Knot Invariants From Gauge Theory In Three, Four, or Five Dimensions

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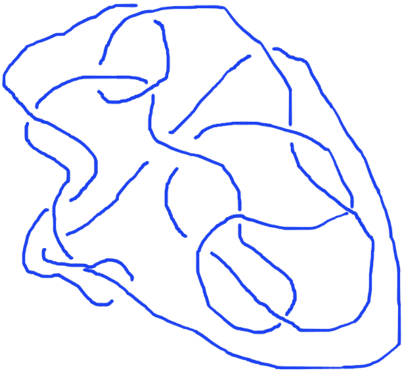
We will be discussing the relations between a sequence of theories in dimension 2-3-4-5:

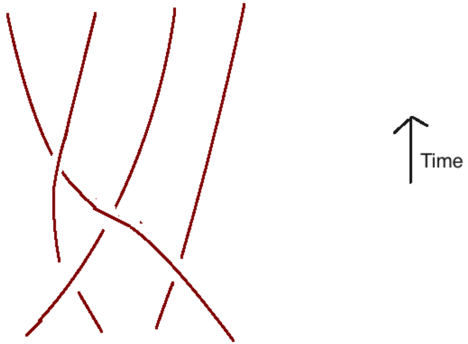
- Conformal blocks of the WZW model in dimension 2.
- The Jones polynomial and Chern-Simons gauge theory in dimension 3.
- $\mathcal{N} = 4$ super Yang-Mills in dimension 4.
- A dimension 5 construction that will lead to a categorification of the Jones polynomial – a candidate for Khovanov homology.

The Jones polynomial is a rather subtle invariant of a knot in $\mathbb{R}^3$:



Ever since Vaughn Jones's original work, constructions of it have generally been related to mathematical physics – in a bewildering variety of ways.

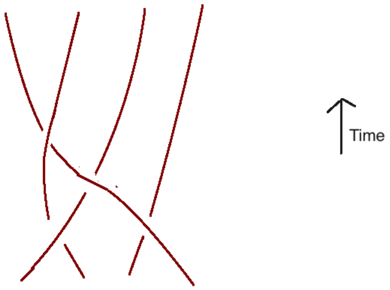
One of Jones's original constructions involved the Jones representations of the braid group. A braid in the mathematical sense is a picture like this:



Points move around in the plane, then return to their starting positions. In this example, I've drawn a braid with 4 strands.

Braids form a group: they can be composed by gluing one braid on top of another. Jones constructed some rather mysterious representations of the braid group. The representation matrices depended on a complex parameter q, so I will call these representations $R_i(q)$ (here i ranges over a finite set, the precise number depending on the number of strands considered).

One of the early definitions of the Jones polynomial of a knot was as follows. We can build a knot by gluing together the top and bottom ends of a braid:



This gluing is a little bit like taking a trace. Let B be a braid. Let us write $\mathcal{R}_{i,q}(B)$ for the matrix that represents this braid in the representation $R_i(q)$. An early definition of the Jones polynomial is that it is a certain linear combination of these traces

$$J(q) = \sum_i c_i(q) \, \mathrm{Tr}_{R_i(q)} \, \mathcal{R}_{i,q}(B).$$

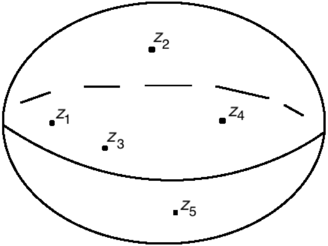
Everything about this formula was a bit mysterious: the construction of the braid group representations $R_i(q)$ was non-obvious, the particular functions $c_i(q)$ were obscure, and – since the same knot can be constructed by taking the “trace” of many different braids – it is not obvious why a knot invariant can be constructed in such a fashion anyway.

I first learned about all these things from Michael Atiyah, who predicted that the mysteries should be unraveled by interpreting the Jones polynomial in quantum field theory.

