T-duality, loop space and Witten gerbe modules

Type IIA $\Longleftrightarrow$ Type IIB

Geometry of Quantum Fields and Strings

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Collaborators and references

[BEM]

Peter Bouwknegt, Jarah Evslin and V. M.,

T-duality: Topology Change from H-flux.

Communications in Mathematical Physics,

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[HM20]

Fei Han and V. M.,

T-Duality, Jacobi Forms and Witten Gerbe Modules

[**arXiv: 2001.00322**]

Outline of talk

1. T-duality in an H-flux: the case of free circle actions.
2. A graded version and the Euler operator
3. A loop space refinement and Jacobi forms
4. Examples of Jacobi forms: Witten gerbe modules
5. Jacobi forms and graded Hori formula
6. An alternate (earlier) approach to T-duality on loopspace

String theory in a background flux

Review: T-duality in an H-flux:

the case of free circle actions

String theory in a background flux

Data for a partial definition for Type II string theory is:

Let Z be spacetime:

1. A **background H-flux** $H \in \Omega^3(Z)$, $dH = 0$ with integral periods. Let $\{U_\alpha\}$ be a good open cover of Z, $B_\alpha \in \Omega^2(U_\alpha)$ such that $dB_\alpha = H|_\alpha$. Let $A_{\alpha\beta} \in \Omega^1(U_{\alpha\beta})$ such that $B_\alpha - B_\beta = dA_{\alpha\beta}$. Then $(H, B_\alpha, A_{\alpha\beta})$ captures integral info.

2. **Ramond-Ramond (RR) fields** $G \in \Omega^{even}(Z)$, Type IIB and $G \in \Omega^{odd}(Z)$, Type IIA satisfying the equations of motion, $(d - H\wedge)G = 0$; $\Rightarrow$ twisted cohomology/twisted K-theory.

3. **Einstein-Maxwell equation for the metric.**

4. **dilaton + axion.**

T-duality - The case of circle bundles

We will be concerned with T-duality between the string theories, **Type IIA** $\Longleftrightarrow$ **Type IIB**, for circle bundle compactifications or free circle actions with an H-flux.

The transformation rules of the low energy effective fields under T-duality, are known as the **Buscher rules**.

However, in cases in which there is a topologically nontrivial NS 3-form H-flux, the Buscher rules only make sense **locally** and do not give global information/rules.

The general formula for the topology and H-flux of the T-dual with respect to any **free** circle action (on smooth spacetime) was presented for the first time in **[BEM]**.

T-duality - The case of circle bundles

In [BEM], compactify spacetime Z as a principal $\mathbb{T}$-bundle over M, with Chern class $c_1(Z) \in H^2(M, \mathbb{Z})$, and flux $H \in H^3(Z, \mathbb{Z})$.

$$\begin{array}{ccc} \mathbb{T} & \longrightarrow & Z \\ & & \Big\downarrow{\scriptstyle \pi} \\ & & M \end{array} \tag{1}$$

The **T-dual** is another principal $\mathbb{T}$-bundle over M, denoted by $\hat{Z}$,

$$\begin{array}{ccc} \hat{\mathbb{T}} & \longrightarrow & \hat{Z} \\ & & \Big\downarrow{\scriptstyle \hat{\pi}} \\ & & M \end{array} \tag{2}$$

which has Chern class $\quad c_1(\hat{Z}) = \pi_* H$.

The Gysin sequence for Z enables us to define a T-dual H-flux $\hat{H} \in H^3(\hat{Z}, \mathbb{Z})$, satisfying $\quad c_1(Z) = \hat{\pi}_* \hat{H}$

T-duality & correspondence spaces

N.B. $\hat{H}$ is not fixed by this data, since any integer degree 3 cohomology class on M that is pulled back to $\hat{Z}$ integrates to zero. However, $[\hat{H}]$ is determined uniquely upon imposing the condition $[H] = [\hat{H}]$ on the correspondence space $Z \times_M \hat{Z}$, otherwise known as the doubled space, $\quad Z \times_M \hat{Z} = \{(x, \hat{x}) \in Z \times \hat{Z} : \pi(x) = \hat{\pi}(\hat{x})\}$.

$$\begin{tikzcd}
 & (Z \times_M \hat{Z}, [H] = [\hat{H}]) \arrow[dl, "p"] \arrow[dr, "\hat{p}"'] & \\
(Z, H) \arrow[dr, "\pi"'] & & (\hat{Z}, \hat{H}) \arrow[dl, "\hat{\pi}"] \\
 & M &
\end{tikzcd}$$

T-duality in a background flux - the slogan

Thus a slogan for T-duality for **circle bundles** is the exchange,

$$\boxed{\textbf{background H-flux} \Longleftrightarrow \textbf{Chern class}}$$

The surprising/striking **new** phenomenon that we discovered is that there is a **change in topology** of spacetime when either

* the background H-flux topologically nontrivial,

* or the Chern class is topologically nontrivial.

T-duality in a background flux - Examples

Example (Lens space)

$L(p,1) = S^3/\mathbb{Z}_p$, where $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ & $\mathbb{Z}_p$ acts on S^3 by

$$\exp(2\pi ik/p).(z_1, z_2) = (z_1, \exp(2\pi ik/p)z_2), \quad k = 0, 1, \ldots, p-1.$$

$L(p,1)$ is the total space of the circle bundle over S^2 with Chern class equal to p times the generator of $H^2(S^2, \mathbb{Z}) \cong \mathbb{Z}$.

FACT: $L(p,1)$ is **never** homeomorphic to $L(q,1)$ if $p \neq q$.

Nevertheless

$$(L(p,1), H = q) \qquad \text{and} \qquad (L(q,1), H = p)\,.$$

are T-dual pairs! Thus T-duality is the interchange

$$\boxed{p \Longleftrightarrow q}$$

T-duality in a background flux - Examples

Since $L(1,1) = S^3$ & $L(0,1) = S^2 \times S^1$, we get the T-dual pairs:

$$(S^2 \times S^1, H = 1) \qquad \text{and} \qquad (S^3, H = 0)$$

A picture (suppressing one dimension) illustrating this is the *doughnut universe* $(H = 1)$ & the *spherical universe* $(H = 0)$

Example (Heisenberg nilmanifolds)

Recall that the Heisenberg group is

$$\mathrm{Heis}_{\mathbb{R}} = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \quad x, y, z \in \mathbb{R}$$

which is a central extension,

$$1 \to \mathbb{R}_z \to \mathrm{Heis}_{\mathbb{R}} \to \mathbb{R}^2_{x,y} \to 1$$

and for each $p \in \mathbb{Z} \setminus \{0\}$, $\mathrm{Heis}_{\mathbb{R}}$ has the lattices

$$\mathrm{Heis}_{\mathbb{Z}}(p) = \begin{pmatrix} 1 & x & z/p \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \quad x, y, z \in \mathbb{Z}$$

Example

which is a central extension,

$$1 \to \mathbb{Z}_z \to \mathrm{Heis}_{\mathbb{Z}}(p) \to \mathbb{Z}^2_{x,y} \to 1$$

It follows that the Nilmanifold $\mathrm{Nil}(p) = \mathrm{Heis}_{\mathbb{R}}/\mathrm{Heis}_{\mathbb{Z}}(p)$ is a principal circle bundle over $\mathbb{T}^2$ with Chern number p.

