooth manifold, and let $f\colon X \to \mathbb{R}^{k+1}$ be a smooth map. The **winding number** of f, denoted $W(f,z)$, around any point $z \in \mathbb{R}^{k+1} \setminus f(X)$ is defined as the degree of the map

$$u\colon X \to S^k,\ x \mapsto \frac{f(x)-z}{|f(x)-z|}.$$

As a formula:

$$W(f,z) = \deg(u).$$

The winding number will be the main tool in the proof of Hopf's theorem. In order to exploit it effectively, we investigate some of its properties:

Step 1

Let $f\colon U \to \mathbb{R}^k$ be a smooth map defined on an open subset U of $\mathbb{R}^k$, and let x be a regular point, with $f(x) = z$. Let B be a sufficiently small closed ball centered at x, and define $\partial f\colon \partial B \to \mathbb{R}^k$ to be the restriction of f to the boundary of B. Then we have

$$W(\partial f, z) = \begin{cases} +1 \text{ if } f \text{ preserves orientation at } x, \\ -1 \text{ if } f \text{ reerses orientation at } x. \end{cases}$$

Proof: After possibly translating things, we can assume $x = 0 = z$, which keeps the notation simpler. We set $A = df_0$. We are going to show that $W(A,0)$ can be used to calculate $W(\partial f,0)$. This will follow if we show that we can choose B small enough such that there is a homotopy $F_t\colon \partial B \times [0,1] \to S^{k-1}$ between $Ax/|Ax|$ and $\partial f(x)/|\partial f(x)|$. For then

$$W(\partial f,0) = \deg\left(\frac{\partial f(x)}{|\partial f(x)|}\right) = \deg\left(\frac{Ax}{|Ax|}\right) = W(A,0).$$

Now we are going to construct the homotopy F_t. By Taylor theory, we can write

(27) $$f(x) = Ax + \epsilon(x), \text{ where } \epsilon(x)/|x| \to 0 \text{ when } x \to 0.$$

We define

$$f_t(x) = Ax + t\epsilon(x) \text{ for } t \in [0,1].$$

Then, f_t is a homotopy from $f_0(x) = Ax$ to $f_1(x) = f(x)$.

Since $x = 0$ is a regularit point, we know that A is an isomorphism. Hence the image of the unit ball in $\mathbb{R}^k$ under A strictly contains a closed ball of some radius $r > 0$. Since every linear isomorphism is a diffeomorphism, we also know that A maps boundaries to boundaries, i.e., S^{k-1} to the boundary of the closed ball of radius r. Hence

$$|Ax| > r \text{ for all } x \in S^{k-1}.$$

As a consequence,

$$|A\frac{x}{|x|}| > c \text{ and thus } |Ax| > |rx| \text{ for all } x \in \mathbb{R}^k \setminus \{0\}.$$

Now we use (27). Since $\epsilon(x)/|x| \to 0$ as $x \to 0$, we can choose a ball B small enough such that

$$\epsilon(x)/|x| < \frac{r}{2} \text{ for all } x \in \partial B.$$

Then we have

$$|f_t(x)| = |Ax| - t|\epsilon(x)| > r|x| - \frac{r}{2}|x| = \frac{r}{2}|x|,$$
$$\textit{i.e.}, |f_t(x)| > 0 \text{ for all } x \in \partial B.$$

Hence we can define the desired homotopy F_t by

$$F_t \colon \partial B \times [0,1] \to S^k,\ x \mapsto \frac{f_t(x)}{|f_t(x)|}.$$

Now we compute $W(A,0)$. Therefor we apply the Linear Isotopy Lemma and get that A is homotopic to the identity if it preserves orientions, and homotopic to the reflection map $(x_1, \ldots, x_k) \mapsto (-x_1, x_2, \ldots, x_k)$ if it reverses orientations. In the former case, we have $W(A,0) = +1$, and in the latter case $W(A,0) = -1$. **QED**

This result determines how local diffeomorphisms can wind. Now we are going to use this information to count preimages.

Step 2

Let $f\colon B \to \mathbb{R}^k$ be a smooth map defined on some closed ball B in $\mathbb{R}^k$. Suppose that z is a regular value of f that has no preimages on the boundary sphere ∂B, and let $\partial f\colon \partial B \to \mathbb{R}^k$ be its restriction to the boundary. Then the number of preimages of z, counted with our usual orientation convention, equals the winding number $W(\partial f, z)$.

Proof: By the Stack of Records Theorem, we know that $f^1(z)$ is a finite set $\{x_1, \ldots, x_n\}$, and we can choose disjoint balls B_i around each x_i. Since $f^1(z)$ is disjoint from ∂B by assumption, we can shrink these balls such that $B_i \cap \partial B = \emptyset$ and so that each B_i is sufficiently small so that Step 1 can be applied.

Let $\partial f_i = f_{|\partial B_i}$. Then **Step 1** implies that the **number of preimage points**, counted with our usual orientation convention, equals $\sum_{i=1}^n W(\partial f_i, z)$.

Let $B' := B \setminus \cup_i B_i$ and consider the map

$$u\colon \partial B \to S^{k-1},\ x \mapsto \frac{f(x) - z}{|f(x) - z|}.$$

Since $f(x) \neq z$ on B', this map extends to all of B'. This implies

$$W(f_{|\partial B'}, z) = \deg(u) = 0.$$

The orientations of the boundaries are related by

$$\partial B' = \partial B \cup_{i=1}^n (-\partial B_i).$$

This implies

$$W(f_{|\partial B'}, z) = W(\partial f, z) - \sum_{i=1}^n W(\partial f_i, z).$$

Hence in total we get $W(\partial f, z) = \sum_{i=1}^n W(\partial f_i, z)$. **QED**

Step 3

Let B be a closed ball in $\mathbb{R}^k$, and let $f\colon \mathbb{R}^k \setminus \mathrm{Int}(B) \to Y$ be a smooth map defined outside the open ball $\mathrm{Int}(B)$. Let $\partial f\colon \partial B \to Y$ be the restriction to the boundary. Assume that ∂f is homotopic to a constant map. Then f extends to a smooth map defined on all of $\mathbb{R}^k$ into Y.

Proof: For simplicity, we assume that B is centered at 0. Then we can write every non-zero point $x \in B$ uniquely as $x = ty$ for some $y \in \partial B$ and some

$t \in [0,1]$. By assumption, there is a homotopy $g_t \colon \partial B \to Y$ with $g_1 = \partial f$ and g_0 being a constant map.

Now we define the map $F \colon \mathbb{R}^k \to Y$ by setting

$$F(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R}^k \setminus \operatorname{Int}(B) \\ g_t(x) & \text{if } x \in B \text{ and } x = ty \text{ for some } y \in \partial B \text{ and } t \in [0,1]. \end{cases}$$

Note that F is well-defined on $R^k \setminus \operatorname{Int}(B)$, since f and g_t agree on $\partial B = B \cap (\mathbb{R}^k \setminus \operatorname{Int}(B))$ where we have $f = \partial f = g_1$. Note also that $F(0)$ is well-defined as the constant value of g_0.

Now it remains to use smooth bump function to turn F into a smooth homotopy (it is already smooth except, possibly, on ∂B). **QED**

The Special Case

Special case

Any smooth map $f \colon S^k \to S^k$ having **degree zero** is homotopic to a **constant map**.