A very important step was taken by Tsuchiya and Kanie. They showed that the Jones representations of the braid group are the ones that arise when one decomposes the correlation functions of the two-dimensional WZW model in conformal blocks, as originally analyzed by Knizhnik and Zamolodchikov. The WZW model is a two-dimensional conformal field theory that depends on the choice of a compact symmetry group G and a positive integer k. It turns out that in the relation to the Jones polynomial, we should take $G = \mathrm{SU}(2)$ and relate k to q by $q = \exp(2\pi\mathrm{i}/(k+2))$. The WZW model has "primary fields" in various representations of G. To make contact with the Jones representations of the braid group, we consider a primary field Φ in the two-dimensional representation.

Consider genus 0 correlation functions of Φ:

$$G(z_1, \bar{z}_1; z_2, \bar{z}_2; \cdots; z_n, \bar{z}_n) = \langle \Phi(z_1, \bar{z}_1)\Phi(z_2, \bar{z}_2)\Phi(z_3, \bar{z}_3) \cdots \Phi(z_n, \bar{z}_n) \rangle.$$



These functions are neither holomorphic nor antiholomorphic, and they cannot be factored as the product of a holomorphic and an antiholomorphic function. But Knizhnik and Zamolodchikov had shown that they are *finite* sums of products of holomorphic and antiholomorphic functions:

$$G(z_1, \bar{z}_1; z_2, \bar{z}_2; \cdots; z_n, \bar{z}_n) = \sum_\alpha f_\alpha(z_1, z_2, \cdots, z_n) \bar{f}_\alpha(\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_n).$$

Here the functions $f_\alpha(z_1, z_2, \cdots, z_n)$ are multivalued holomorphic functions. For each n, we can define a vector bundle V_n over the configuration space of n distinct points $z_1, z_2, \cdots, z_n$ with a basis given by the f_α. Single-valuedness of the original correlation functions $G(z_1, \bar{z}_1; z_2, \bar{z}_2; \cdots; z_n, \bar{z}_n)$ implies that the V_n are flat vector bundles, so their *monodromies* when the points move around give representations of the braid group. The observation of Tsuchiya and Kanie was (simplifying slightly) that these are the Jones representations.

As understood by physicists at the time, the WZW model is a purely two-dimensional quantum field theory, with no particular connection to three dimensions. But the relation of the conformal blocks of the WZW model to the Jones representations of the braid group showed that the WZW model somehow had an unexpected relation to three dimensions. (Another somewhat similar clue in that direction came from work in this period of E. Verlinde, and there was important work on the conformal blocks by G. Moore and N. Seiberg. I won't have time to recall those matters.)

In three dimensions, there actually is a quantum field theory that depends on precisely the same data as the 2d WZW model, namely a gauge group G and a nonzero integer k. We simply do Yang-Mills theory in three spacetime dimensions, with a compact gauge group G, but we choose the action to be the Chern-Simons function

$$I = \frac{k}{4\pi} \int_W \mathrm{d}^3 x \, \epsilon^{ijk} \mathrm{Tr} \left(A_i \partial_j A_k + \frac{2}{3} A_i A_j A_k \right)$$

and not the usual Yang-Mills action. The function I is only gauge-invariant mod $2\pi k$. In quantum theory, the action must be gauge invariant mod $2\pi\mathbb{Z}$ so that the argument $\exp(\mathrm{i}I)$ of the Feynman path integral will be well-defined. So k must be an integer. (This argument, due originally to Deser, Jackiw, and Templeton, is somewhat similar to the argument showing integrality of k in the WZW model in two dimensions.)

The importance of using the Chern-Simons function rather than a standard Yang-Mills action is that it can be defined on an oriented three-manifold W with no choice of metric tensor on W, so therefore the Feynman path integral of this theory

$$Z_W(k) = \int DA \exp(iI)$$

doesn't depend on anything except W itself – in other words if it makes sense at all, it will give a topological invariant of W.