FACT: $\mathrm{Nil}(p)$ is **never** homeomorphic to $\mathrm{Nil}(q)$ if $p \neq q$.

Nevertheless $(\mathrm{Nil}(p), H = q)$ and $(\mathrm{Nil}(q), H = p)$.

are T-dual pairs! Thus T-duality is the interchange

$$\boxed{p \Longleftrightarrow q}$$

Similarly for Seifert fibred spaces over Riemann surfaces.

T-duality gives rise to a map inducing degree-shifting isomorphisms between the

* H-twisted cohomology of Z and $\widehat{H}$-twisted cohomology of $\widehat{Z}$;
* H-twisted K-theory of Z and $\widehat{H}$-twisted K-theory of $\widehat{Z}$;

where charges of RR-fields in background 3-flux fields live.

These are **twisted** generalizations of the smooth analog of the

Fourier-Mukai transform = a geometric Fourier transform.

T-duality map is assumed to be an isometry, relating

radius R circle fibres of $Z \Leftrightarrow$ radius $1/R$ circle fibres of $\widehat{Z}$,

a salient feature of T-duality.

T-duality in a background flux - cohomology

Choosing connection 1-forms A and $\hat{A}$, on the $\mathbb{T}$-bundles Z and $\hat{Z}$, respectively, the rules for transforming the RR fields can be encoded in the **[BEM]** is a Twisted Fourier-Mukai transform,

$$T_* G = \int_{\mathbb{T}} e^{A \wedge \hat{A}}\, G\,, \tag{3}$$

where $G \in \Omega^{\bullet}(Z)^{\mathbb{T}}$ is the total RR fieldstrength,

$$\begin{aligned} G &\in \Omega^{even}(Z)^{\mathbb{T}} \quad \text{for } \underline{\textbf{Type IIA}}; \\ G &\in \Omega^{odd}(Z)^{\mathbb{T}} \quad \text{for } \underline{\textbf{Type IIB}}, \end{aligned}$$

and where the right hand side of (3) is an invariant differential form on $Z \times_M \hat{Z}$, and the integration is along the $\mathbb{T}$-fiber of Z.

T-duality in a background flux

Let $F = dA$ and $\hat{F} = d\hat{A}$ be the curvatures of the connections, and we can assume wlog that H is $\mathbb{T}$-invariant. Then on Z

$$H = A \wedge \hat{F} - \Omega\,, \tag{4}$$

for some $\Omega \in \Omega^3(M)$, while the T-dual $\hat{H}$ on $\hat{Z}$ is given by

$$\hat{H} = F \wedge \hat{A} - \Omega\,. \tag{5}$$

We note that

$$d(A \wedge \hat{A}) = -H + \hat{H}\,, \tag{6}$$

T_* indeed maps d_H-closed forms G to $d_{\hat{H}}$-closed forms $T_* G$. Twisted cohomology (first defined by Rohm-Witten) is

$$H^\bullet(Z, H) = H^\bullet(\Omega^\bullet(Z), d_H = d - H\wedge).$$

So T-duality T_* induces a map on twisted cohomologies,

$$T_* : H^\bullet(Z, H) \to H^{\bullet-1}(\hat{Z}, \hat{H}).$$

T-duality in a background flux

We define the Riemannian metrics on Z and $\hat{Z}$ by

$$g_Z = \pi^* g_M + R^2 A \odot A, \qquad g_{\hat{Z}} = \hat{\pi}^* g_M + R^{-2} \hat{A} \odot \hat{A}.$$

Theorem (Bouwknegt, Evslin, V.M., 2004, V.M., Siye Wu 2011)

Under the above choices of Riemannian metrics and flux forms,

$$T \colon \Omega^{\bar{k}}(Z)^{\mathbb{T}} \to \Omega^{\overline{k+1}}(\hat{Z})^{\hat{\mathbb{T}}}, \tag{7}$$

for $k = 0, 1$, are isometries, inducing isometries on the spaces of twisted harmonic forms and twisted cohomology groups.

The circle fibre radius R of Z goes to circle fibre radius $1/R$ of $\hat{Z}$ and there is an induced degree-shifting isomorphism

$$T_* : H^\bullet(Z, H) \cong H^{\bullet+1}(\hat{Z}, \hat{H}).$$

A graded version and the Euler operator

String theory in a background flux

Recall that Hori maps for $k = 0, 1$, are isometries.

$$T\colon \Omega^{\bar{k}}(Z)^{\mathbb{T}} \to \Omega^{\overline{k+1}}(\hat{Z})^{\hat{\mathbb{T}}}, \tag{8}$$

Since $\Omega^{\bar{k}}(Z)^{\mathbb{T}} \cong \Omega^{\bar{k}}(M) \oplus \Omega^{k\bar{+}1}(M) \wedge A$ so that

$$T(F + G \wedge A) = \int_{\mathbb{T}} e^{A\wedge\hat{A}}(F + G \wedge A) = (-1)^{k+1}(G + F \wedge \hat{A})$$

This is the dimensionally reduced form of T-duality.

For $m \in \mathbb{Z}$, define the **level** m **Hori map** by

$$T_{*,m}(G) = \int_{\mathbb{T}} e^{-mA\wedge\hat{A}}G, \tag{9}$$

for G is an $\mathbb{T}$-invariant form on Z and $(d + mH)G = 0$. Since

$$m\hat{H} = mH + d(mA \wedge \hat{A}), \tag{10}$$

it is not hard to see that $T_{*,m}G$ is a $\hat{\mathbb{T}}$-invariant form on $\hat{Z}$ and

$$(d + m\hat{H})(T_{*,m}(G)) = 0.$$

String theory in a background flux

Let $\Omega^{\bar{k}}(Z)^{\mathbb{T}}_{(d+mH)-cl}$ denote the $\mathbb{T}$-invariant $(d+mH)$-closed forms on Z with degree parity $\bar{k}$. Then the level m Hori map,

$$T_{*,m} : \Omega^{\bar{k}}(Z)^{\mathbb{T}}_{(d+mH)-cl} \longrightarrow \Omega^{\overline{k+1}}(\hat{Z})^{\hat{\mathbb{T}}}_{(d+m\hat{H})-cl}$$

