Quantum equivariant volumes

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Based on

1. M. Del Zotto, N. Nekrasov, N. Piazzalunga and M. Zabzine, "Playing With the Index of M-Theory," Commun. Math. Phys. **396** (2022) no.2, 817-865 [arXiv:2103.10271]
2. N. Nekrasov, N. Piazzalunga and M. Zabzine, "Shifts of prepotentials (with an appendix by Michele Vergne)," SciPost Phys. **12** (2022) no.5, 177 [arXiv:2111.07663]
3. L. Cassia, N. Piazzalunga and M. Zabzine, "From equivariant volumes to equivariant periods," Adv. Theor. Math. Phys. **27** (2023) no.4, 961-1064 [arXiv:2211.13269]
4. L. Cassia, P. Longhi and M. Zabzine, "Symplectic Cuts and Open/Closed Strings I," Commun. Math. Phys. **406** (2025) no.1, 15 [arXiv:2306.07329]
5. L. Cassia, P. Longhi and M. Zabzine, "Symplectic cuts and open/closed strings II," to appear in Annales Henri Poincaré [arXiv:2410.10960]

But this is mainly the review, not going into technicalities

Outline

1. Volumes, equivariant volumes
   definition and properties, symplectic cuts

2. Deformations of equivariant volumes

3. Quantum volumes, closed GW invariants

4. Quantum symplectic cuts, open GW invariants

5. Summary

Volumes

volumes of compact manifolds are numbers:

$$M \quad \leftrightarrow \quad \mathrm{vol}(M) = \int \Omega$$

$$M = M_1 \cup M_2 \quad \leftrightarrow \quad \mathrm{vol}(M) = \mathrm{vol}(M_1) + \mathrm{vol}(M_2)$$

Not very interesting, non-compact manifolds are problematic

Equivariant volumes

$$T^k \times M \to M$$

$$d_{eq} = d + \epsilon_a \iota_{v^a}$$

$$\Omega_{\rm eq} = \Omega + ...$$

For symplectic spaces we get

$$\int e^{\omega + \epsilon_a H^a} = {\rm vol}_{\rm eq}(\epsilon, ...)$$

Localization formulas, compact vs non-compact (for example, $\mathbb{C}^N$)

Equivariant volumes, example

$\mathbb{P}^1$-example:

$$\mathrm{vol}_{\mathbb{P}^1}(\epsilon_1, \epsilon_2, t) = \int d^2z_1 d^2z_2 \;\; e^{-\epsilon_1|z_1|^2-\epsilon_2|z_2|^2}\delta(|z_1|^2+|z_2|^2-t)$$

$$= \int d\phi \int d^2z_1 d^2z_2 \;\; e^{-\epsilon_1|z_1|^2-\epsilon_2|z_2|^2+\phi(|z_1|^2+|z_2|^2-t)}$$

$$= \frac{1}{2\pi i}\oint e^{\phi t}\frac{1}{(\phi+\epsilon_1)(\phi+\epsilon_2)}$$

Equivariant volumes, setup

Kähler Toric manifolds

$$X_{\mathbf{t}} = \mu^{-1}(\mathbf{t})/T^r = (\mathbb{C}^N)^{\text{stable}}/(\mathbb{C}^x)^r$$

$$\mu^a = \sum_{i=1}^{N} Q_i^a |z^i|^2 = t^a$$

see $\mathbb{P}^1$ example: $\mathbb{C}^2$, $\mu = |z^1|^2 + |z^2|^2 = t$

$$\mathbb{P}^1 = \Big(\mathbb{C}^2 - (0,0)\Big)/\mathbb{C}^x$$

Equivariant volumes, properties

$\mathrm{vol}_{\mathrm{eq}}(\mathbf{t}, \epsilon)$ is expressed as contour integrals with JK prescription

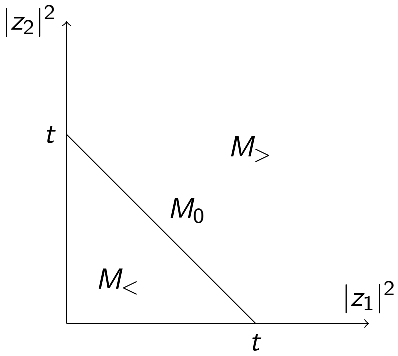
not continuous function of t, chamber structure

equivariant cohomology, for example for $\mathbb{P}^1$

$$\mathbb{C}[\phi, \epsilon_1, \epsilon_2]/(\phi + \epsilon_1)(\phi_2 + \epsilon_2)$$

$$(\partial_t + \epsilon_1)(\partial_t + \epsilon_2)\mathrm{vol}_{\mathrm{eq}}(\epsilon_1, \epsilon_2) = 0$$