We can include a knot or embedded circle $K \subset W$ by including a Wilson loop operator

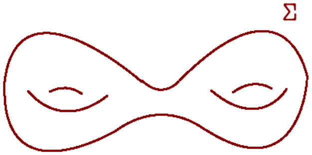
$$\mathcal{W}_R(K) = \mathrm{Tr}_R P \exp \oint_K A,$$

where R is a representation of G and the symbol P represents "holonomy." Now the path integral

$$\int DA \, \exp(iI) \, \mathcal{W}_R(K)$$

depends on K (and R) as well as on W, but nothing else. So (taking $W = \mathbb{R}^3$) this will potentially give an invariant of a knot.

The basic relation between the 3d Chern-Simons theory and the 2d WZW model is that the space of physical states of the 3d theory is the same as the space of conformal blocks of the 2d theory. For example, quantize the 3d theory on $\mathbb{R} \times \Sigma$ where Σ is a Riemann surface, say of genus g:

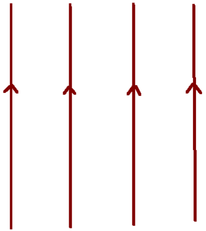


To construct the physical Hilbert space $\mathcal{H}_\Sigma$, we have to quantize the appropriate classical phase space.

The classical phase space is the space of classical solutions of the theory on $\Sigma \times \mathbb{R}$. Since the classical field equation is just $F = 0$, i.e. vanishing of the curvature $F = \mathrm{d}A + A \wedge A$, the classical phase space $\mathcal{M}_\Sigma$ that has to be quantized is the moduli space of flat G-bundles over Σ. This can be quantized by picking a complex structure on Σ, which then induces a complex structure on $\mathcal{M}_\Sigma$. The "prequantum line bundle" is then $\mathcal{L}^k$, where $\mathcal{L}$ is the fundamental holomorphic line bundle over $\mathcal{M}_\Sigma$ (the "determinant line bundle") and the physical Hilbert space is the space of holomorphic sections $H^0(\mathcal{M}_\Sigma, \mathcal{L}^k)$.

I've drastically shortened what is actually a long story, but the point is that the answer $H^0(\mathcal{M}_\Sigma, \mathcal{L}^k)$ coincides with a known and in a sense standard – though rather abstract – description of the space of conformal blocks on a genus g surface. (I probably learned this description from Graeme Segal. It is widely used – in a more general form – in research on the geometric Langlands program.) This is the basic link between 2 and 3 dimensions.

To get the Jones representations of the braid group, we replace Σ by $\mathbb{R}^2$ (or $\mathbb{CP}^1$, in a slightly different approach) but with parallel Wilson lines:



Quantizing in the presence of these Wilson lines, we get the Jones representations of the braid group. With some further arguments, one can obtain a formula for the expectation value of a Wilson operator

$$J(q) = \sum_i c_i(q)\, \mathrm{Tr}_{R_i(q)}\, \mathcal{R}_{i,q}(B)$$

that coincides precisely with Jones's formula for the Jones polynomial in terms of traces.

In this presentation, the theory is really defined only for an integer k, or for q for the form $q = \exp(2\pi \mathrm{i}/(k+2))$, $k \in \mathbb{Z}$. If one wants a theory that is defined on an arbitrary three-manifold W, that is the right answer. But if one considers just knots in $\mathbb{R}^3$, then one can analytically continue the knot invariant $J(q)$ (and its analogs for other groups and representations) to functions of a complex variable q. More specifically $J(q)$ (and its generalizations) is a Laurent polynomial in q, which is why the invariants that Jones discovered are described as a knot "polynomial." This is mysterious from the 3d Chern-Simons point of view, but the interpretation in terms of 2d conformal blocks makes it clear: it follows from simple properties of the Knizhnik-Zamolodchikov connection.

So the approach that makes topological invariance clear does not seem to directly explain why the invariants are a "polynomial." For about 20 years, I accepted this state of affairs, but by around 2007-8, new developments notably involving the "volume conjecture" (Kashaev,....; Gukov) motivated me to look for a new explanation of why the 3d Chern-Simons path integral for knots in $\mathbb{R}^3$ can be analytically continued away from integer k. It turns out that this involved a link between 3 and 4 dimensions that I think is just as interesting as the link between 2 and 3 dimensions.