Now let the formal variable y encode the level. That is, define

$$T_{*,y} : \bigoplus_{m\in\mathbb{Z}} \Omega^{\bar{k}}(Z)^{\mathbb{T}}_{(d+mH)-cl}\, y^m \longrightarrow \bigoplus_{m\in\mathbb{Z}} \Omega^{\overline{k+1}}(\hat{Z})^{\hat{\mathbb{T}}}_{(d+m\hat{H})-cl}\, y^m$$

Take a representative

$$\sum_{m\in\mathbb{Z}} \omega_m y^m, \tag{11}$$

with $\omega_m \in \Omega^{\bar{k}}(Z)^{\mathbb{T}}_{(d+mH)-cl}$, $m \in \mathbb{Z}$. Then each ω_m must be of the form

$$F_m + G_m \wedge A, \tag{12}$$

with F_m, G_m being valued in $\Omega^{\bar{k}}(M)$ and $\Omega^{\overline{k+1}}(M)$.

String theory in a background flux

Applying the level m Hori's formula, we get

$$\begin{aligned} &T_{*,m}(F_m + G_m \wedge A) \\ =&\int_{\mathbb{T}} e^{-mA\wedge\hat{A}}(F_m + G_m \wedge A) \\ =&(-1)^{k+1}(G_m + mF_m \wedge \hat{A}). \end{aligned} \tag{13}$$

Applying the reverse level m Hori's formula, we get

$$\begin{aligned} &\hat{T}_{*,m}((-1)^{k+1}(G_m + mF_m \wedge \hat{A})) \\ =&(-1)^{k+1}\int_{\hat{\mathbb{T}}} e^{mA\wedge\hat{A}}(G_m + mF_m \wedge \hat{A})) \\ =&(-1)^{k+1}((-1)^k mG_m + (-1)^k mF_m) \\ =&-m(F_m + G_m \wedge A). \end{aligned} \tag{14}$$

String theory in a background flux

Therefore we see that

$$
\begin{aligned}
&\widehat{T_{\cdot,y_*}} \circ T_{\cdot,y_*}\left(\sum_{m\in\mathbb{Z}} \omega_m y^m\right) \\
&= \sum_{m\in\mathbb{Z}} \hat{T}_{*,m} \circ T_{*,m}(\omega_m) y^m \\
&= -\sum_{m\in\mathbb{Z}} m\omega_m y^m \\
&= -y\frac{\partial}{\partial y}\left(\sum_{m\in\mathbb{Z}} \omega_m y^m\right)
\end{aligned}
\tag{15}
$$

That is,

$$
\widehat{T_{\cdot,y_*}} \circ T_{\cdot,y_*} = -y\frac{\partial}{\partial y} \tag{16}
$$

A loop space refinement and Jacobi forms

String theory and loopspaces

Motivated by string theory, people have been attempting to generalize many concepts like vector bundles, Dirac operators, the Atiyah-Singer index theory and so on to free loop spaces. Let V be a rank r complex vector bundle on M and $\widetilde{V} = V - \mathbb{C}^r$ in the K-group of M. In the theory of elliptic genera, one considers the **Witten bundles** $\Theta_2(V)$ and $\Theta_3(V)$, elements in $K(M)[[q^{1/2}]]$, as follows:

$$\Theta_2(V) := \bigotimes_{j=1}^{\infty} \Lambda_{-q^{j-1/2}}(\widetilde{V}) \otimes \bigotimes_{j=1}^{\infty} \Lambda_{-q^{j-1/2}}(\widetilde{\bar{V}}), \quad \Theta_3(V) := \bigotimes_{j=1}^{\infty} \Lambda_{q^{j-1/2}}(\widetilde{V}) \otimes \bigotimes_{j=1}^{\infty} \Lambda_{q^{j-1/2}}(\widetilde{\bar{V}}). \quad (17)$$

where $q = \exp(2\pi\sqrt{-1}\tau)$ and $\tau \in \mathbb{H}$. They are formally viewed as vector bundles over (small) loop space LM. Physically, they arise in heterotic string theory.

Recall here that for an indeterminate t,

$$\Lambda_t(E) = \mathbb{C}|_M + tE + t^2\wedge^2(E) + \cdots\,, \quad S_t(E) = \mathbb{C}|_M + tE + t^2S^2(E) + \cdots\,, \tag{18}$$

are the total exterior and symmetric powers of E respectively. The following relations between these two operations hold,

$$S_t(E) = \frac{1}{\Lambda_{-t}(E)}, \quad \Lambda_t(E - F) = \frac{\Lambda_t(E)}{\Lambda_t(F)}. \tag{19}$$

Let $\{2\pi i x_i\}$, $1 \leq i \leq r$, be the formal Chern roots of V and $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$, the upper half plane. In terms of **Jacobi theta functions**, the Chern characters of the **Witten bundles** $\Theta_2(V)$ and $\Theta_3(V)$ are

String theory and loopspaces

$$ch(\Theta_2(V)) = \prod_{i=1}^{r} \frac{\theta_2(x_i,\tau)}{\theta_2(0,\tau)} \in H^{even}(M)[[q^{1/2}]], \quad ch(\Theta_3(V)) = \prod_{i=1}^{r} \frac{\theta_3(x_i,\tau)}{\theta_3(0,\tau)} \in H^{even}(M)[[q^{1/2}]]. \tag{20}$$

Equip V with a connection ∇^V, then the Chern characters above can be represented by holomorphic functions on $\mathbb{H}$, taking values in $\Omega^{even}(M)[[q^{1/2}]]$. Suppose one has

$$0 = \frac{1}{2}p_1(V) = c_1(V)^2 - 2c_2(V) = ch^{[4]}(V), \tag{21}$$

where $p_1(V), c_1(V), c_2(V)$ and $ch(V)$ stand for the first Pontryagin class, the first and second Chern class and the Chern character respectively.

Under these **anomaly vanishing conditions**, the degree p (with p even) components $ch^{[p]}(\Theta_2(V))$ and $ch^{[p]}(\Theta_3(V))$ are modular forms of weight $\frac{p}{2}$ over $\Gamma_0(2)$ and $\Gamma_\theta(2)$ respectively.

Examples of Jacobi forms: Witten gerbe modules

Examples of Jacobi forms: Witten gerbe modules

Gerbe modules and connections

The next goal is to generalise this construction from finite dimensional vector bundles, to infinite dimensional Hilbert bundle gerbe modules.