The special case implies:

Corollary

Any smooth map $f \colon S^k \to \mathbb{R}^{k+1} \setminus \{0\}$ having winding number zero with respect to the origin is homotopic to a constant map.

Proof of the Corollary: By assumption, the degree of the map $\frac{f}{|f|}$ is zero. By the special case, this implies that $\frac{f}{|f|}$ is homotopic to a constant map. But $\frac{f}{|f|}$ and f are homotopic via the homotopy

$$F \colon S^k \times [0,1] \to \mathbb{R}^{k+1} \setminus \{0\},\ (x,t) \mapsto tf(x) + (1-t)\frac{f}{|f|}$$

Since homotopy is a transitive relation, f is also homotopic to a constant map. **QED**

Proof of the special case:

The proof is by induction on the dimension k. We have established the case $k = 1$ in a previous lecture. So we assume the special case being true for $k - 1$ and want to deduce it for k.

We need to prove a lemma first:

A lemma

Let $f\colon \mathbb{R}^k \to \mathbb{R}^k$ be a smooth map with 0 as a regular value. Suppose that $f^{-1}(0)$ is finite and that the number of preimage points in $f^{-1}(0)$ is zero when counted with the usual orientation convention. Assuming the special case in dimension $k - 1$. Then there exists a map $g\colon \mathbb{R}^k \to \mathbb{R}^k \setminus \{0\}$ such that $g = f$ outside a compact set.
In particular, the homotopy $tf + (1t)g$ from g to f is constant outside this compact set.

Proof: Since $f^{-1}(0)$ is a finite, we can choose a ball B centered at the origin with $f^{-1}(0) \subset \text{Int}(B)$. By assumption, the number of preimages is zero when counted with the usual orientation convention. By Step 2, the map $\partial f\colon \partial B \to \mathbb{R}^k \setminus \{0\}$ has winding number zero. Since ∂B is diffeomorphic to S^{k-1}, so ∂f is a map from ${}^{k-1}$ to $\mathbb{R}^k \setminus \{0\}$.

Since we are assuming the special case being true in dimension $k - 1$, we can apply its corollary in that dimension. Thus, ∂f is homotopic to a constant map. Hence

$$f_{|\mathbb{R}^k \setminus \text{Int}(B)}\colon \mathbb{R}^k \setminus \text{Int}(B) \to \mathbb{R}^k \setminus \{0\}$$

is a map to which we can apply Step 3. This implies that f extends to a smooth map $g\colon \mathbb{R}^k \to \mathbb{R}^k \setminus \{0\}$ with $f = g$ outside the compact space B. **QED**

Now we get back to the proof of the special case, and we are given a smooth map $f\colon S^k \to S^k$ with $\deg(f) = 0$.

The **idea of the proof** is to show that f is homotopic to a map $h\colon S^k \to S^k \setminus \{b\}$, where b is some point in S^k. But $S^k \setminus \{b\}$ is diffeomorphic to $\mathbb{R}^k$ via stereographic projection (from b). Since $\mathbb{R}^k$ is **contractible**, this implies h is homotopic to a constant map. Then f is also homotopic to a constant map.

So we need to show:

Claim: f is homotopic to a smooth map $g\colon S^k \to S^k \setminus \{b\}$.

By Sards Theorem, we can choose distinct regular values a and b of f. By the Stack of Records Theorem, the preimage sets are finite, say $f^{-1}(a) = \{a_1, \dots, a_n\}$ and $f^{-1}(b) = \{b_1, \dots, b_m\}$.

Moreover, we can find an open neighborhood U of a_1 such that U is diffeomorphic to $\mathbb{R}^k$ via a diffeomorphism $\alpha\colon \mathbb{R}^k \to U$ and such that $b_i \notin U$ for all $i = 1, \dots, m$.

Since $k > 1$, we can apply the corollary of the Isotopy Lemma to the points $\{a_2, \dots, a_n\}$ in $Y := S^k \setminus \{b\}$ to get a diffeomorphism which is isotopic to the identity, compactly supported, and moves the points a_i into U.

Since homotopy is a transitive relation, we can therefore assume that U is an open neighborhood of $f^{-1}(a)$ with $b \notin f(U)$.

Now let $\beta\colon S^k \setminus \{b\} \to \mathbb{R}^k$ be a diffeomorphism with $\beta(a) = 0$. Then

$$\beta \circ f \circ \alpha\colon \mathbb{R}^k \xrightarrow{\alpha} U \xrightarrow{f} S^k \setminus \{b\} \xrightarrow{\beta} \mathbb{R}^k$$

is a smooth map from $\mathbb{R}^k$ to $\mathbb{R}^k$. Since a is a regular value of f, 0 is a regular value of $\beta \circ f \circ \alpha$. Moreover, since $f^{-1}(a)$ is finite, $(\beta \circ f \circ \alpha)^{-1}(0)$ is finite as well.

Now we use the **assumption** $\deg(f) = 0$. For this means that the **number of preimages of a under f is zero** when counted with our usual orientation convention. Hence the number of **preimages of 0 under $\beta \circ f \circ \alpha$ is zero** when counted with the usual orientation convention.

Thus, we can **apply the lemma** to $\beta \circ f \circ \alpha\colon \mathbb{R}^k \to \mathbb{R}^k$ and get a map $g\colon \mathbb{R}^k \to \mathbb{R}^k \setminus \{0\}$ such that $g = \beta \circ f \circ \alpha$ outside a compact set B and g is homotopic to $\beta \circ f \circ \alpha$ on $\mathbb{R}^k$.

Since α and β are **diffeomorphisms**, this implies that f is homotopic to $\beta^{-1} \circ g \circ \alpha^{-1}$ as a map from U to $S^k \setminus \{b\}$.

Since $g = \beta \circ f \circ \alpha$ outside B, we have

$$\beta^{-1} \circ g \circ \alpha^{-1} = f \text{ on } U \setminus \alpha^{-1}(B).$$

Thus, the map

$$h\colon S^k \to S^k \setminus \{b\}$$

defined by setting

$$h = \begin{cases} f & \text{on } S^k \setminus \alpha^{-1}(B) \\ \beta^{-1} \circ g \circ \alpha^{-1} & \text{on } \alpha^{-1}(B) \end{cases}$$

is smooth, and h is the desired map homotopic to f. This proves the special case. **QED**

Towards proof of Hopf's theorem

Now we are almost ready to prove Hopf's result.

Extending maps to Euclidean spaces

Let W be a compact smooth manifold with boundary, and let $f\colon \partial W \to \mathbb{R}^k$ be a smooth map. Then f can be extended to a globally defined map $F\colon W \to \mathbb{R}^k$.

Proof: As always we assume that W is a subset of some $\mathbb{R}^N$. Since W is compact, it is a closed subset of $\mathbb{R}^N$, and so is ∂W. Since f is a smooth map defined on a closed subset of $\mathbb{R}^N$, it may be locally extended to a smooth map on open sets. Since ∂W is compact and boundaryless, we can apply the **ϵ-Neighborhood Theorem** to extend f to a map F defined on a neighborhood U of ∂W in $\mathbb{R}^N$.

Now we choose a smooth bump function ρ that is constant 1 on ∂W and 0 outside some compact subset of U.