Before tackling path integrals, let us start with an ordinary integral in 1 dimension:

$$Z(a) = \int_{-\infty}^{\infty} \mathrm{d}x \exp(-x^4 + ax^2).$$

To improve the analogy between this integral and a Feynman path integral, let us derive a Ward identity:

$$0 = \int_{-\infty}^{\infty} \mathrm{d}x \frac{\mathrm{d}}{\mathrm{d}x} \left(x \exp(-x^4 + ax^2) \right),$$

or

$$\left(1 - 4\frac{\mathrm{d}^2}{\mathrm{d}a^2} + 2a\frac{\mathrm{d}}{\mathrm{d}a} \right) Z(a) = 0.$$

When we generalize the path integral, we want to preserve the Ward identity.

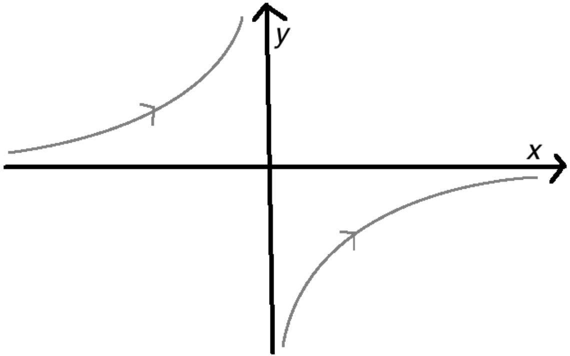
To generalize the original integral, we analytically continue from x to a complex variable $z = x + iy$. We can write the original integral as

$$\int_\Gamma \mathrm{d}z \exp(-z^4 + az^2),$$

where Γ is an integration cycle that happens to be the real axis. We will generalize the integral by considering some other Γ in the "same" integral.

What sort of Γ can we use? A closed cycle is no use, as the integral will vanish by Cauchy's theorem. Nor can we let Γ have a boundary, for then the Ward identity (whose proof involved integration by parts) will fail. The only option is for Γ to run from infinity to infinity, in such a way that the integral converges.

Here are a couple of examples of possible contours:



I've presented this in one dimension, but it should be clear that we can treat an n-dimensional integral

$$\int_{\mathbb{R}^n} \mathrm{d}x_1 \ldots \mathrm{d}x_n \exp(-F(x_1, \ldots, x_n)),$$

where F is a suitable polynomial, in much the same way. We first analytically continue from real variables x_k to complex variables $z_k = x_k + \mathrm{i}y_k$. Then we consider an integral over a suitable cycle $\Gamma \subset \mathbb{C}^n$:

$$\int_\Gamma \mathrm{d}z_1 \ldots \mathrm{d}z_n \exp(-F(z_1, \ldots, z_n)).$$

The properties we want for Γ are the following: (i) It must be middle-dimensional and without boundary; (ii) the function $h = -\mathrm{Re}\, F$ must go to $-\infty$ at infinity along Γ, so that the integral over Γ will converge.

For every such Γ, we get a generalization of the original integral, such that all "Ward identities" are obeyed. It turns out – though I won't explain all the details today – that there is a nice theory of the possible Γ's, given by Morse theory, and that this gives a good framework for understanding analytic continuation of such integrals. What we want to do today is to place the Chern-Simons path integral in the same framework. A quantum mechanical path integral in $0 + 1$ dimensions can, by the way, be studied in a very similar fashion.

The Chern-Simons path integral

$$\int_{\mathcal{U}} DA \exp\left(i\frac{k}{4\pi}\int_W \mathrm{Tr}(A\wedge \mathrm{d}A+\frac{2}{3}A\wedge A\wedge A)\right)$$