Let M be an oriented closed smooth manifold of dimension $2r$. Let H be a closed 3-form on M with integral periods. Let $B_\alpha \in \Omega^2(U_\alpha)$ such that $dB_\alpha = H|_\alpha$. Let $A_{\alpha\beta} \in \Omega(U_{\alpha\beta})$ such that $B_\alpha - B_\beta = dA_{\alpha\beta}$. Let $\{(L_{\alpha\beta}, d + A_{\alpha\beta})\}$ be geometric realization of a **gerbe (with connection)**. Then we have

$$(\nabla^L_{\alpha\beta})^2 = F^L_{\alpha\beta} = B_\beta - B_\alpha. \qquad (22)$$

Gerbe modules

Let $E = \{E_\alpha\}$ be a collection of (infinite dimensional) separable Hilbert bundles $E_\alpha \to U_\alpha$ whose structure group is reduced to $U_\mathfrak{I}$, which are unitary operators on the model Hilbert space $\mathfrak{H}$ of the form (identity + trace class operator). Here $\mathfrak{I}$ denotes the Lie algebra of $U_\mathfrak{I}$, the trace class operators on $\mathfrak{H}$. In addition, assume that on the overlaps $U_{\alpha\beta}$ there are isomorphisms

$$\phi_{\alpha\beta} : L_{\alpha\beta} \otimes E_\beta \cong E_\alpha, \tag{23}$$

which are consistently defined on triple overlaps because of the gerbe property. Then $\{E_\alpha\}$ is said to be a **gerbe module** for the gerbe $\{L_{\alpha\beta}\}$.

Gerbe modules with connection

A **gerbe module connection** ∇^E is a collection of (local) connections $\{\nabla^E_\alpha\}$ is of the form $\nabla^E_\alpha = d + A^E_\alpha$ where $A^E_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{I}$ whose curvature F^{E_α} on the overlaps $U_{\alpha\beta}$ satisfies

$$\phi^{-1}_{\alpha\beta}(F^{E_\alpha})\phi_{\alpha\beta} = F^{L_{\alpha\beta}} I + F^{E_\beta}. \tag{24}$$

Using equation (42), this becomes

$$\phi^{-1}_{\alpha\beta}(B_\alpha I + F^E_\alpha)\phi_{\alpha\beta} = B_\beta I + F^E_\beta. \tag{25}$$

It follows that $\exp(-B)\,\mathrm{Tr}\left(\exp(-F^E) - I\right)$ is a globally well defined differential form on M of even degree. Notice that $\mathrm{Tr}(I) = \infty$ which is why we need to consider the subtraction.

Anomaly vanishing conditions

Suppose that $\nabla^E, \nabla^{E'}$ are gerbe module connections on the gerbe modules E, E' respectively. Then the **twisted Chern character** is

$$\begin{aligned} &Ch_H : K^0(Z, \mathcal{G}) \to H^{even}(Z, H) \\ &Ch_H(E, E') = \exp(-B)\,\mathrm{Tr}\left(\exp(-F^E) - \exp(-F^{E'})\right) \end{aligned} \tag{26}$$

That this is a well defined homomorphism is explained in [BCMMS]. The degree 0 term of $Ch_H(E, E')$ is 0, and

$$Ch_H^{[2]}(E, E') = \mathrm{Tr}[F^E - F^{E'}] = \{\mathrm{Tr}[F^{E_\alpha} - F^{E'_\alpha}]\} \in H^2(Z) \tag{27}$$

The degree 4 term is

$$Ch_H^{[4]}(E, E') = \frac{\mathrm{Tr}[(B+F)^2 - (B+F')^2]}{2} \in H^4(Z, H) \tag{28}$$

The **anomaly vanishing conditions** in the twisted case is, $Ch_H^{[2]}(E, E') = 0$ and $Ch_H^{[4]}(E, E') = 0$.

Anomaly vanishing conditions

On U_α, define

$$\Theta(E_\alpha) = \bigotimes_{u=1}^{\infty} \Lambda_{-q^u}(E_\alpha) \otimes \bigotimes_{u=1}^{\infty} \Lambda_{-q^u}(\bar{E}_\alpha). \tag{29}$$

It turns out that $\Theta(E) := \{\Theta(E_\alpha)\}$ defines a globally defined gerbe module, which we call the **Witten gerbe module.** Since it involves tensor products of E, $\Theta(E)$ is a gerbe module for the gerbe $\{\bigoplus_{m\in\mathbb{Z}} L_{\alpha\beta}^{\otimes m}\}$ and its induced connection.

Anomaly vanishing conditions

To compute the graded twisted chern character $GCh_H(\Theta(E))$, one expresses $\Theta(E)$ as a sum

$$GCh\left(\frac{\Theta(E)}{\Theta(E')}\right) = \sum_{m\in\mathbb{Z}}(\sum_{n=0}^{\infty} Ch_{mH}(W_{m,n}(E,E'))q^n)y^m \qquad (30)$$

where $\{W_{m,n}(E,E'\}$ is a gerbe module for the gerbe $(mH, mB_\alpha, mA_{\alpha\beta})$ for each $m \in \mathbb{Z}$, and hence the expression above makes sense.

Under the twisted anomaly vanishing condition discussed earlier, it turns out that the graded twisted chern character $Gch_H(\Theta(E))$ is a **Jacobi form** (which will be defined next).

Jacobi theta functions

Recall that

$$SL_2(\mathbb{Z}) := \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \middle| \, a, b, c, d \in \mathbb{Z}, \; ad - bc = 1 \right\}$$

is the modular group. Let

$$S = \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right), \quad T = \left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)$$

be the two generators of $SL_2(\mathbb{Z})$. Their actions on $\mathbb{H}$ are given by

$$S : \tau \to -\frac{1}{\tau}, \quad T : \tau \to \tau + 1.$$

Jacobi theta functions

Let

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| \, c \equiv 0 \pmod 2 \right\},$$

$$\Gamma^0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| \, b \equiv 0 \pmod 2 \right\}$$

$$\Gamma_\theta = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{or} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod 2 \right\}$$

be the three modular subgroups of $SL_2(\mathbb{Z})$. It is known that the generators of $\Gamma_0(2)$ are T, ST^2ST, the generators of $\Gamma^0(2)$ are STS, T^2STS and the generators of Γ_θ are S, T^2.