Then we can extend f to all of W by letting it be

$$\rho \cdot F \text{ on } U, \text{ and } 0 \text{ outside of } U.$$

This is a smooth function defined on all of $\mathbb{R}^N$ with values in $\mathbb{R}^k$ and being $f = 1 \cdot F$ on ∂W. **QED**

Now we apply this lemma to maps with values in spheres:

Extension Theorem

Let W be a compact, connected, oriented $k+1$-dimensional smooth manifold with boundary, and let $f\colon \partial W \to S^k$ be a smooth map. Then f **extends** to a **globally** defined map $F\colon W \to S^k$ with $\partial F = f$ if and only if $\deg(f) = 0$.

Proof: We already know that if f can be extended to all of W, then $\deg(f) = 0$. It remains to show the opposite direction.

So let f be as in the thereom, and assume $\deg(f) = 0$. By the previous lemma, we can extend f to a smooth map $F\colon W \to \mathbb{R}^{k+1}$. By the **Transversality Extension Theorem**, we can assume that 0 is a regular value of F. Since W

is compact of dimension $k+1$, we know that $F^{-1}(0)$ is a finite set. Hence we can apply the corollary to the Isotopy Lemma to this finite set, and move $F^{-1}(0)$ inside $\mathrm{Int}(B)$ where B is a closed ball contained $\mathrm{Int}(W)$.

In particular, since $F^{-1}(0) \subset \mathrm{Int}(B)$, the map $\frac{F}{|F|}$ extends to $W' := W \backslash \mathrm{Int}(B)$. Hence

$$W\left(\frac{F}{|F|},0\right) = \deg\left(\frac{F}{|F|}\right) = 0.$$

On the other hand, we know by our assumtpion that

$$W(F_{|\partial W},0) = W(f,0) = \deg(f) = 0,$$

where we use $f = f/|f|$, since f has values in S^k.

Now let

$$\partial F = F_{\partial B} \colon \partial B \to \mathbb{R}^{k+1} \setminus \{0\}$$

be the restriction to the boundary. By the definition of W' and boundary orientations, we have

$$\partial W' = (\partial W) \cup (-\partial B).$$

Hence we get

$$W(F_{|\partial W'},0) = W(F_{|\partial W},0) - W(F_{|\partial B},0)$$

and therefore $W(F_{|\partial B},0)$ by our previous observations.

Now the corollary to the special case implies that ∂F is homotopic to a constant map. By **Step 3**, this implies that ∂F extends to a map $G \colon W \to \mathbb{R}^{k+1} \backslash \{0\}$. Then the map $\frac{G}{|G|} \colon W \to S^k$ is the global extension of f. **QED**

And, finally, the last step:

Proof the Hopf Degree Theorem: Let f_0 and f_1 be two maps $X \to S^k$ and let $W := X \times [0,1]$. We define a map $f \colon \partial W \to S^k$ by setting

$$f = \begin{cases} f_0 & \text{on } X \times \{0\} \\ f_1 & \text{on } X \times \{1\}. \end{cases}$$

By the **Extension Theorem**, f extends to a map on all of W if and only if $\deg(f) = 0$. By definition, such an extension would be a homotopy between f_0 and f_1. Thus we have

$$f_0 \sim f_1 \iff \deg(f) = 0.$$

It remains to relate $\deg(f)$ to $\deg(f_0)$ and $\deg(f_1)$. But, since $\partial W = (X \times \{1\}) \cup (X \times \{0\})$ with the opposite orientation on $X \times \{0\}$, it follows that

$$\deg(f) = \deg(f_1) - \deg(f_0).$$

Thus

$$f_0 \sim f_1 \iff \deg(f_1) = \deg(f_0).$$

QED

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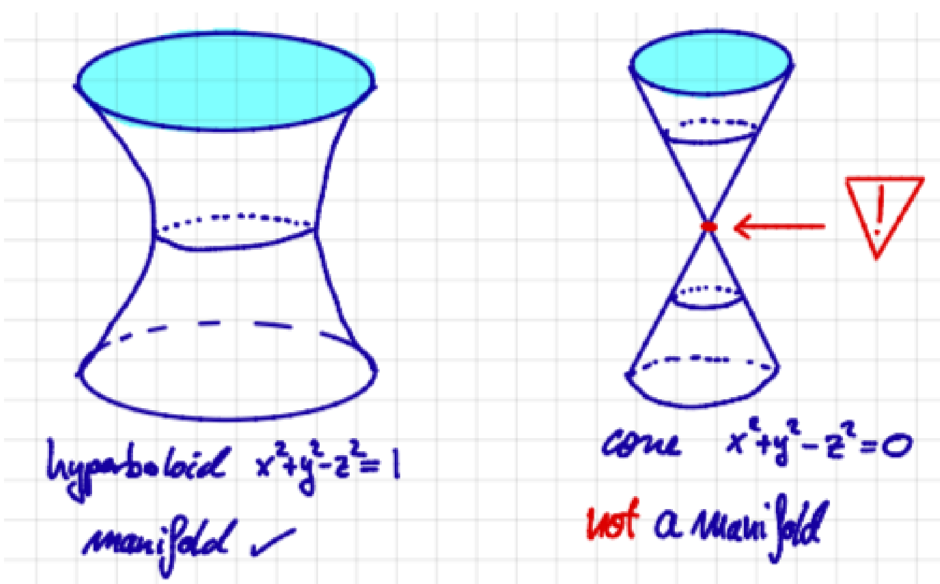
APPENDIX A

Exercises

1. Exercises after Lecture 3

[1] Show that every k-dimensional vector subspace V of $\mathbb{R}^N$ is a manifold diffeomorphic to $\mathbb{R}^k$ and that any linear map $V \to \mathbb{R}^m$ is smooth. (Remember that choosing a basis for V corresponds to choosing a linear isomorphism $\phi\colon \mathbb{R}^k \to V$. Expressing a vector in V in terms of this basis, means to attach coordinates to this vector. Since ϕ is linear, we refer to the corresponding coordinates as linear coordinates.)

[2] **a)** Prove that the subspace of $\mathbb{R}^3$, defined by $x^2 + y^2 - z^2 = a$, is a manifold if $a > 0$.
b) Explain why $x^2 + y^2 - z^2 = 0$ does not define a manifold.



3 The torus $T(a,b)$ is the set of points in $\mathbb{R}^3$ at distance b from the circle of radius a in the xy-plane, where $0 < b < a$. Prove that each $T(a,b)$ is diffeomorphic to $S^1 \times S^1 \subset \mathbb{R}^4$. What happens when $b = a$?