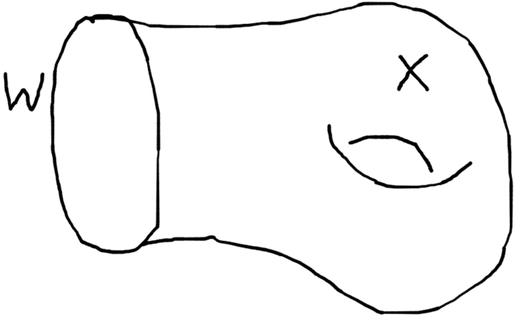
is an integral over the infinite-dimensional space $\mathcal{U}$ of all gauge fields A for gauge group G. The integrand is the exponential of a polynomial, so we are in roughly the situation that I just described. There is no problem formally in analytic continuation of the integrand. A is replaced by a complex-valued connection $\mathcal{A} = A + i\phi$, with the gauge group now being the complexification $G_{\mathbb{C}}$ of the original gauge group. (For example, $SU(2) \to SL(2,\mathbb{C})$.) Also the exponent of the path integral is a polynomial in A that can be analytically continued to the "same" polynomial in $\mathcal{A}$. And $\mathcal{U}$ is replaced by the space $\mathcal{U}_{\mathbb{C}}$ of complex-valued connections. The only catch is that the integration has to run over a middle-dimensional subspace $\Gamma \subset \mathcal{U}_{\mathbb{C}}$.

We need to find a subspace $\Gamma \subset \mathcal{U}_{\mathbb{C}}$ that *(i)* is middle-dimensional, and *(ii)* has the property that the analytically continued path integral

$$\int_{\Gamma} D\mathcal{A} \, \exp\left(i \frac{k}{4\pi} \int_{W} \mathrm{Tr}(\mathcal{A} \wedge \mathrm{d}\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}) \right)$$

converges. How are we supposed to do that?

There is a standard way to handle the first part. Let $\mathcal{U}_{\mathbb{C}}$ be any set of fields on a manifold W. Pick a manifold X of one dimension higher whose boundary is W.



Pick "any" (elliptic) differential equation on X such that the desired fields on W give local "boundary values" for a solution of those equations. Then define Γ to be the subspace of fields on W consisting of boundary values of global solutions on X. This subspace is always within a finite-dimensional amount of being middle-dimensional – the difference being given by an index.

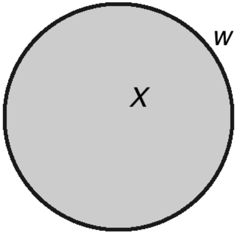
Example: suppose that W is a circle and $\mathcal{U}_{\mathbb{C}}$ is the space of complex-valued scalar fields ϕ. By hand, we can pick a middle-dimensional subspace. We make a Fourier expansion

$$\phi = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}$$

and then we define Γ by setting to zero half of the Fourier components

$$a_n = 0, \quad n < 0.$$

But we could also put this in our picture: we regard the circle W as the boundary of a unit disc X in the complex z plane. We write $z = re^{i\theta}$. The picture is like this:



Now we regard the scalar field ϕ on W as boundary values of a scalar field on X, which we also call ϕ and we ask that this extended scalar field should obey

$$\frac{\partial}{\partial \bar{z}}\phi = 0.$$

A general solution can be expanded

$$\phi = \sum_{n \geq 0} a_n z^n$$

so when restricted to $|z| = 1$ it is

$$\phi = \sum_{n \geq 0} a_n e^{in\theta}.$$

In other words, the space of boundary values of solutions on X is the middle-dimensional subspace Γ that we defined by hand at the beginning.

Going back to our problem with the complex-valued connection $\mathcal{A} = A + i\phi$ on a three-manifold W, we now set W to be the boundary of a four-manifold X, and on X we consider some differential equation

$$P(A, \phi) = 0.$$

More or less any (elliptic) P will do, if all we want is a middle-dimensional subspace Γ in the space of complex-valued gauge fields. However, P is essentially uniquely determined if we want the path integral to converge.

Just as in the one-dimensional example we started with, the path integral is in danger of diverging because after analytic continuation the integrand of the path integral

$$\int_\Gamma D\mathcal{A}\,\exp\left(i\frac{k}{4\pi}\int_W \mathrm{Tr}(\mathcal{A}\wedge \mathrm{d}\mathcal{A}+\frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A})\right)$$

has a real part that is unbounded above.

