Searching For New Invariants Of Four-Manifolds And Knots

Edward Witten

Geometry of Quantum Fields and Strings, Auckland

- New invariants of smooth four-manifolds are needed
- Donaldson and Seiberg-Witten invariants do not distinguish smooth four-manifolds that may well be different
- Something may be needed that goes beyond counting solutions

- Recall Donaldson invariants are based on counting solutions of the instanton equation for an SU(2) gauge field:

$$F_{\mu\nu}^{+} = 0$$

- To define the usual Seiberg-Witten equations, we replace SU(2) by U(1) but we add a charged "hypermultiplet" field M. Resulting equations are schematically

$$F_{\mu\nu}^{+} + \bar{M}\Gamma_{\mu\nu}^{+}M = 0, \quad \not{D}M = 0.$$

  M is a positive chirality spinor of charge 1.
- Many variants of these equations can be obtained by replacing U(1) with some other gauge group G and replacing M with a spinor field valued in some other representation R.
- We can try to define invariants by counting solutions of these equations.

Two basic problems:

- Unless the representation R is very small, there are compactness issues such as those discussed by Taubes
- In particular, as we vary the metric on a four-manifold X, a solution may run off to $M = \infty$ and disappear
- There is also a problem of a completely different kind: to the extent that one can define such invariants, Seiberg-Witten theory of the Coulomb branch tends to suggest that they will not contain new four-manifold information.
- I want to stress that the last point is not bullet-proof and some experts disagree, or at least are skeptical.

Today I will focus on a particular case that is related to $\mathcal{N} = 4$ super Yang-Mills theory. We will consider any gauge group, but SU(2) is a typical example. The hypermultiplet will be valued in the adjoint representation. For that particular case, there are more options in the "twisting" that formally produces a topological field theory. The case I want to focus on today are what are sometimes called the Vafa-Witten (VW) equations (C. Vafa and EW, hep-th/9408074).

- After twisting, the fields are a gauge field A, which is a connection on a G-bundle $E \to X$; a field B that is a self-dual two-form valued in the adjoint representation of G, in other words B is a section of $\Omega^{2,+}(X) \otimes \mathrm{ad}(E)$, and a scalar C valued in the adjoint representation, in other words C is a section of $\mathrm{ad}(E)$.

- The equations are schematically

$$F^+ + \frac{1}{2}[C, B] + \frac{1}{4}[B, B] = 0, \qquad D^\mu B_{\mu\nu} + D_\nu C = 0.$$

- One can show that in any irreducible solution on a compact four-manifold, $C = 0$. However, including C is important for the ellipticity of the equations.

- Based on an index theorem, the expected dimension of the moduli space of solutions is 0.
- So imitating Donaldson, the natural (formal) invariant is the "number" of solutions, for a given choice of the instanton number.
- However, solutions can actually naturally occur in families because some reductions of the equation lead to a nontrivial expected dimension.
- The most obvious such reduction is to set $B = C = 0$, whence the equations reduce to the instanton equations and the solutions of those equations occur in families of positive dimension.

Vafa and I were not really interested in defining four-manifold invariants. We were trying to test the S-duality conjecture of $\mathcal{N} = 4$ super Yang-Mills theory, which was hard to test because computations for strong coupling are difficult. We showed that if X satisfies a very strong condition that its Ricci tensor is non-negative, $R_{\mu\nu} \geq 0$, then all solutions have $B = C = 0$ and are instantons. Moreover, a formal argument then shows that the invariant a_k that "counts" solutions for instanton number k is the Euler characteristic of $\mathcal{M}_k$, the instanton number k moduli space. S-duality predicts that the function

$$F(q) = \sum_k a_k q^k$$

should have modular properties.

Not too many four-manifolds have $R_{\mu\nu} \geq 0$, but luckily for a few cases that do (a K3 surface, $\mathbb{CP}^2$, and an ALE space) the values of the a_k could be extracted from results of mathematicians (Klyachko, Yoshioka, Nakajima, et. al.). From these examples, it was possible to make some interesting tests of the S-duality prediction.

Much more has been done in this direction more recently (for example, R. Thomas, arXiv:1810.00078).