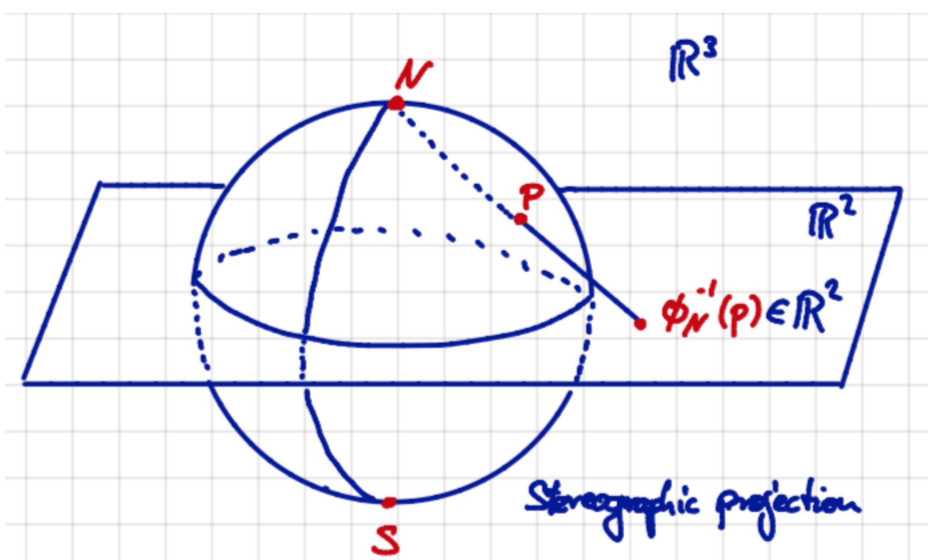
4 Let $N = (0,\ldots,0,1) \in S^k$ be the "north pole" on the k-dimensional sphere. The stereographic projection ϕ_N^{-1} from $S^k \setminus \{N\}$ onto $\mathbb{R}^k$ is the map which sends a point p to the point at which the line through N and p intersects the subspace in $\mathbb{R}^{k+1}$ defined by $x_{k+1} = 0$. (See the picture for $k = 2$.)

a) Show that ϕ_N^{-1} is given by the formula

$$(x_1,\ldots,x_{k+1}) \mapsto \frac{1}{1 - x_{k+1}}(x_1,\ldots,x_k).$$

b) Find a formula for the inverse ϕ_N of ϕ_N^{-1}, and check that both maps are smooth.

c) Let $S = (0,\ldots,0,-1) \in S^k$ be the "south pole". Describe the parametrization using the stereographic prjoction starting in S instead of N, and conclude that S^k is a k-dimensional manifold.



2. Exercises after Lecture 4

1 Let V be a vector subspace of $\mathbb{R}^N$. Show that $T_x(V) = V$ for $x \in V$.

2 Determine the tangent space to the torus $S^1 \times S^1 \subset \mathbb{R}^4$ at an arbitrary point p. Recall the description of the torus $T(a,b) \subset \mathbb{R}^3$ from the previous exercise set. Can you describe the tangent space at a point in $T(a,b) \subset \mathbb{R}^3$?

3 Determine the tangent space to the subspace of $\mathbb{R}^3$ defined by $x^2 + y^2 - z^2 = a$ at $(\sqrt{a},0,0)$ for $a > 0$.

4 The graph of a map $f\colon X \to Y$ is the subset of $X \times Y$ defined by

$$\Gamma(f) = \{(x,f(x)) \in X \times Y : x \in X\}.$$

Define $F\colon X \to \Gamma(f)$ by $F(x) = (x,f(x))$. We assume that X and Y are smooth manifolds and f is a smooth map.

a) Show F is a diffeomorphism, and conclude that $\Gamma(f)$ is a smooth manifold.

b) We also write F for the composite map $F\colon X \to X \times Y$, $x \mapsto (x,f(x))$. Show that $dF_x(v) = (v, df_x(v))$. (You can use $T_{(x,y)}(X \times Y) = T_x(X) \times T_y(Y)$.)

c) Show that the tangent space to $\Gamma(f)$ at the point $(x,f(x))$ is the graph of $df_x\colon T_x(X) \to T_{f(x)}(Y)$.

5 A curve in a manifold X is a smooth map $t \mapsto c(t)$ of an open interval of $\mathbb{R}$ into X. The velocity vector of the curve c at time t_0 in $x_0 = c(t_0)$ -denoted simply $dc/dt(t_0)$ - is defined to be the vector $dc_{t_0}(1) \in T_{x_0}(X)$, where $dc_{t_0}\colon \mathbb{R}^1 \to T_{x_0}(X)$.

a) For $X = \mathbb{R}^k$ and $c(t) = (c_1(t), \ldots, c_k(t))$, show that

$$\frac{dc}{dt}(t_0) = dc_{t_0}(1) = (c_1'(t_0), \ldots, c_k'(t_0)) \in T_{x_0}\mathbb{R}^k.$$

b) For an arbitrary k-dimensional smooth manifold, use the above observation and local parametrizations to prove that every vector in $T_{x_0}(X)$ is the velocity vector of some curve in X.

Aside: This shows that there is a unique correspondence between tangent vectors at $x_0 \in X$ and velocity vectors at t_0 of curves $c\colon I \to X$ with $c(t_0) = x_0$. Note that two curves $c_1\colon I \to X$ and $c_2\colon J \to X$, with I and J open in $\mathbb{R}$, have the same velocity vector in $c_1(t_1) = x_0 = c_2(t_2)$ if $d(c_1)_{t_1}(1) = d(c_2)_{t_2}(1) \in T_{x_0}(X)$. One can show that having the same velocity vector in a point of X is an equivalence relation the set of curves through x_0 in X. Using this relation, we have shown that there is a unique correspondence between tangent vectors at X in x and equivalence classes of smooth curves through x_0 in X.

3. Exercises after Lecture 6

1 Let $A\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear map, and $b \in \mathbb{R}^n$. Show that the mapping

$$f\colon \mathbb{R}^n \to \mathbb{R}^n,\ x \mapsto Ax + b$$

is a diffeomorphism of $\mathbb{R}^n$ if and only if A is invertible.

2 Show that the map

$$f\colon \mathbb{R}^2 \to \mathbb{R}^3,\ (s,t) \mapsto ((2+\cos(2\pi s))\cos(2\pi t), (2+\cos(2\pi s))\sin(2\pi t), \sin(2\pi s))$$

is an immersion. Is it an embedding?

3 Let γ_α be the curve on the torus defined by

$$\gamma_\alpha\colon \mathbb{R} \to S^1 \times S^1,\ t \mapsto (e^{2\pi it}, e^{2\pi i\alpha t})$$

where we consider S^1 as a subset of $\mathbb{C} \cong \mathbb{R}^2$. Show that γ_α factors through an embedding $S^1 \to S^1 \times S^1$ when α is rational, i.e. find a map $g_\alpha\colon S^1 \to S^1 \times S^1$ which is an embedding such that γ_α is the composite of the map $\mathbb{R} \to S^1$, $t \mapsto e^{2\pi it}$, followed by g_α.

4 Consider the map $f\colon (0,3\pi/4) \to \mathbb{R}^2$, $t \mapsto \sin(2t)(\cos t, \sin t)$.

a) Show that f is an immersion.

b) Let $\mathrm{Im}\,(f) = f((0,3\pi/4)) \subset \mathbb{R}^2$ be the image of f (considered as a subspace in $\mathbb{R}^2$). Show that $f\colon (0,3\pi/4) \to \mathrm{Im}\,(f)$ is not a homeomorphism. (Draw a picture of the image of f.)

c) To test your understanding answer the following questions (and give reasons for your answer):

- What is the difference between $\mathrm{Im}\,(f)$ and the graph $\Gamma(f)$?
- Is the map $F\colon (0,3\pi/4) \to (0,3\pi/4) \times \mathbb{R}^2$, $t \mapsto (t, f(t))$, an embedding?
- Would f be an embedding if it was defined on the closed interval $[0,3\pi/4]$?
- Is the map $g\colon (0,3\pi/4) \to \mathbb{R}^3$, $t \mapsto \sin(2t)(\cos t, \sin t, t)$ an embedding?
- Is the map $h\colon [0,3\pi/4] \to \mathbb{R}^3$, $t \mapsto (\sin(2t)\cos t, \sin(2t)\sin t, 2t)$ an embedding?