Jacobi theta functions

The Jacobi theta-function (and its variants) defined by infinite products are

$$\theta(v,\tau) = 2q^{1/8}\sin(\pi v)\prod_{j=1}^{\infty}[(1-q^j)(1-e^{2\pi\sqrt{-1}v}q^j)(1-e^{-2\pi\sqrt{-1}v}q^j)], \tag{31}$$

It is a holomorphic function for $(v,\tau)\in\mathbb{C}\times\mathbb{H}$, where $\mathbb{C}$ is the complex plane and $\mathbb{H}$ is the upper half plane.
The theta function satisfies the the following transformation law

$$\theta(v,\tau+1) = e^{\pi\frac{\sqrt{-1}}{4}}\theta(v,\tau), \quad \theta\left(v,-1/\tau\right) = \frac{1}{\sqrt{-1}}\left(\frac{\tau}{\sqrt{-1}}\right)^{1/2} e^{\pi\sqrt{-1}\tau v^2}\theta\left(\tau v,\tau\right)\ ; \tag{32}$$

$$\theta(z+1,\tau) = -\theta(z,\tau),\ \theta(z+\tau,\tau) = -e^{-\pi\sqrt{-1}(\tau+2z)}\theta(z,\tau), \tag{33}$$

Jacobi forms and graded Hori formula

Jacobi forms and graded Hori formula

Let Γ be a subgroup of $SL(2,\mathbb{Z})$ of finite index. Let L be an integral lattice in $\mathbb{C}$ preserved by Γ. Denote $\mathbb{H}$ the upper half plane. A **(meromorphic) Jacobi form** of weight s and index l over $L \rtimes \Gamma$ is a (meromorphic) function $J(z,\tau)$ on $\mathbb{C}\times\mathbb{H}$ such that

(i) $J\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^s e^{2\pi\sqrt{-1}l(cz^2/(c\tau+d))} J(z,\tau)$;

(ii) $J(z+\lambda\tau+\mu,\tau) = e^{-2\pi\sqrt{-1}l(\lambda^2\tau+2\lambda z)} J(a,\tau)$, where

$$(\lambda,\mu)\in L, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

We will use a slight extension of the above definition of Jacobi forms, namely, (i) we will allow $J(z,\tau)$ to take values in the differential forms on a manifold M; (ii) as $J(z,\tau)$ takes values in differential forms, we don't require the singular points be poles but only be undefined.

Jacobi forms and graded Hori formula

Let M be a manifold with H-flux. Let $\mathcal{A}^{\bar{k}}(M)^{\mathbb{T}}_{(d+mH)-cl}$ denote the space of holomorphic functions on $\mathbb{H}$ except for a set of isolated points, which take values in $\Omega^{\bar{k}}(Z)_{(d+mH)-cl}$, the $\mathbb{T}$-invariant $(d+mH)$-closed forms on M with degree parity $\bar{k}$. Let $\mathcal{H}^{\bar{k}}(M, mH)$ denote the space of holomorphic functions on $\mathbb{H}$ except for a set of isolated points, which take values in $H^{\bar{k}}(M, mH)$.

Denote $q = e^{2\pi\sqrt{-1}\tau}, \tau \in \mathbb{H}$ and $y = e^{-2\pi\sqrt{-1}z}$, $z \in \mathbb{C}$. On the spacetime M, further consider the 2-variable series

$$\omega(z,\tau) \in \bigoplus_{m\in\mathbb{Z}} \mathcal{H}^{\bar{k}}(M, mH) \cdot y^m$$

Jacobi forms and graded Hori formula

with the following properties: $\omega(z, \tau)$ is represented by

$$\sum_{m \in \mathbb{Z}} \omega_m(\tau) y^m, \tag{34}$$

with $\omega_m(\tau) \in \mathcal{A}^{\bar{k}}(M)^{\mathbb{T}}_{(d+mH)-cl}$, $m \in \mathbb{Z}$ such that the degree p (with $\bar{p} = \bar{k}$) component

$$\sum_{m \in \mathbb{Z}} \omega_m(\tau)^{[p]} y^m \tag{35}$$

is the expansion at $y = 0$ of a Jacobi form of weight $\frac{p+\bar{k}}{2}$ and index 0 over $L \rtimes \Gamma$. Denote the abelian group of all such $\omega(z, \tau)$ by $\mathcal{J}_0^{\bar{k}}(M, H; L, \Gamma)$.

Jacobi forms and graded Hori formula

Now consider the situation of T-duality with pair $(Z,H),(\hat{Z},\hat{H})$ as before. For $m\in\mathbb{Z}$,recall the **level** m **Hori map** by

$$T_{*,m}(G)=\int_{\mathbb{T}}e^{-mA\wedge\hat{A}}G, \tag{36}$$

for G is an $\mathbb{T}$-invariant form on Z and $(d+mH)G=0$.

Define the **graded Hori map** of Jacobi forms,

$$LT_*:\bigoplus_{m\in\mathbb{Z}}\mathcal{A}^{\bar{k}}(Z)^{\mathbb{T}}_{(d+mH)-cl}\cdot y^m\to\bigoplus_{m\in\mathbb{Z}}\mathcal{A}^{\overline{k+1}}(\hat{Z})^{\hat{\mathbb{T}}}_{(d+m\hat{H})-cl}\cdot y^m \tag{37}$$

by

$$LT_*\left(\sum_{m\in\mathbb{Z}}\omega_m(\tau)y^m\right)=\sum_{m\in\mathbb{Z}}T_{*,m}(\omega_m(\tau))y^m, \tag{38}$$

for

$$\sum_{m\in\mathbb{Z}}\omega_m(\tau)y^m\in\bigoplus_{m\in\mathbb{Z}}\mathcal{A}^{\bar{k}}(Z)^{\mathbb{T}}_{(d+mH)-cl}\cdot y^m.$$

Jacobi forms and graded Hori formula

Theorem (T-duality for Jacobi forms)

Let $H(\mathbb{H})$ denote the space of holomorphic functions on $\mathbb{H}$. The following statements hold:

(i) LT and $\widehat{LT}$ are both isomorphisms of $H(\mathbb{H})$ modules under the restriction that the coefficient of y^0 is zero; moreover

$$\widehat{LT} \circ LT = -y\frac{\partial}{\partial y}, \quad LT \circ \widehat{LT} = -y\frac{\partial}{\partial y}; \tag{39}$$

(ii) After restriction, we have

$$LT\left(\mathcal{J}_0^{\bar{k}}(Z, H; L, \Gamma))\right) \subseteq \mathcal{J}_0^{\overline{k+1}}(\hat{Z}, \hat{H}; L, \Gamma) \tag{40}$$

and therefore get a morphism of abelian groups,

$$LT : \mathcal{J}_0^{\bar{k}}(Z, H; L, \Gamma)) \to \mathcal{J}_0^{\overline{k+1}}(\hat{Z}, \hat{H}; L, \Gamma); \tag{41}$$

Alternate approach to T-duality on loop space

Alternate approach to T-duality on loop space

[HM15]

Fei Han and V. M.,
Exotic twisted equivariant cohomology of loop spaces, twisted Bismut-Chern character and T-duality.
Communications in Mathematical Physics,
337, No. 1, (2015) 127-150.