What will save the day is an identity

$$\mathrm{Re}\,\left(i\frac{k}{4\pi}\int_W \mathrm{Tr}(\mathcal{A}\wedge \mathrm{d}\mathcal{A}+\frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A})\right)$$
$$=\int_X P^2-\int_X \mathrm{d}^4x\,\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}+\sum_{\mu,\nu}(D_\mu\phi_\nu)^2+\sum_{\mu,\nu}[\phi_\mu,\phi_\nu]^2\right).$$

This identity says that the dangerous left hand side is not negative in general, but is negative when $P=0$, which is what we need.

On glancing back at the last slide, a quantum field theorist might notice something: the negative term on the right hand side is closely related to (minus) the bosonic part of the action of $\mathcal{N} = 4$ super Yang-Mills theory. One can think of ϕ_μ as four scalar fields of $\mathcal{N} = 4$ super Yang-Mills theory, topologically twisted to turn them into a one-form. The other two scalar fields have been set to zero.

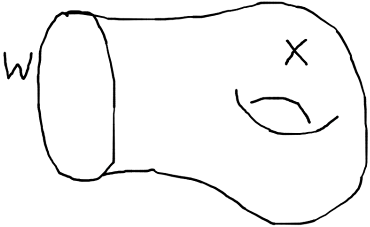
Related to this, the equations $P = 0$, which explicitly read

$$F_{\mu\nu} - [\phi_\mu, \phi_\nu] - \epsilon_{\mu\nu\alpha\beta} D^\alpha \phi^\beta = 0$$

$$D_\mu \phi^\mu = 0$$

are BPS equations for a twisted version of $\mathcal{N} = 4$ super Yang-Mills; these equations were studied by Kapustin and me in our work on gauge theory applied to geometric Langlands. (We will also use the same equations on Sunday in discussing four-manifolds.) We will soon see why this is useful, but for the moment, let us leave aside the relation to super Yang-Mills theory.

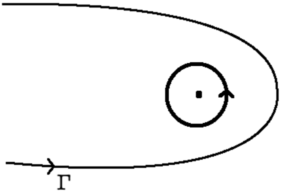
Thus the picture is as follows. Let W be a closed oriented three-manifold. For any choice of a four-manifold X of boundary W,



one gets an integration cycle for an analytically continued version of the Chern-Simons path integral on W. The integration cycle $\mathcal{U}$ consists of all complex gauge fields $\mathcal{A}$ on W that are boundary values of a solution of the equation $P = 0$ on X.

For an integration cycle of this kind, there is never any integrality condition. A 1-dimensional analog is the analytic continuation of the Bessel function by changing the integration cycle

$$\oint \frac{\mathrm{d}z}{z^{k+1}} \exp(t(z+z^{-1})) \to \int_\Gamma \frac{\mathrm{d}z}{z^{k+1}} \exp(t(z+z^{-1})).$$



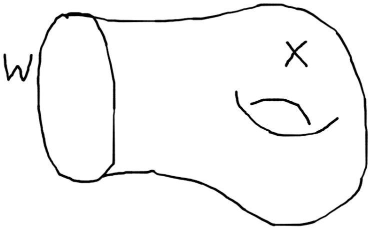
If we want to integrate on the unit circle, k has to be an integer. If k is an integer, it is equivalent to integrate over Γ, and if we do that, k no longer has to be an integer.

For the special case of knots in $\mathbb{R}^3$, one can argue using Morse theory that all integration cycles are equivalent. So the standard integration cycle of Chern-Simons theory can be replaced by one for which k has no reason to be an integer. Therefore, we get a new explanation of why the Jones polynomial can be analytically continued away from integer k.

In fact, because of the relation to super Yang-Mills theory, we get much more. The equation $P = 0$ is an equation of "supersymmetric localization" for $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions. In other words, the theory has a particular supercharge Q, satisfying $Q^2 = 0$, and such that if we pass to the cohomology of Q, the Feynman integral of the four-dimensional gauge theory "localizes" on solutions of $P = 0$.