5 Let X be an n-dimensional smooth manifold, Z be a k-dimensional smooth submanifold of X, and let $z \in Z$. Show that there exists a local coordinate system $(x_1, \ldots, x_n)$ defined in a neighborhood U of z in X such that $Z \cap U$ is defined by the equations $x_{k+1} = 0, \ldots, x_n = 0$, i.e. $Z \cap U$ is the subset of points in U for which the functions $x_{k+1}, \ldots, x_n$ all vanish.

4. Exercises after Lecture 8

1 Let $f\colon X \to Y$ be a submersion and U an open subset of X. Show that $f(U)$ is open in Y. (In other words, submersions are open maps.)

2 **a)** If X is compact and Y connected, show that every (nontrivial) submersion $f\colon X \to Y$ is surjective. (Recall that a space Y is called connected if Y cannot be written as the union of two nonempty disjoint open subsets; or equivalently, if Y and $\emptyset$ are the only subsets which are both open and closed in Y).

b) Show that there exist no submersions of compact manifolds into $\mathbb{R}^n$ for any n.

3 Show that the orthogonal group $O(n)$ is compact. (Hint: Show that if $A = (a_{ij})$ lies in $O(n)$, then for each i, $\sum_j a_{ij}^2 = 1$.)

4 Show that the tangent space to $O(n)$ at the identity matrix I is the vector space of skew symmetric $n \times n$-matrices, i.e. matrices B satisfying $B^t = -B$.

5 Prove that the set R_1 of all 2×2-matrices of rank 1 is a three-dimensional submanifold of $\mathbb{R}^4 = M(2)$. (Hint: Show that the determinant function is a submersion on the manifold of nonzero 2×2-matrices $M(2) \setminus \{0\}$.)

5. Exercises after Lecture 10

1 **a)** Show that a local diffeomorphism $f\colon X \to Y$ which is bijective is a diffeomorphism.

b) Show that a local diffeomorphism $f\colon X \to Y$ which is one-to-one is a diffeomorphism of X onto an open subset of Y.

c) Show that a bijective smooth map $f\colon X \to Y$ of constant rank is a diffeomorphism.
(Comment: You can assume that f is a submersion to simplify things. If you want to challenge yourself, you could only assume that X is compact. Showing that f also is a submersion in general requires the use of Baire's category theorem.)

d) Show that a bijective Lie group homomorphism is a Lie group isomorphism.

2 Show that an open subgroup H, i.e. a subgroup which is also an open subset, of a connected Lie group G is equal to G.

3 Let G be a Lie group and let $e \in G$ be the identity element.

a) Let $\mu\colon G \times G \to G$ denote the multiplication map, and let $g,h \in G$. Recall that we denote by L_g the left translation in G by g, and by R_h the right translation by h. Using the identification $T_{(g,h)}(G \times G) = T_g(G) \times T_h(G)$, show that the differential of μ at (g,h)

$$d\mu_{(g,h)}\colon T_g(G) \times T_h(G) \to T_{gh}(G)$$

is given by

$$d\mu_{(g,h)}(X,Y) = d\mu_{(g,h)}(X,0) + d\mu_{(g,h)}(0,Y) = d(R_h)_g(X) + d(L_g)_h(Y).$$

(Hint: Calculate $d\mu_{(g,h)}(X,0)$ and $d\mu_{(g,h)}(0,Y)$ separately.)

b) Let $\iota\colon G \to G$ denote the inversion map. Show that $d\iota_e\colon T_e(G) \to T_e(G)$ is given by $d\iota_e(X) = -X$.

c) Use the previous point to show that, for any $g \in G$, the derivative of ι at g is given by

$$d\iota_g\colon T_g(G) \to T_{g^{-1}},\ Y \mapsto -d(R_{g^{-1}})_e(d(L_{g^{-1}})_g(Y)) \text{ for all } Y \in T_g(G).$$

4 Show that for any Lie group G, the multiplication map $\mu\colon G\times \to G$ is a submersion.

5 Show that the differential of the determinant map $\det\colon GL(n,\mathbb{R}) \to \mathbb{R}$ at $A \in GL(n,\mathbb{R})$ is given by

$$d(\det)_A(B) = (\det A) \cdot (\operatorname{tr} A^{-1}B) \text{ for all } B \in M(n).$$

In particular, $d(\det)_A(AB) = (\det A) \cdot (\operatorname{tr} AB)$ for all $B \in M(n)$.

6. Exercises after Lecture 12

1 As a first test of our understanding of transversality, answer the following questions:

a) Let $z = (a,b) \in S^1 \subseteq \mathbb{R}^2$ and let $N_z = \{(a,y) : y \in \mathbb{R}\}$ be the vertical line intersecting the circle at z. When is $S^1 \subseteq \mathbb{R}^2$ transverse to $N_z \subseteq \mathbb{R}^2$?

b) Which of the following linear spaces intersect transversally?
- The plane spanned by $\{(1,0,0),(2,1,1)\}$ and the y-axis in $\mathbb{R}^3$.
- $\mathbb{R}^k \times \{0\}$ and $\{0\} \times \mathbb{R}^l$ in $\mathbb{R}^n$. (The answer depends on k, l, and n.)
- $V \times \{0\}$ and the diagonal in $V \times V$, for a real vector space V.
- The spaces of symmetric ($A^t = A$) and skew symmetric ($A^t = -A$) matrices in $M(n)$.

c) Do $SL(n)$ and $O(n)$ meet transversally in $M(n)$?

2 **a)** Let $f\colon X \to Y$ be a map transversal to a submanifold Z in Y. Then we know that $W = f^{-1}(Z)$ is a submanifold of X. Prove that $T_x(W)$ is the preimage of $T_{f(x)}(Z)$ under the linear map $df_x\colon T_x(X) \to T_{f(x)}(Y)$.

b) Let X and Z be transversal submanifolds of Y. Deduce from the previous point that, for every $y \in X \cap Z$,

$$T_y(X \cap Z) = T_y(X) \cap T_y(Z).$$

3 Let V be a vector space, and let Δ be the diagonal of $V \times V$. For a linear map $A\colon V \to V$, consider the graph $\Gamma(A) = \{(v, Av) : v \in V\}$. Show that $\Gamma(A) \pitchfork \Delta$ if and only if $+1$ is not an eigenvalue of A.

4 Let $f\colon X \to X$ be a map, and let x be a fixed point of f, i.e. $f(x) = x$. If $+1$ is not an eigenvalue of $df_x\colon T_x(X) \to T_x(X)$, then x is called a *Lefschetz fixed point* of f. The map f is called a *Lefschetz map* if all its fixed points are Lefschetz. Prove that if X is compact and f is Lefschetz, then f has only finitely many fixed points.

(Hint: Show that the intersection of the graph of f and the diagonal of X is a 0-dimensional submanifold of $X \times X$.)

5 Consider the following intersections in $\mathbb{C}^5 \setminus \{0\}$:

$$S_k^7 = \{z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6k-1} = 0\} \cap \{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2 = 1\}.$$

Show S_k^7 is a 7-dimensional submanifold by showing that the intersection is transversal in $\mathbb{C}^5 \setminus \{0\}$.