- But what if we want to define four-manifold invariants, rather than testing S-duality? Then we have the two problems that I already mentioned:
- The equations are not known to have useful compactness properties (lecture by Taubes)
- Seiberg-Witten theory tends to suggest that if we can define invariants that count the solutions, they will not contain new four-manifold information.

An optimistic idea about the second problem:

- Formally, the VW invariants can be categorified.
- That is because the solutions of the VW equations are the critical points of a certain functional, and the gradient flow equation for this functional is an elliptic PDE in five dimensions.

Once we set $C = 0$, the VW equations for the other fields are the equations for a critical point of a certain functional

$$W(A, B) = \int_X \mathrm{d}^4x \sqrt{g} \left(B^{\mu\nu} F^+_{\mu\nu} - \frac{1}{3} B_{\mu\nu} [B_{\nu\lambda}, B_{\lambda\mu}] \right).$$

Since this is the case, we can add a fifth dimension and look at the gradient flow equation for this functional

$$\frac{\partial \Phi}{\partial t} = -\frac{\delta W}{\delta \Phi},$$

where Φ schematically represents the pair (A, B) and the gradient is defined using the obvious metric on the space of fields

$$|\Phi|^2 = -\int_X \mathrm{d}^4x \sqrt{g} \, \mathrm{Tr} \left(\delta A^2 + \delta B^2 \right).$$

Fortuitously, this gradient flow equation (sometimes called the Haydys-Witten or HW equation) is elliptic. It does NOT have five-dimensional rotation symmetry, so it is only defined on a five-manifold such as $X \times \mathbb{R}$ with a preferred "time" direction. But since it is elliptic, one can sensibly count its solutions. So one can do Morse or Floer theory for the functional Γ. In this way, one would define a "categorified" version of the VW invariants: the invariants would be vector spaces associated to a four-manifold rather than numerical invariants. To the extent that the VW invariants are Euler characteristics of instanton moduli spaces, the categorified invariants would be the cohomology groups of instanton moduli space – vector spaces rather than numbers.

The physical interpretation would be in terms of $4+1$-dimensional super Yang-Mills theory. On a five-manifold of the particular form $X \times \mathbb{R}$, this theory has a "twisted" version that is topological on X, but not on $\mathbb{R}$. It has two unbroken supercharges, say Q and $\bar{Q}$, satisfying $Q^2 = \bar{Q}^2 = 0$ and $\{Q, \bar{Q}\} = H$, where H is the Hamiltonian. The "categorified" invariant is the cohomology of Q. To the extent that the solutions of the VW equations all have $B = 0$ and we do not have to worry about singularities, the classical ground states in the sector with Pontraygin number k are instantons, the operator Q is the de Rham differential of instanton moduli space $\mathcal{M}_k$, and the categorified invariant is the cohomology of $\mathcal{M}_k$. The categorified theory is doubly-graded by k, that is by the instanton number, and by the degree d of a differential form.

Let us discuss a few facts about the VW solutions. The VW equations have a $\mathbb{Z}_2$ symmetry $\tau : B \to -B$ (and $C \to -C$). The obvious τ-invariant solutions are instantons, with $B = C = 0$. There is another branch of τ-invariant solutions in which the gauge field A is $\mathfrak{u}(1)$-valued (not $\mathfrak{su}(2)$-valued) and B is nonzero, but one can compensate for $B \to -B$ by a $U(1)$ gauge transformation. Using a standard basis t_1, t_2, t_3 of $\mathfrak{su}(2)$, this happens if A is proportional to t_1 and B is a linear combination of t_2 and t_3. Let us write $\mathcal{M}_k$ and $\mathcal{W}_k$ for the moduli spaces of τ-invariant solutions with Pontryagin number (instanton number) k.