(This type of localization, which is familiar to quantum field theorists, is a generalization of Duistermaat-Heckman/Atiyah-Bott localization in equivariant cohomology.)

Thus what I have said can be stated in a more physical way: Let X be a four-manifold with boundary W. Assuming X is not flat, we pick particular couplings to preserve the supercharge Q, and we also pick a particular boundary condition along W. With this done, the statement is that $\mathcal{N} = 4$ super Yang-Mills theory on X gives an analytically continued version of Chern-Simons theory on W.



The boundary condition we need has a simple description in terms of string theory "branes": it arises from the D3-NS5 system with a nonzero theta-angle.

But it is actually a rather unusual boundary condition from a topological field theory point of view. From the point of view of the second order Yang-Mills equations of the underlying $\mathcal{N} = 4$ theory, it is a standard elliptic boundary condition. But in the theory localized on solutions of $P = 0$, the boundary condition becomes trivial: it just says that $P = 0$ along the boundary. So it looks very unusual in the twisted topological field theory. (An analog of this boundary condition in dimension 2 is the one associated to the "coisotropic branes" of Kapustin and Orlov.)

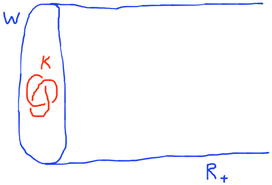
This step in going from 3 to 4 dimensions is relatively non-standard, but once we get this far, we can get a lot farther using more standard arguments.

First we apply electric-magnetic duality to the four-dimensional $\mathcal{N} = 4$ theory. In terms of branes this just changes a D3-NS5 system with a θ-angle to a D3-D5 system, still with a θ-angle.

In terms of gauge theory, what happens is the following:

- The gauge group G is mapped to the Langlands-GNO dual group $G^\vee$.
- The q parameter $q = \exp(2\pi\mathrm{i}/(k+2))$ of the Jones polynomial is mapped to $q = e^{\mathrm{i}\theta}$ where θ is the gauge theory θ-angle.
- This means that a field of instanton number N is going to be weighted by $\exp(\mathrm{i}N\theta) = q^N$.
- In terms of the bulk localization equations, nothing happens. Localization is still on solutions of the equation $P = 0$.
- However, because of the flip NS5$\to$D5, the boundary condition is now completely different. It now determines an elliptic boundary condition (but an unusual one, as explained shortly) on the equation $P = 0$.
- Because the equation and boundary condition are elliptic, it makes sense (modulo compactness issues which we may hear about in the next lecture by Cliff Taubes) to count, modulo signs, the solutions of the equation $P = 0$ that satisfy the boundary condition.

If a_N is the "number" of solutions of $P = 0$ with instanton number N that satisfy the boundary condition



then supersymmetric localization tells us that the partition function is

$$J(q) = \sum_N a_N q^N$$

and this is then a dual formula for the Jones polynomial.

(To be precise, to get the Jones polynomial, the boundary three-manifold W should be just $W = \mathbb{R}^3$, and the four-manifold X of boundary W should be just $X = \mathbb{R}^3 \times \mathbb{R}_+$ where $\mathbb{R}_+$ is a half-line.)

The boundary condition on the equation $P=0$ that comes from the D3-D5 system has all the general properties of an elliptic boundary condition, but it is constructed in an unusual way by requiring the fields to have a certain sort of singularity along the boundary. (This boundary condition was studied in R. Mazzeo and EW, arXiv:1311.3167 and 1712.00835; it may enter in the next lecture by Taubes.) Pick coordinates $\vec{x}$ along the boundary W and a coordinate y normal to the boundary (so the boundary is at $y=0$). Write $\phi = \vec{\phi}\cdot \mathrm{d}\vec{x} + \phi_y \mathrm{d}y$. The boundary condition is defined by requiring $\vec{\phi}$ to have a "Nahm pole":

$$\vec{\phi} \sim \frac{\vec{t}}{y} \quad y \to 0,$$

where $\vec{t}$ are a standard set of generators of $\mathfrak{su}(2)$.