Motivation for some constructions on loop space

Jones-Petrack showed that a completed version of equivariant cohomology of loopspace LZ with respect to the rotation circle action, localises to the ordinary cohomology of Z, that is,

$$h^\bullet_{\mathbb{T}}(LZ) \overset{res}{\cong} H^\bullet(Z)[u, u^{-1}]]$$

[HM15] is concerned with the analog of this result is for **twisted cohomology**, $H^\bullet(Z, H)$ where H is a closed degree 3 form on Z with integral periods, i.e. $[H] \in H^3(Z; \mathbb{Z})$.
Here $H^\bullet(Z, H) = H^\bullet(\Omega^{odd/even}(Z), d + H\wedge)$ is a $\mathbb{Z}_2$-graded cohomology theory, coinciding with $H^\bullet(Z)$ when $H = 0$.

It was first studied by Rohm-Witten (1986), and arose in String Theory as the **charge group** classifying D-brane charges at least rationally. It has many applications in mathematics such as twisted eta invariants, twisted analytic torsion, etc.

Motivation for some constructions on loop space

In [HM15], we defined an **exotic equivariant cohomology**.
A key innovation is the construction of a canonical S^1**-flat superconnection** on the the holonomy line bundle of a gerbe with connection, satisfying the **localisation formula**

$$h^\bullet_{\mathbb{T}}(LZ, \nabla^{\mathcal{L}^B} : \bar{H}) \overset{res}{\cong} H^\bullet(Z, H)[u, u^{-1}]]$$

where *res* is the localisation map.

Gerbes

Consider a pair (Z, H), where Z is a spacetime and H is a background flux, i.e. a closed 3-form on Z with $\mathbb{Z}$ periods.

We want to study open covers $\{U_\alpha\}$ of Z such that the space of loops $\{LU_\alpha\}$ is an open cover of $LZ = C^\infty(S^1, Z)$.
The usual Cech open cover of Z consisting of a convex open cover of Z does **not** satisfy this property.
Suppose that $\{U_\alpha\}$ is a maximal open cover of Z with the property that $H^i(U_{\alpha_I}) = 0$ for $i = 2, 3$ where $U_{\alpha_I} = \bigcap_{i \in I} U_{\alpha_i}$, $|I| < \infty$. Such an open cover is a **Brylinski open cover** of Z.
It is easy to see that $\{LU_\alpha\}$ is an open cover of LZ.

Let H a closed 3-form on Z with integral periods. Then $H\big|_{U_\alpha} = dB_\alpha$ since $H^3(U_\alpha) = 0$ where $B_\alpha \in \Omega^2(U_\alpha)$. Also $B_\beta - B_\alpha = dA_{\alpha\beta}$ since $H^2(U_\alpha \cap U_\beta) = 0$. Then (H, B, A) defines a connective structure (or connection) for a **gerbe** $\mathcal{G}_B$ on Z.

Gerbes

More precisely, a **gerbe** $\mathcal{G}$ on Z is a collection of line bundles $\{L_{\alpha\beta}\}$ on double overlaps, $L_{\alpha\beta} \to U_{\alpha\beta} = U_\alpha \cap U_\beta$ such that on triple overlaps $U_{\alpha\beta\gamma}$ there is a trivialization

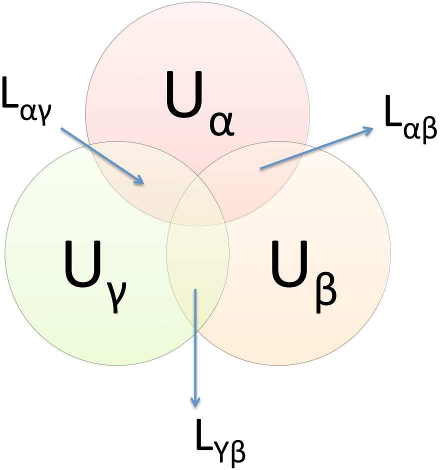
$$\phi_{\alpha\beta\gamma} : L_{\alpha\beta} \otimes L_{\beta\gamma} \otimes L_{\gamma\alpha} \xrightarrow{\cong} \mathbb{C}$$

Then $\{\phi_{\alpha\beta\gamma}\}$ is a $U(1)$-valued Cech 2-cocycle representing the **Dixmier-Douady invariant** of the gerbe in $H^3(Z, \mathbb{Z})$.
Upto equivalence, gerbes on Z are classified by $H^3(Z, \mathbb{Z})$.

A **trivial gerbe** $\{L_{\alpha\beta}\}$ is of the form $L_{\alpha\beta} = L_\alpha \otimes L^*_\beta$, where $\{L_\alpha \to U_\alpha\}$ is a collection of line bundles.

Gerbes

GERBE



Example: Spin$^{\mathbb{C}}$-gerbes

Let $\{g_{\alpha\beta} : U_{\alpha\beta} \to SO(n)\}$ denote the set of transition functions for the oriented orthonormal frame bundle of Z,

$$U(1) \to Spin^{\mathbb{C}}(n) \to SO(n)$$

is the defining nontrivial central extension. Let $L \to SO(n)$ be the associated line bundle, $L = Spin^{\mathbb{C}}(n) \times_{U(1)} \mathbb{C}$. Then the gerbe $\{L_{\alpha\beta} = g^*_{\alpha\beta}(L)\}$ is called the **Spin$^{\mathbb{C}}$-gerbe** of Z. The Dixmier-Douady class of this gerbe is equal to $W_3(Z)$, the 3rd integral Stiefel-Whitney class of Z. ***So every oriented manifold has a Spin$^{\mathbb{C}}$-gerbe.***

This construction also works for the oriented orthonormal frame bundle of any oriented vector bundle E over Z.

Example: *PU*-gerbes

Let $\{g_{\alpha\beta} : U_{\alpha\beta} \to PU\}$ denote the set of transition functions for a principal *PU*-bundle *P* over *Z*,

$$U(1) \to U \to PU$$

is the defining nontrivial central extension.

Let $L \to PU$ be the associated line bundle, $L = U \times_{U(1)} \mathbb{C}$. Then the gerbe $\{L_{\alpha\beta} = g^*_{\alpha\beta}(L)\}$ is called the *PU*-gerbe of *P* over *Z*.

The Dixmier-Douady class of this gerbe is equal to $DD(P)$.