(Hint: At some point you may want to show that, at a point $z = (z_1, \ldots, z_5)$, the vector $w := (\frac{m}{2} z_1, \frac{m}{2} z_2, \frac{m}{2} z_3, \frac{m}{3} z_4, \frac{m}{6k-1} z_5)$, with $m := 2 \cdot 3 \cdot (6k-1)$, lies in one of the tangent spaces but not in the other.)

7. Exercises after Lecture 13

1 A manifold X is *contractible* if its identity map is homotopic to some constant map $X \to \{x\}$ where x is any point of X.

a) Show that if X is contractible, then all maps of an arbitrary manifold Y into X are homotopic.

b) Conversely, show that if all maps of an arbitrary manifold Y into X are homotopic, then X is contractible.

c) Show that $\mathbb{R}^k$ is contractible.

2 A manifold X is *simply connected* if it is connected and if every smooth map from the circle S^1 into X is homotopic to a constant map. Show that all contractible spaces are simply connected. (Note that the converse is false.)

3 Show that the antipodal map $S^k \to S^k$, $x \mapsto -x$, is homotopic to the identity if k is odd. (We will see later that this is not true if n is even.)

(Hint: Start off with $k = 1$ by using the linear maps defined by

$$[0{,}1] \to M(2),\ t \mapsto \begin{pmatrix} \cos(\pi t) & -\sin(\pi t) \\ \sin(\pi t) & \cos(\pi t) \end{pmatrix}.)$$

4 Show that every connected manifold X is path-connected, i.e. given any two points $x_0,x_1 \in X$, there exists a smooth curve $f\colon [0{,}1] \to X$ with $f(0) = x_0$ and $f(1) = x_1$.

(Hint: Use the fact that homotopy is an equivalence relation to show that the relation "x_0 and x_1 can be joined by a smooth curve" is an equivalence relation on X. Then show that the equivalence classes are both open and closed subsets of X.)

8. Exercises after Lecture 15

1 Recall that a manifold X is *simply connected* if it is connected and if every smooth map of the circle S^1 into X is homotopic to a constant map. Prove that the sphere S^k is simply connected if $k > 1$. (Hint: If $f\colon S^1 \to S^k$ and $k > 1$, Sard's Theorem gives us a point $p \notin f(S^1)$. Now use stereographic projection.)

2 Show that the determinant function on $M(n)$ is a Morse function if $n = 2$, but not if $n > 2$. (Hint: To find the partial derivatives of det, one can use Laplace's formula for the determinant: for any fixed j,

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \cdot \det(A_{ij})$$

where A_{ij} is the submatrix of A with ith row and jth column removed. Check if the zero matrix is nondegenerate.)

3 Show that the "height function" $h\colon S^k \to \mathbb{R}$, $(x_1, \ldots, x_{k+1}) \mapsto x_{k+1}$ on the k-sphere S^k is a Morse function with two critical points, one of which is a maximum and the other a minimum.

4 A vector field on X is a smooth section of $\pi\colon T(X) \to X$, i.e. a smooth map $\sigma\colon X \to T(X)$ such that $\pi \circ \sigma = \mathrm{Id}_X$. An equivalent way to describe such a section is to give a map $s\colon X \to \mathbb{R}^N$ such that $s(x) \in T_x(X)$ for all x (with correspinding $\sigma(x) = (x, s(x))$). A point $x \in X$ is a zero of the vector field σ if $\sigma(x) = (x, 0)$ or equivalently $s(x) = 0$.

a) Show that if k is odd, there exists a vector field on S^k having no zeros.
(Hint: For $k = 1$, use $(x_1, x_2) \mapsto (-x_2, x_1)$.)

b) Prove that if S^k has a vector field which has no zeros, then its antipodal map $x \mapsto -x$ is homotopic to the identity.
(Hint: Show that you may assume $|s(x)| = 1$ everywhere. Now contemplate about $(\cos(\pi t))x + (\sin(\pi t))s(x)$ when t varies from 0 to 1.)

c) Show that if k is even, then the antipodal map on S^k is homotopic to the reflection map

$$r\colon S^k \to S^k, (x_1, \ldots, x_{k+1}) \mapsto (-x_1, x_2, \ldots, x_{k+1}).$$

(Hint: Consider also the reflections $r_i(x_1,\ldots,x_{k+1}) = (x_1,\ldots,-x_i,\ldots,x_{k+1})$. Show that $r_i \circ r_{i+1}$ is homotopic to the identity on S^k.)

5 Let X be the set of all straight lines in $\mathbb{R}^2$ (not just lines through the origin).

a) Show that X is an abstract smooth 2-manifold by showing that we can identify X with an open subset of the real projective plane $\mathbb{R}\mathrm{P}^2$. (Here we use that open subsets of abstract smooth k-manifolds are again abstract smooth k-manifolds.)

b) Show that there is a bijection bewtween X and the set of equivalence classes

$$(S^1 \times \mathbb{R})/\sim$$

where $\sim$ is the equivalence relation defined by

$$(s,x) \sim (y,t) \iff t = \pm s \text{ and } y = x.$$

9. Exercises after Lecture 17

1 If $U \subset \mathbb{R}^k$ and $V \subset \mathbb{H}^k$ are open neighborhoods of 0, prove that there exists no diffeomorphism of V with U. (Hint: Inverse Function Theorem.)

2 Prove that if $f\colon X \to Y$ is a diffeomorphism of manifolds with boundary, then ∂f maps ∂X diffeomorphically onto ∂Y. (Hint: Inverse Function Theorem.)

3 We define the smooth maps

$$F\colon \mathbb{R} \times [-1/2,1/2] \to \mathbb{R}^3,\ (t,s) \mapsto (\cos t, \sin t, s), \text{ and}$$
$$G\colon \mathbb{R} \times [-1/2,1/2] \to \mathbb{R}^3,\ (t,s) \mapsto ((1 + s\cos(t/2))\cos t, (1 + s\cos(t/2))\sin t, s\sin(t/2)).$$

We define X to be the image of F in $\mathbb{R}^3$, and Y to be the image of G in $\mathbb{R}^3$.

a) Show that X is a 2-dimensional manifold with boundary whose boundary is diffeomorphic to the disjoint union of two copies of the unit circle. (Convince yourself that X is a cylinder obtained by starting with a rectangular surface and then glueing two opposite edges together.)

b) Show that Y is a 2-dimensional manifold with boundary whose boundary is diffeomorphic to just one copy of the unit circle. (Convince yourself that Y is a Möbius band obtained by starting with a rectangular surface and then glueing two opposite edges after twisting one edge once. If you do not get through all the formulae, make sure you understand the answer visually at least.)