Finally:

From this starting point, it is straightforward to "categorify" the Jones polynomial and to get a candidate for the Khovanov homology of a knot. What we get is closely related to work of Gukov, Schwarz, and Vafa (2004), whose starting point was earlier work of Ooguri and Vafa. I believe we will be hearing another perspective from Aganagic.

In terms of branes, categorification just means replacing the D3-D5 system by a D4-D6 system. Starting with D3-branes on a four-manifold X, the chain is

$$X \to X \times S^1 \to X \times \mathbb{R}.$$

Replacing X with $X \times S^1$ does nothing from a topological field theory point of view, but replacing $X \times S^1$ by $X \times \mathbb{R}$ is "categorification." It means that one constructs a physical Hilbert space $\mathcal{H}$, rather than purely numerical inariants. $\mathcal{H}$ is $\mathbb{Z} \times \mathbb{Z}$ graded by instanton number N and "fermion number" F. (The fermion number is given by an Atiyah-Patodi-Singer η-invariant.) If one "decategorifies" by replacing $\mathbb{R}$ by S^1, the natural thing to calculate is the partition function, which is a supertrace or index, giving back the Jones polynomial:

$$J(q) = \mathrm{Tr}_{\mathcal{H}}(-1)^F q^N.$$

Mathematically, without mentioning branes, what happens is the following. Remember that the Jones polynomial comes from counting solutions of an equation $P = 0$. In general, the set of solutions of an equation cannot be categorified (as far as I know). However, according to Morse theory, the set of critical points of a function f on a manifold M can be categorified: the categorification of the set of solutions of $\mathrm{d}f = 0$ is the cohomology of M, $H^*(M, \mathbb{R})$.

So we want the set of solutions of the equation $P = 0$ to be the critical points of some functional. This is actually not true on a generic four-manifold, but it is true for a four-manifold of the special form $X = W \times \mathbb{R}_+$ (which we use in studying the Jones polynomial) or more generally $W \times L$ where L is a 1-manifold. For such X, the equation $P = 0$ is schematically

$$\frac{\delta \Gamma}{\delta \Phi} = 0,$$

for some "action" or "energy" function Γ, where $\Phi = (A, \phi)$ is the full set of fields that appear in the equation $P = 0$.

Categorification means that one adds another dimension, replacing X by $Y = X \times \mathbb{R}$, with $\mathbb{R}$ parametrized by the "time" t, and one introduces the "gradient flow" equation

$$\frac{\partial \Phi}{\partial t} = -\frac{\delta \Gamma}{\delta \Phi}.$$

"Luckily," this equation turns out to be elliptic (we will study the same equation on Sunday in another guise) so it makes sense to count its solutions, assuming that the relevant compactness issues (which we may hear about in the next lecture) can be overcome. Assuming this, we can construct an analog of Floer cohomology and this is the categorification of the Jones polynomial.

Concretely, this means the following.

- The solutions of $P = 0$ are the "classical vacua."
- We make a Hilbert space $\mathcal{H}_0$ with one basis vector for each classical vacuum.
- The instanton number and fermion number (η-invariant) of the classical vacua are two conserved charges N and F, giving a $\mathbb{Z} \times \mathbb{Z}$ grading of $\mathcal{H}_0$.
- The physical Hilbert space $\mathcal{H}$ (which is expected to be an invariant of the knot) is not $\mathcal{H}_0$ but rather it is the cohomology of a supercharge Q that acts on $\mathcal{H}_0$.
- The matrix elements of Q between different basis vectors in $\mathcal{H}_0$ are computed by counting the gradient flow lines between different critical points, as usual in the supersymmetric approach to Morse theory.

In summary, we have discussed the relations between a chain of theories in dimensions 2-3-4-5:

- In dimension 2, we have the conformal blocks of the WZW model.
- In dimension 3, we have the Jones polynomial and its description by Chern-Simons theory.
- In dimension 4, we relate this to $\mathcal{N} = 4$ super Yang-Mills theory,
- In the last (?) step, we go to dimension 5 and get a categorification of the Jones polynomial.