An index theorem says that the expected dimension of the moduli space of VW solutions is 0, so one expects a "typical" solution to be isolated. But τ-invariant solutions come in families $\mathcal{M}_k$ and $\mathcal{W}_k$. The index theorem predicts for $\mathcal{M}_k$ the dimension

$$d_k = 8k - 3\Delta, \qquad \Delta = 1 - b_1 + b_2^+$$

and for $\mathcal{W}_k$ the dimension

$$\widetilde{d}_k = -8k + \Delta.$$

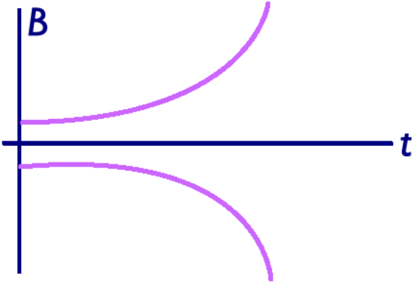
So $\mathcal{M}_k$ is generically empty unless k is sufficiently positive, and $\mathcal{W}_k$ is generically empty unless k is sufficiently negative. (Later, when convenient we take k sufficiently positive so we can focus on instantons and not worry about $\mathcal{W}_k$.)

It can be shown that the grading of the physical Hilbert space by the degree of a differential form is odd under $B \to -B$, which acts as the Hodge $\star$ operator on the cohomology of instanton moduli space. (If this moduli space has degree d, the natural grading ranges from $-d/2$ to $d/2$, shifted from usual by $-d/2$ so as to be symmetric around degree 0.)

Now let us discuss the solutions that are not τ-invariant:

- They occur in pairs related by $B \to -B$ and these pairs have opposite cohomological grading.
- They are generically expected to be isolated.
- Without a solution of the compactness issue, they can disappear when we vary the metric of X.
- This last point is why VW theory and its categorification are hard.

The fact that the solutions might disappear when we vary the metric might be unfamiliar so let me go in a little detail. We imagine a family of metrics on X depending on a parameter t. For $t < t_0$, there is a pair of solutions related by $B \leftrightarrow -B$. But for $t \to t_0$, they may disappear:

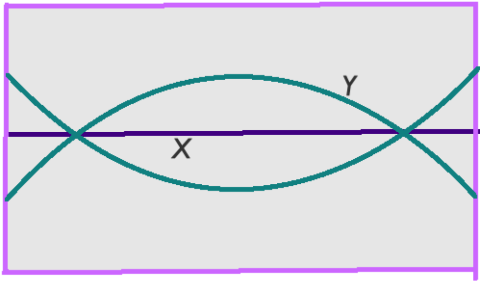


When this happens, the VW "invariants" and their categorification will jump. Thus some sort of bound keeping B from going to infinity is needed in order to really define invariants.

There is a physical picture that can shed some light. We consider Type IIA superstring theory on $\mathbb{R} \times \Omega^+_2(X) \times \mathbb{R}^2$ with N D4-branes wrapped on $\mathbb{R} \times X \times \{0\}$, where X is embedded in $\Omega^+_2(X)$ as the zero-section and $\{0\}$ is the origin in $\mathbb{R}^2$. The low energy physics is the twisted $\mathcal{N} = 4$ super Yang-Mills (with gauge group $\mathrm{U}(N)$) that formally leads to categorified VW invariants. Compactness issues have to do with the possiblity that a brane could separate from $X \times \{0\} \subset \Omega^+_2(X) \times \mathbb{R}^2$. Reducible solutions correspond to branes separating in the $\mathbb{R}^2$ directions. This can be avoided by standard topological conditions. B becoming large correspond to branes moving in the fiber direction of $\Omega^+_2(X) \to X$. There is no known cure for this, or satisfactory understanding of how it happens.

We can think of $\Omega_2^+(X)$ as a manifold with a (generally unintegrable) G_2 structure. B is analogous to the "Higgs field" in Hitchin's description of Higgs bundles in 2 dimensions: its eigenvalues represent points in the fiber of $\Omega_2^+(X)$. When B is large, a semiclassical picture should emerge. The different components of B will have to commute with each other because of the $\mathrm{Tr}[B, B]^2$ term in the Yang-Mills action. As in Hitchin's theory of Higgs bundles, their common eigenvalues represent a point in the fiber of the fibration $\Omega_2^+(X) \to X$. For $G = \mathrm{SU}(2)$, there are two such points in the fiber over any point $p \in X$. They are equal and opposite because an $\mathfrak{su}(2)$ matrix is traceless.