4 Suppose that X is a manifold with boundary and $x \in \partial X$. Let $\phi\colon U \to X$ be a local parametrization with $\phi(0) = x$, where U is an open subset of $\mathbb{H}^k$. Then $d\phi_0\colon \mathbb{R}^k \to T_x(X)$ is an isomorphism. Define the upper halfspace $H_x(X)$ in $T_x(X)$ to be the image of $\mathbb{H}^k$ under $d\phi_0$, $H_x(X) := d\phi_0(\mathbb{H}^k)$.

a) Prove that $H_x(X)$ does not depend on the choice of local parametrization.

b) Show that there are precisely two unit vectors in $T_x(X)$ that are perpendicular to $T_x(\partial X)$ and that one lies inside $H_x(X)$, the other outside. The one in $H_x(X)$ is called the inward unit normal vector to the boundary, and the other is the outward unit normal vector to the boundary. Denote the outward unit normal vector by $n(x)$.

c) If $X \subset \mathbb{R}^N$, we consider $n(x)$ as an element in $\mathbb{R}^N$ and get a map $n : \partial X \to \mathbb{R}^N$. Show that n is smooth.

5 Let $X = \{(x,y) \in \mathbb{R}^2 : x \geq -1\}$, $Y = \mathbb{R}$ and

$$f\colon X \to Y,\ (x,y) \mapsto x^2 + y^2.$$

a) What is the boundary of X? Show that 1 is a regular value of f. Is 1 a regular value of ∂f?

b) Determine $f^{-1}(1)$, $\partial(f^{-1}(1))$ and $f^{-1}(1) \cap \partial X$. Why does the answer not contradict the assertion of the Preimage Theorem for manifolds with boundary?

10. Exercises after Lecture 19

1 Prove the Theorem of Perron-Frobenius: An $n \times n$-matrix A with only nonnegative entries, must have a real nonnegative eigenvalue.

(Hint: It suffices to assume A nonsingular, otherwise O is an eigenvalue. Let A also denote the associated linear map of $\mathbb{R}^n$, and consider the map $v \to Av/|Av|$ restricted to $S^{n-1} \to S^{n-1}$. Show that this maps the first quadrant

$$Q = \{(x_l, \ldots, x_n) \in S^{n-1} : \text{ all } x_i \geq 0\}$$

into itself. Now use the fact that there is a homeomorphism $B^{n-1} \to Q$, to get a continuous map $B^{n-1} \to B^{n-1}$.)

2 Let X and Y be submanifolds of $\mathbb{R}^N$. Show that for almost every $a \in \mathbb{R}^N$ the translate $X + a$ intersects Y transversally.

3 **a)** Let Y be a compact submanifold of $\mathbb{R}^M$, and $w \in \mathbb{R}^M$. Show that there exists a (not necessarily unique) point $y \in Y$ closest to w, and prove that $w - y \in N_y(Y)$. (Hint: If $c(t)$ is a curve on Y with $c(0) = y$, then the smooth function $|w - c(t)|^2$ has a minimum at 0. Now use that we have shown on Exercise Set 2 that there is a unique correspondence between tangent vectors at y and velocity vectors at 0 of curves $c\colon (-a,a) \to Y$ with $c(0) = y$.)

b) Use the previous point to show: Let Y be a compact submanifold of $\mathbb{R}^M$, and $w \in \mathbb{R}^M$. Let $h\colon N(Y) \to \mathbb{R}^M$, $h(y,v) = y + v$, be the map used in the proof of the ϵ-Neighborhood Theorem in the lecture. We know that h maps a neighborhood of Y in $N(Y)$ diffeomorphically onto $Y^\epsilon \subset \mathbb{R}^M$, where $\epsilon > 0$ is constant. Prove that if $w \in Y^\epsilon$, then $\pi(w)$ is the unique point of Y closest to w, where $\pi = \sigma \circ h^{-1}$.

4 Let X be a submanifold of $\mathbb{R}^N$. Show that “almost every” vector space V of any fixed dimension k in R^N intersects X transversally, i.e.

$$V + T_x(X) = \mathbb{R}^N \text{ for every } x \in X.$$

(Hint: Use the fact that the set $S \subset \left(\mathbb{R}^N\right)^k$ consisting of all linearly independent k-tuples of vectors in $\mathbb{R}^N$ is open in R^{Nk}. Show that the map $\mathbb{R}^k \times S \to \mathbb{R}^N$ defined by

$$((t_1, \ldots, t_k), v_1, \ldots, v_k) \mapsto t_1 v_1 + \cdots + t_k v_k$$

is a submersion, and apply the results of the lecture.)

5 This is a harder problem, but it is an interesting application of the Transversality Theorem and ϵ-neighborhoods. So try it!

a) Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map with $n > 1$, and let $K \subset \mathbb{R}^n$ be compact and $\epsilon > 0$. Show that there exists a map $g\colon \mathbb{R}^n \to \mathbb{R}^n$ such that dg_x is never 0, and $|f(x) - g(x)| < \epsilon$ for all $x \in K$.
(Hint: Let $M(n)$ be the space of $n \times n$-matrices. Show that the map $F\colon R^n \times M(n) \to M(n)$, defined by $F(x,A) = df_x + A$, is a submersion. Pick A so that $F_A \pitchfork \{0\}$ for $F_A\colon x \mapsto (x,A)$ as in the lecture. Now use this knowledge to construct g. At some point along this way you will have used $n > 1$. Make sure you see where and how it has been used.)

b) Show that this result is false for $n = 1$ (i.e., find f, ϵ, $K \subset \mathbb{R}$ such that we cannot find such a g).
(Hint: You could contemplate on the Mean Value Theory.)

11. Exercises after Lecture 21

1 Show that there exists a complex number z such that

$$z^7 + \cos(|z|^2)(1 + 93z^4) = 0.$$

2
- **a)** Assume $\dim X \geq 1$: Show hat if $f\colon X \to Y$ is homotopic to a constant map, then $I_2(f,Z) = 0$ for all complementary dimensional closed submanifolds Z in Y.
  (Hint: Show that if $\dim Z < \dim Y$, then f is homotopic to a constant $X \to \{y\}$, where $y \notin Z$.
- **b)** For $\dim X = 0$, show that this assertion is wrong. (If X is one point, for which Z will $I_2(f,Z) \neq 0$?)
- **c)** Show that S^1 is not simply-connected. (Recall that we call a manifold X simply-connected if it is connected and if every map of the circle S^1 into X is homotopic to a constant map.)
  (Hint: Consider the identity map.)

3
- **a)** Show that intersection theory is trivial in contractible boundaryless manifolds: if Y is boundaryless and contractible (i.e. its identity map is homotopic to a constant map) and $\dim Y > 0$, then $I_2(f,Z) = 0$ for every $f\colon X \to Y$, X compact and Z closed, $\dim X + \dim Z = \dim Y$. In particular, intersection theory is trivial in Euclidean space.
- **b)** Prove that no compact boundaryless manifold - other than the one-point space - is contractible.
  (Hint: Apply the previous point to the identity map.)