Gerbes, connections and their holonomy line bundle

A **connection** on the gerbe $\mathcal{G}_B$ is $\{(L_{\alpha\beta}, \nabla^L_{\alpha\beta})\}$, a collection of line bundles $L_{\alpha\beta} \to U_{\alpha\beta}$ such that there is an isomorphism $L_{\alpha\beta} \otimes L_{\beta\gamma} \cong L_{\alpha\gamma}$ on $U_{\alpha\beta\gamma}$ and collection of connections $\{\nabla^L_{\alpha\beta}\}$ such that $\nabla^L_{\alpha\beta} = d + A_{\alpha\beta}$ (note that as $H^2(U_\alpha \cap U_\beta) = 0$, the bundle $L_{\alpha\beta}$ is trivial). Then we have

$$(\nabla^L_{\alpha\beta})^2 = F^L_{\alpha\beta} = B_\beta - B_\alpha. \tag{42}$$

The **holonomy** of this gerbe is a line bundle $\mathcal{L}^B \to LZ$ over the loop space LZ. $\mathcal{L}^B$ has $\mathbb{T}$-invariant Brylinski local sections $\{\sigma_\alpha\}$ with respect to $\{LU_\alpha\}$ such that the transition functions are $\{e^{-\sqrt{-1}\tau(A_{\alpha\beta})}\}$, i.e. $\qquad \sigma_\alpha = e^{-\sqrt{-1}\tau(A_{\alpha\beta})}\sigma_\beta,$

$\tau : \Omega^\bullet(U_{\alpha_I}) \longrightarrow \Omega^{\bullet-1}(LU_{\alpha_I})$ is the transgression map defined as $\tau(\xi_I) = \int_{\mathbb{T}} ev^*(\xi_I), \qquad \xi_I \in \Omega^\bullet(U_{\alpha_I})$. Here ev is the evaluation map $ev : \mathbb{T} \times LU_{\alpha_I} \to U_{\alpha_I} : (t, \gamma) \to \gamma(t)$.

Gerbes and their holonomy line bundle

The holonomy line bundle $\mathcal{L}^B$ on loopspace LZ comes with a natural connection, whose definition with respect to the basis $\{\sigma_\alpha\}$ is $\nabla^{\mathcal{L}^B} = d - \sqrt{-1}\tau(B_\alpha)$. The curvature of the connection $\nabla^{\mathcal{L}^B}$ is $F_B = (\nabla^{\mathcal{L}^B})^2 = -\sqrt{-1}\tau(H)$ is the transgression of the minus i x 3-curvature H of the gerbe $\mathcal{G}_B$.

Observe that $\mathcal{L}^B$ is never flat if $H \neq 0$.

Consider $\Omega^\bullet(LZ, \mathcal{L}^B) =$ the space of differential forms on loop space LZ with values in the holonomy line bundle $\mathcal{L}^B \to LZ$ of the gerbe $\mathcal{G}_B$ on Z.

Induced tensors on loop space

Let $\omega \in \Omega^i(Z)$. Define $\hat{\omega}_s \in \Omega^i(LZ)$ for $s \in [0,1]$ by

$$\hat{\omega}_s(X_1, \ldots, X_i)(\gamma) = \omega(X_1\big|_{\gamma(s)}, \ldots, X_i\big|_{\gamma(s)})$$

for $\gamma \in LZ$ and $X_1, \ldots, X_i$ are vector fields on LZ defined near γ. Then one checks that $d\hat{\omega}_s = \widehat{d\omega}_s$.

The i-form

$$\bar{\omega} = \int_0^1 \hat{\omega}_s ds \quad \in \Omega^i(LZ)$$

is the extension of ω on Z, to LZ. Then $\bar{\omega} =$ is $\mathbb{T}$-invariant, that is, $L_K(\bar{\omega}) = 0$ and $d\bar{\omega} = \overline{d\omega}$.
Moreover $\tau(\omega) = i_K\bar{\omega}$ and that $\bar{\omega}$ restricts to ω on the submanifold of constant loops.

Exotic twisted equivariant cohomology of loop space

Let H be as before and $\bar{H} \in \Omega^3(LZ)$ be the associated closed 3-form on LZ. Define $D_{\bar{H}} = \nabla^{\mathcal{L}^B} - i_K + \bar{H}$. Then we compute,

Lemma

$(D_{\bar{H}})^2 = 0$ *on* $\Omega^\bullet(LZ, \mathcal{L}^B)^{\mathbb{T}}$.

Proof.

Let $\{U_\alpha\}$ be a Brylinski open cover of Z. Then $\bar{H}\Big|_{LU_\alpha} = d\bar{B}_\alpha$ on LU_α. On LU_α, we have

$$
\begin{aligned}
(D_{\bar{H}})^2 =& (\nabla^{\mathcal{L}^B} - i_K + \bar{H})^2 && (43)\\
=& (d - i_K\bar{B}_\alpha - i_K + \bar{H})^2 && (44)\\
=& \big((d - i_K) + (d - i_K)\bar{B}_\alpha\big)^2 && (45)\\
=& \big(\exp(-\bar{B}_\alpha)(d - i_K)\exp(\bar{B}_\alpha)\big)^2 && (46)\\
=& -L_K - (L_K\bar{B}_\alpha) = -L_K, && (47)
\end{aligned}
$$

Proof.

where L_K denotes the Lie derivative of the vector field K. As the Brylinski sections are invariant, we have $L_K = L_K^{\mathcal{L}^B}$ on LU_α. So $(D_{\bar{H}})^2 = -L_K^{\mathcal{L}^B}$, which vanishes on $\Omega^\bullet(LZ, \mathcal{L}^B)^{\mathbb{T}}$ as claimed. $\square$

Notice that $D_{\bar{H}} = \nabla^{\mathcal{L}^B} - i_K + \bar{H}$ is a **flat $\mathbb{T}$-equivariant superconnection** (in the sense of Quillen) on $\Omega^\bullet(LZ, \mathcal{L}^B)^{\mathbb{T}}$. Therefore $\left(\Omega^\bullet(LZ, \mathcal{L}^B)^{\mathbb{T}}, D_{\bar{H}}\right)$ is a $\mathbb{Z}_2$-graded complex. We call the cohomology of this complex the **exotic twisted $\mathbb{T}$-equivariant cohomology** of loop space, denoted by $H^\bullet_{\mathbb{T}}(LZ, \nabla^{\mathcal{L}^B} : \bar{H})$.

Completed exotic twisted equivariant cohomology of loop space

Define the **completed periodic exotic twisted $\mathbb{T}$-equivariant cohomology** $h^*_{\mathbb{T}}(LZ, \nabla^{\mathcal{L}^B} : \bar{H})$ to be the cohomology of the complex

$$(\Omega^\bullet(LZ, \mathcal{L}^B)^{\mathbb{T}}[u, u^{-1}]], \nabla^{\mathcal{L}^B} - ui_K + u^{-1}\bar{H}).$$

NB the holonomy line bundle $\mathcal{L}^B$ is trivial when restricted to Z, the constant loop space, we have

Theorem (Localisation)

The restriction to the constant loops

$$res : h^*_{\mathbb{T}}(LZ, \nabla^{\mathcal{L}^B} : \bar{H}) \cong H^*(Z, H)[u, u^{-1}]]$$

is an isomorphism.

loop space proposal

This justifies the following 2 proposals:

RR fields in type II String Theory in a background H-flux, are exotic differential forms in $\Omega^\bullet(LZ, \mathcal{L}^B)^{S^1}$ *and are closed wrt the exotic differential* $D_{\bar{H}}$. *(EOM)*

It also includes massive RR-fields.