Thus when B is large, the picture looks something like this:



Drawn is X (horizontal line) inside $\Omega_2^+(X)$. Motion of branes in the fiber direction gives (for $G = \mathrm{SU}(2)$) a submanifold $Y \subset \Omega_2^+(X)$, which projects to a double cover Y of X, as shown. Y is invariant under $B \to -B$ because B is traceless. For supersymmetry, one expects Y to be a "coassociative cycle" (a four-cycle on which the three-form of the G_2 structure restricts to 0) endowed with a line bundle $\mathcal{L} \to Y$ that will have a U(1) instanton connection a.

The moduli space of pairs (Y, a) has virtual dimension 0 according to an index theorem, so it is reasonable to expect to find isolated solutions. The compactness issue in this language has to do with whether as we vary the metric on X and thus the G_2 structure on $\Omega_2^+(X)$, Y can go to infinity. Such issues have been analyzed in the gauge theory language by Taubes. Hopefully it is possible to match that description with what I've said.

Continuing the search for new invariants of four-manifiolds, consider this:

- Standard arguments show that for a generic metric g on a four-manifold X (of $b_2^+ > 1$ to avoid reducible solutions), the moduli space $\mathcal{M}_k$ of instanton number k is a smooth manifold.
- The topological type of $\mathcal{M}_k$ is *not* a four-manifold invariant, because when one interpolates between two different metrics g and g', one generically will pass through a singularity.
- However, when one changes from g to g', $\mathcal{M}_k$ changes by a cobordism.
- Can cobordism invariants of $\mathcal{M}_k$ such as Pontryagin numbers be viewed as four-manifold invariants of X?

There are at least two reasons that this last question has not been much studied:

- Technically, these Pontryagin numbers are hard to define because of the singularity associated with instanton "bubbling" (shrinking to zero size).
- Also, a formal argument predicts that if one could define them, these numbers would not be essentially new four-manifold invariants but could be expressed in terms of Donaldson or Seiberg-Witten invariants.

I am going to suggest that not all Pontryagin numbers but precisely those Pontryagin numbers that appear in the q-expansion of the elliptic genus of Ochanine, Landweber, and Stong (OLS) can be naturally defined as four-manifold invariants. I actually do not claim that they are essentially new four-manifold invariants. However understanding why they are invariants will help us to define invariants that may really be new.

Instead of Type IIA superstring theory on $\mathbb{R} \times \Omega_2^+(X) \times \mathbb{R}^2$, we are going to replace $\mathbb{R}^2$ by $\mathbb{R} \times S^1$ and consider Type IIB superstring theory on $\mathbb{R}^2 \times \Omega_2^+(X) \times \mathbb{R}$. In this theory, we consider N D5-branes wrapped on $\mathbb{R}^2 \times X$ (times a point $0 \in \mathbb{R}$). The resulting theory has $(1,1)$ supersymmetry on $\mathbb{R}^2$. The fields A, B of the VW equations on X are promoted to superfields on $\mathbb{R}^2$. They are governed by a superpotential which is the same functional we studied before whose critical points are the solutions of the VW equations:

$$W(A,B) = \int_X \mathrm{d}^4 x \sqrt{g} \left(B^{\mu\nu} F^+_{\mu\nu} - \frac{1}{3} B_{\mu\nu} [B_{\nu\lambda}, B_{\lambda\mu}] \right).$$

The low energy theory is, formally a sigma-model whose target is the VW moduli space.

Let us recall that the VW equations have a symmetry $\tau : B \to -B$. In the brane picture on $\mathbb{R}^2 \times \Omega^2_+(X) \times \mathbb{R}$, this symmetry acts as -1 on the fiber of $\Omega_2^+ \to X$ and on the last factor $\mathbb{R}$. In other words, it acts as -1 on the normal bundle to the D5-branes. The superpotential is odd under this symmetry, which means that τ is a "discrete R symmetry" in the (1,1) supersymmetric theory on $\mathbb{R}^2$.