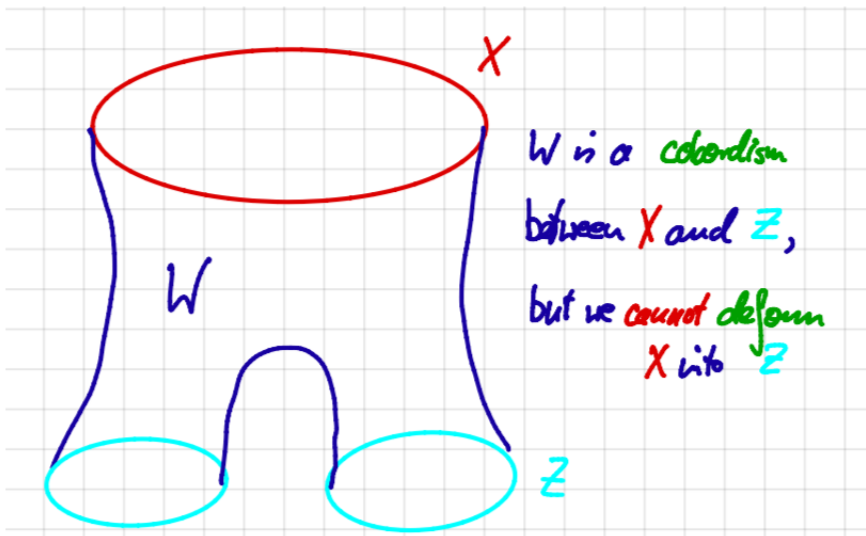
4
- **a)** Let $f\colon X \to S^k$ be a smooth map with X compact and $0 < \dim X < k$. Show that, for all closed submanifolds $Z \subset S^k$ of dimension complementary to X, $I_2(f,Z) = 0$.
  (Hint: Use Sard's Theorem to show that there exists a $p \notin f(X) \cap Z$. Now use stereographic projection and the previous exercises.)
- **b)** Show that S^2 and the torus $T = S^1 \times S^1$ are not diffeomorphic.

5
- **a)** Two compact manifolds X and Z of the same dimension in Y are called **cobordant** in Y if there exists a compact manifold with

boundary $W \subset Y \times [0,1]$ such that

$$\partial W = X \times \{0\} \cup Z \times \{1\}.$$

The manifold W is also called a **cobordism** between X and Z.
Show that if we can deform X into Z, i.e. if there is a smooth homotopy from the embedding $i_0\colon X \hookrightarrow Y$ of X in Y to an embedding $i_1\colon X \hookrightarrow Y$ with $i_1(X) = Z$ such that each i_t is an embedding, then X and Z are cobordant.
Note that the standard image of a cobordism, a pair of pants, illustrates that the converse is false: X and Z are cobordant, but we cannot deform X into Z, since X has one connected component whereas Z has two.



b) Show that if X and Z are cobordant in Y, then for every compact submanifold C in Y with dimension complementary to X and Z, i.e. $\dim X + \dim C = \dim Z + \dim C = \dim Y$ (where $\dim X = \dim Z$ because they are cobordant), we have

$$I_2(C,X) = I_2(C,Z).$$

(Hint: Let f be the restriction to W of the projection map $Y \times [0,1] \to Y$, and use the Boundary Theorem.)

6 Let $p_1, \ldots, p_n$ be real polynomials in $n+1$ variables. Assume each p_i is homogeneous of odd order, i.e. there is an odd number m_i such that $p_i(\lambda x) = \lambda^{m_i} p_i(x)$ for all $\lambda \in \mathbb{R}$. We consider each p_i also as a smooth function $\mathbb{R}^{n+1} \to \mathbb{R}$ by sending x to $p_i(x)$.

Show that there is a line through the origin in $\mathbb{R}^{n+1}$ on which all the p_i's simultaneously vanish.

(Hint: Read Lecture 21 carefully.)

7 Let $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be the unit circle and $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the two-dimensional sphere.

Show that there is no continuous map $f\colon S^2 \to S^1$ with $f(-p) = -f(p)$ for all $p \in S^2$.

Hint: Assume such a map f existed. Then we could define the continuous map

$$g\colon B^2 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \to S^1,\ g(x,y) := f(x,y,\sqrt{1 - x^2 - y^2}).$$

Show that g satisfies $g(-q) = -g(q)$ for all $q \in S^1 = \partial B^2$. What is the degree modulo 2 of g? Conclude that f cannot exist.

12. Exercises after Lecture 22

1 Let $\beta = (v_1, \ldots, v_k)$ be an ordered basis of a vector space V.

a) Show that replacing one v_i by a multiple cv_i yields an equivalently oriented ordered basis if $c > 0$, and an oppositely oriented one if $c < 0$.

b) Show that transposing two elements, i.e., interchanging the places of v_i and v_j for $i \neq j$, yields an oppositely oriented ordered basis.

c) Show that subtracting from one v_i a linear combination of the others yields an equivalently oriented ordered basis.

d) Suppose that V is the direct sum of V_1 and V_2. Show that the direct sum orientation of V from $V_1 \oplus V_2$ equals $(-1)^{(\dim V_1)(\dim V_2)}$ times the orientation from $V_2 \oplus V_1$.

2 The upper half space $\mathbb{H}^k$ is oriented by the standard orientation of $\mathbb{R}^k$. Thus $\partial\mathbb{H}^k$ acquires a boundary orientation. But $\partial\mathbb{H}^k$ may be identified with $\mathbb{R}^{k-1}$. Show that the boundary orientation agrees with the standard orientation of $\mathbb{R}^{k-1}$ if and only if k is even.

3 **a)** Write down the orientation of S^2 as the boundary of the closed unit ball B^3 in $\mathbb{R}^3$, by specifying a positively oriented ordered basis for the tangent space at each $(a,b,c) \in S^2$.

b) Show that the boundary orientation of S^k equals the orientation of $S^k = g^{-1}(1)$ as the preimage under the map

$$g\colon \mathbb{R}^{k+1} \to \mathbb{R},\ x \mapsto |x|^2.$$

4 Suppose that $f\colon X \to Y$ is a diffeomorphism of connected oriented manifolds with boundary. Show that if $df_x\colon T_x(X) \to T_{f(x)}(Y)$ preserves orientation at one point x, then f preserves orientation globally.

5 Let X and Z be transversal submanifolds in Y and assume X, Z and Y are oriented. Let $i\colon X \hookrightarrow Y$ be the inclusion of X into Y, $j\colon Z \hookrightarrow Y$ be the inclusion of Z into Y. We orient the intersection $X \cap Z$ as the preimage $i^{-1}(Z)$, and the intersection $Z \cap X$ as the preimage $j^{-1}(X)$. Show that the orientations of $X \cap Z$ and $Z \cap X$ are related by

$$X \cap Z = (-1)^{(\operatorname{codim} X)(\operatorname{codim} Z)} Z \cap X.$$

(Hint: Show that the orientation of $S = X \cap Z$ at any y is induced by the direct sum

$$(N_y(S,X) \oplus N_y(S,Z)) \oplus T_y(S) = T_y(Y).$$

What happens when you consider $Z \cap X$ instead?)

6 **a)** Let V be a vector space. Show that both orientations on V define the same product orientation on $V \times V$.

b) Let X be an orientable manifold. Show that the product orientation on $X \times X$ is the same for all choices of orientation on X.

c) Suppose that X is not orientable. Show that $X \times Y$ is never orientable, no matter what manifold Y may be. In particular, $X \times X$ is not orientable.
(Hint: First show that $X \times \mathbb{R}^m$ is not orientable, and then use that every Y has an open subset diffeomorphic to $\mathbb{R}^m$.)

d) Prove that there exists a natural orientation on some neighborhood of the diagonal Δ in $X \times X$, whether or not X can be oriented.
But note that Δ itself is orientable if and only if $X \times X$ is orientable. Why?
(Hint: Cover a neighborhood of Δ by local parametrizations $\phi \times \phi\colon U \times U \to X \times X$, where $\phi\colon U \to X$ is a local parametrization of X, then apply the previous observations.)