Also

Over the rationals, D-brane charges on space-time Z in a background H-flux, take values in $h^*_{\mathbb{T}}(LZ, \nabla^B : \bar{H})$.

Path ordered exponential

Let $\mathcal{A}$ be a unital Banach algebra and $a : [0,1] \longrightarrow \mathcal{A}$ be a continuous function. Define the **path ordered exponential**, denoted $\mathcal{T}(t) = \mathcal{T}(\exp(\int_0^1 a(s)ds))$ as the unique solution to

$$\frac{d}{dt}\mathcal{T}(t) = a(t)\mathcal{T}(t)$$
$$\mathcal{T}(0) = 1$$

Then it has a convergent power series expansion

$$\mathcal{T}(t) = 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(t)} a(s_1)\cdots a(s_n)ds_1\cdots ds_n$$

where $\Delta_n(t)$ is the n-simplex of size t, ie

$$\Delta_n(t) = \{0 \leq s_1 \leq \cdots \leq s_n \leq t\}.$$

Twisted Bismut-Chern character

Via the path ordered exponential method, lift the twisted Chern character of [BCMMS] to loop space LZ by defining $BCh_{H,\alpha}(\nabla^E, \nabla^{E'}) \in \Omega^\bullet(LU_\alpha, \mathcal{L}^B)^{\mathbb{T}}[u, u^{-1}]]$ by

$$BCh_{H,\alpha}(\nabla^E, \nabla^{E'}) =$$

$$\left(1 + \sum_{n=1}^{\infty} (-u)^{-n} \int_{\Delta_n(1)} \widehat{B}_{\alpha\, s_1} \cdots \widehat{B}_{\alpha\, s_n}\right) \left(BCh_\alpha(\nabla^E) - BCh_\alpha(\nabla^{E'})\right) \sigma_\alpha$$

$$= \mathcal{T}\left(exp\left(\frac{-1}{u} \int_0^1 \widehat{B}_{\alpha\, s} ds\right)\right) \left(BCh_\alpha(\nabla^E) - BCh_\alpha(\nabla^{E'})\right) \sigma_\alpha$$

$BCh_\alpha(\nabla^E)$ is the path ordered exponential lift of the Chern chracter to loop space LZ due to Bismut. Since the curvature of ∇^E is vector valued therefore parallel transport wrt ∇^E has to be inserted into the curvature factors before taking the trace.

Cartan model for equivariant cohomology

Define the **twisted Bismut-Chern character form** $BCh_H(\nabla^E, \nabla^{E'}) \in \Omega^\bullet(LZ, \mathcal{L}^B)^{\mathbb{T}}[u, u^{-1}]]$ to be the global form patched together from the local forms constructed above.

Theorem

(i) We have $(\nabla^{\mathcal{L}^B} - ui_K + u^{-1}\bar{H})BCh_H(\nabla^E, \nabla^{E'}) = 0$;
(ii) The exotic twisted $\mathbb{T}$*-equivariant cohomology class* $[BCh_H(\nabla^E, \nabla^{E'})]$ *does not depend on the choice of connections* $\nabla^E, \nabla^{E'}$.
(iii) One has a commutative diagram

$$\begin{tikzcd} K^\bullet(Z,H) \arrow[rr, "BCh_H"] \arrow[dr, "Ch_H"'] & & h^\bullet_{\mathbb{T}}(LZ, \nabla^{\mathcal{L}^B} : \bar{H}) \arrow[dl, "res"] \\ & H^\bullet(Z,H)[u,u^{-1}]] & \end{tikzcd}$$

T-duality: a loop space perspective

Consider

$$\begin{tikzcd} (Z,H) \arrow[dr, "p"'] & & (\widehat{Z},\widehat{H}) \arrow[dl, "\widehat{p}"] \\ & X & \end{tikzcd}$$

where $Z, \widehat{Z}$ are principal circle bundles over a base X with fluxes H and $\widehat{H}$, respectively, satisfying $p_*(H) = c_1(\widehat{Z})$, $\widehat{p}_*(\widehat{H}) = c_1(Z)$ and $H - \widehat{H}$ is exact on the correspondence space $Z \times_X \widehat{Z}$. The T-duality Theorem for circle bundles states that there is an isomorphism of twisted K-theories $K^\bullet(Z,H) \cong K^{\bullet+1}(\widehat{Z},\widehat{H})$ and an isomorphism of twisted cohomology theories, $H^\bullet(Z,H) \cong H^{\bullet+1}(\widehat{Z},\widehat{H})$,

As a consequence of our Localisation Theorem, properties of the twisted Bismut-Chern character, T-duality Theorem for circle

T-duality: a loop space perspective

Theorem (T-duality : a loop space perspective)

In the notation above, there is an isomorphism

$$T : h^{\bullet}_{\mathbb{T}}(LZ, \nabla^{\mathcal{L}^B} : \bar{H}) \xrightarrow{\cong} h^{\bullet+1}_{\mathbb{T}}(L\widehat{Z}, \nabla^{\mathcal{L}^{\widehat{B}}} : \bar{\widehat{H}}),$$

such that the following diagram commutes,

$$\begin{tikzcd}
K^{\bullet}(Z, H) \arrow[r, "T", "\cong"'] \arrow[d, "BCh_H"'] \arrow[dd, bend right=60, "Ch_H"'] & K^{\bullet+1}(\hat{Z}, \hat{H}) \arrow[d, "BCh_{\hat{H}}"] \arrow[dd, bend left=60, "Ch_{\hat{H}}"] \\
h^{\bullet}_{\mathbb{T}}(LZ, \nabla^{\mathcal{L}^B} : \bar{H}) \arrow[r, "T"'] \arrow[d, "res"', "\cong"] & h^{\bullet+1}_{\mathbb{T}}(L\widehat{Z}, \nabla^{\mathcal{L}^{\widehat{B}}} : \bar{\widehat{H}}) \arrow[d, "\cong"', "res"] \\
H^{\bullet}(Z, H)[u, u^{-1}]] \arrow[r, "\cong", "T"'] & H^{\bullet+1}(\widehat{Z}, \widehat{H})[u, u^{-1}]]
\end{tikzcd} \tag{48}$$