Let us now ask: what are interesting topological invariants that we can extract from a $(1,1)$ supersymmetric theory in two dimensions? Provided the theory has a discrete R-symmetry – such as our τ – one can define the "elliptic genus" of OLS. In terms of qft, it is defined by replacing $\mathbb{R}^2$ with $\mathbb{R} \times S^1$, and then introducing a monodromy by τ in going around the S^1. Thus the geometry is symbolically $\mathbb{R} \times S^1 \times \widetilde{\Omega}_2^+(X) \times \widetilde{\mathbb{R}}$, where I write $\widetilde{\Omega}_2^+$ and $\widetilde{\mathbb{R}}$ to indicate the -1 monodromy in going around the S^1.

The effect of the monodromy is that the fields (A, B) on X have to be τ-invariant in order to maintain supersymmetry. This means that we only have to consider the solutions parametrized by $\mathcal{M}_k$ (instanton moduli space) or $\mathcal{W}_k$ (solutions in which A is abelian and B is odd under a certain gauge transformation). Moreover, the two types of solution exist for different ranges of instanton number: $\mathcal{M}_k$ for $k > 0$ and $\mathcal{W}_k$ for $k < 0$. So if the instanton number is sufficiently positive, we will only see $\mathcal{M}_k$.

When we went from D4-branes on $\mathbb{R} \times X \subset \mathbb{R} \times \Omega_2^+(X) \times \mathbb{R}^2$ to D5-branes on $\mathbb{R}^2 \times X \subset \Omega_2^+(X) \times \mathbb{R}$, we lost a U(1) symmetry – rotation of $\mathbb{R}^2$ – that led to the cohomological grading in the attempt to "categorify the VW invariants." Hence the theory we are discussing now is only graded by $\mathbb{Z} \times \mathbb{Z}_2$ (instanton number times $(-1)^F$), not $\mathbb{Z} \times \mathbb{Z}$. Also, once we make the monodromy by τ to define the elliptic genus, we only have one unbroken supercharge Q. Since $Q^2 \neq 0$, we cannot define cohomology groups. But we can still define the *index* of Q, which in the present context is really the elliptic genus

$$F(q) = \mathrm{Tr}\,(-1)^F q^P,$$

where P is the momentum along S^1.

Now we ask: Is $F(q)$ really a topological invariant, independent of the Riemannian metric of X? I claim Yes:

- As a physicist, one would say that the answer is "yes," provided the target space of the sigma-model is effectively compact.
- Here compactness means that the branes cannot effectively separate from each other.
- The important issue is the instanton bubbling singularity.
- In brane physics, instanton bubbling at a point $p \in X$ means that an instanton turns into a D1-brane wrapped on $\mathbb{R} \times S^1 \times p$.
- The question of compactness is then whether this brane can separate from the remaining brane system.
- This is prevented if there is a nonzero theta-angle for the U(1) gauge field on the D1-brane worldvolume.
- That in turn will be the case if in the underlying Type IIB description in ten dimensions, we turn on a suitable RR potential.

That explanation will probably sound mysterious if one is not a string theorist. So I will explain a nontrivial special case just in terms of PDE's. Here I will consider only single-instanton bubbling. If X is a simply-connected four-manifold with b_2^+ odd, then it admits an almost complex structure, which in turn means that on X, there exists an everywhere nonzero selfdual two-form ω. Any choice of such an ω gives a way to resolve the instanton-bubbling singularity. This follows from the ADHM construction (the following was interpreted in terms of noncommutative geometry by Nekrasov and Schwarz). The singularity when a single instanton shrinks to a point is modeled by $\mathbb{R}^8///U(1)$ where $\mathbb{R}^8$ is viewed as a flat hyper-Kahler manifold and $///$ represents the hyper-Kahler quotient. In other words, the singularity is described by

$$\vec{\mu}^{-1}(0)/\mathrm{U}(1),$$

where $\vec{\mu}$ is the hyper-Kahler moment map. Given ω, we can resolve the singularity by replacing $\vec{\mu}^{-1}(0)/U(1)$ by

$$\vec{\mu}^{-1}(\omega)/\mathrm{U}(1).$$

This is smooth, so (if single-instanton bubbling is all we have to worry about) all the Pontryagin numbers of $\mathcal{M}_k$ are well-defined once we pick an almost complex structure and therefore a homotopy class of ω's. But what happens if we change the almost complex structure? There are a couple of moves to consider, but the main one is a change in the first Chern class of the almost complex structure. By further study of the ADHM construction, one can show that the change in the topology of $\mathcal{M}_k$ in such a move is a "classical flop" (originally studied in Atiyah (1958)).

To describe a classical flop, consider complex variables $z_1, \ldots, z_4$ that satisfy an equation

$$|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = b$$

with constant b, and divide this space by U(1) acting by common phase rotations on all the z_i. The quotient, for $b \neq 0$, is a smooth six-manifold. But it passes through a singularity, with a jump in topology, in passing through $b = 0$. It turns out that the Pontryagin numbers that appear in the elliptic genus are invariant under a classical flop and no other Pontryagin numbers have this property (Totaro, arXiv:math/0003240). (The fact that the Pontraygin numbers that do appear in the elliptic genus are invariant under a classical flop is a generalization of facts about sigma-models with Calabi-Yau targets (Aspinwall, Morrison, and Greene, 1993; EW, 1993).)

So I believe that the elliptic genus of $\mathcal{M}_k$ is a topological invariant of X. But I do not believe that this invariant contains essentially new information about four-manifolds. The reason that I do not is that one can study the matter by Seiberg-Witten theory. To be specific, we look at the Coulomb branch of a system of D5-branes on $S^1 \times S^1 \times R^4$ (with a discrete R-symmetry twist in going around one of the two S^1's). The Coulomb branch in this case is described by a particular K3 surface – an elliptic fibration over a base space that in this example is a copy of $\mathbb{CP}^1$. Though I do not have a complete description of this particular fibration, it is possible to see that it only has singularities of a standard sort and therefore that the resulting four-manifold invariants can be expressed in terms of the standard ones.

- Though the elliptic genus is invariant under classical flops (and many other operations) on any manifold Y, it is really a much more natural invariant if Y is spin.
- Otherwise the sigma-model that would compute the elliptic genus of Y is anomalous.
- Therefore, to proceed further, we want the case that $\mathcal{M}_k(X)$ is spin.
- This can be analyzed by the same methods Donaldson used to show that $\mathcal{M}_k(X)$ is orientable.
- The upshot is to show that $\mathcal{M}_k(X)$ is spin if X is.
- This is related to an anomaly of the brane construction if X is not spin.

I continue assuming that X is spin.

- In this case, we can use the sigma-model with target $\mathcal{M}_k(X)$ to define some invariants that go beyond the elliptic genus.
- In particular, if the dimension of $\mathcal{M}_k(X)$ is of the form $8r+1$ or $8r+2$, then we can define the "mod 2 index" of the Dirac operator on $\mathcal{M}_k(X)$ acting on any of the representations that appear in the elliptic genus.
- These should all be topological invariants of X by the same arguments as for the elliptic genus.
- For example, the mod 2 index valued in each of these representations will be invariant under classical flops (and lots of other operations).

The representations that arise in the elliptic genus at successive "mass levels" are the trivial representation; the n-dimensional vector representation V of $SO(n)$; $\wedge^2 V$; etc.

So I am talking about the mod 2 index of the Dirac operator on $\mathcal{M}_k(X)$ acting on sections of S, $S \otimes T$, $S \otimes \wedge^2 T$, $\dots$, where S is the spin bundle and T is the tangent bundle of $\mathcal{M}_k(X)$. (Only finitely many of these are independent, the number depending on the dimension $8r+1$ or $8r+2$ of $\mathcal{M}_k(X)$.)

To conclude:

- This is what I can offer that *might* be an essentially new four-manifold invariant: the mod 2 index of the Dirac operator on $\mathcal{M}_k(X)$, with values in any of the representations appearing in the elliptic genus.
- I don't really know if these invariants are essentially new, but it does not seem possible to use any standard physics argument to argue that they are not.
- Even if they are new, I don't know if they are useful.
- The essential difference (relative to VW invariants or the elliptic genus of $\mathcal{M}_k(X)$) is that these invariants cannot be computed by a four-dimensional path integral.
- The "categorified VW invariants" – if they make any sense – also cannot be computed by a four-dimensional path integral so could potentially be new if the compactness problem can be solved.