Compact vs non-compact, singularities in ϵ's cut $\mathbb{C}^2$ to $\mathbb{P}^2$ and $O(-1)$-bundle over $\mathbb{P}^1$



$$\frac{1}{\epsilon_1\epsilon_2} = \mathrm{vol}_{\mathbb{C}^2} = \mathrm{vol}_{\mathbb{P}^2}(t,\epsilon_1,\epsilon_2) + \mathrm{vol}_{O(-1)}(t,\epsilon_1,\epsilon_2)$$

where we have

$$\mathrm{vol}_{\mathbb{P}^2}(t,\epsilon_1,\epsilon_2) = \int d\phi \frac{e^{\phi t}}{(\phi+\epsilon_1)(\phi+\epsilon_2)\phi} = \frac{1}{\epsilon_1\epsilon_2} + \frac{e^{-\epsilon_1 t}}{(\epsilon_2-\epsilon_1)(-\epsilon_1)} + \frac{e^{-\epsilon_2 t}}{(\epsilon_1-\epsilon_2)(-\epsilon_2)}$$

$$\mathrm{vol}_{O(-1)}(\epsilon_1,\epsilon_2,t) = \int d\phi \frac{e^{\phi t}}{(\phi+\epsilon_1)(\phi+\epsilon_2)(-\phi)} = \frac{e^{-\epsilon_1 t}}{(\epsilon_2-\epsilon_1)\epsilon_1} + \frac{e^{-\epsilon_2 t}}{(\epsilon_1-\epsilon_2)\epsilon_2}$$

$$\mathrm{vol}_{\mathbb{P}^2}(\epsilon_1, \epsilon_2, t) = \int_0^t dc \; \mathrm{vol}_{\mathbb{P}^1}(\epsilon_1, \epsilon_2, c)$$

$$\mathrm{vol}_{\mathbb{P}^2}(\epsilon_1, \epsilon_2, t) = \int_0^t dc \; e^{\frac{1}{2}(\epsilon_1+\epsilon_2)c} \mathrm{vol}_{\mathbb{P}^1}(\frac{1}{2}(\epsilon_2 - \epsilon_1), c)$$

where

$$\mathrm{vol}_{\mathbb{P}^1}(\epsilon, c) = \frac{e^{\epsilon c}}{\epsilon} + \frac{e^{-\epsilon c}}{-\epsilon}$$

Also we have

$$\mathrm{vol}_{O(-1)}(\epsilon_1, \epsilon_2, t) = \int_t^\infty dc \; \mathrm{vol}_{\mathbb{P}^1}(\epsilon_1, \epsilon_2, c)$$

Deformation of equivariant volumes

there are different deformations of equivariant volumes:
Quantum mechanical deformation, just counting integer points inside of momentum polytope

$$Z(T, q_1, q_2) = \sum_{n_1+n_2=T} q_1^{n_1} q_2^{n_2}$$

$q_i = e^{\hbar \epsilon_i}$, $t = \hbar T$

$$Z(T, q_1, q_2) = \hbar^{-2} \mathrm{vol}(t, \epsilon_1, \epsilon_2) + \ldots$$

$$\left(1 - e^{\hbar(\partial_t + \epsilon_1)}\right)\left(1 - e^{\hbar(\partial_t + \epsilon_2)}\right) Z(T, q_1, q_2) = 0$$

Deformation of equivariant volumes

$$Z_{\mathbb{C}^2}(q_1, q_2) = Z_{\mathbb{P}^2}(T, q_1, q_2) + Z_{O(-1)}(T+1, q_1 q_2)$$

$$\sum_{d_1, d_2 = 0}^{\infty} e^{\hbar \epsilon_1 d_1} e^{\hbar \epsilon_2 d_2} = \sum_{d_1 + d_2 \leq T} e^{\hbar \epsilon_1 d_1} e^{\hbar \epsilon_2 d_2} + \sum_{T+1 \leq d_1 + d_2} e^{\hbar \epsilon_1 d_1} e^{\hbar \epsilon_2 d_2}$$

Deformation of equivariant volumes

Quantum equivariant cohomology

$$\mathbb{C}[\phi, \epsilon_1, \epsilon_2] / \Big((\phi + \epsilon_1)(\phi_2 + \epsilon_2) - e^{-\lambda t} \Big)$$

$$\Big((\partial_t + \epsilon_1)(\partial_t + \epsilon_2) - e^{-\lambda t} \Big) \mathcal{F}^D = 0$$

for large t behaves as equivariant volume

One can mix two deformations (I will comment later)

properties:
PDE= quantum equivariant cohomology relations (Picard-Fuchs equation)

$$\mathcal{F}^D(\mathbf{t}, \epsilon)$$

nice functions, no chamber structure

but different semi-classical expansions in different chambers

$$\mathcal{F}^D(\mathbf{t}, \epsilon) = \mathrm{vol}_{eq}(\mathbf{t}, \epsilon) + \ldots$$

non-compact examples and non-equivariant limit, singularities in ϵ's (explain)

Enumerative geometry

genus zero Gromov-Witten invariants (counting holomorphic spheres)

$$\mathcal{F}^D(\mathbf{t}, \epsilon)$$

the explicit relation is tricky, quasi-maps

Symplectic cuts and quantum volumes

CY symplectic cut (Lagrangian submanifold)

$$\mathcal{F}^D(t,\epsilon) = \int_{-\infty}^{+\infty} dc\ \mathcal{H}^D(t,c,\epsilon)$$

$$\partial_c W(t,c,\epsilon) = \frac{1}{2\pi i}\Big(\mathcal{H}^D(t,c+i\pi,\epsilon) - \mathcal{H}^D(t,c-i\pi,\epsilon)\Big)$$

$$W(t,c,\epsilon) = \frac{1}{2\pi i}\int_{c-i\pi}^{c+i\pi} ds\ \mathcal{H}^D(t,s,\epsilon)$$

Symplectic cuts and quantum volumes

$\mathbb{C}^3$, cut is given by $(0, 1, -1)$

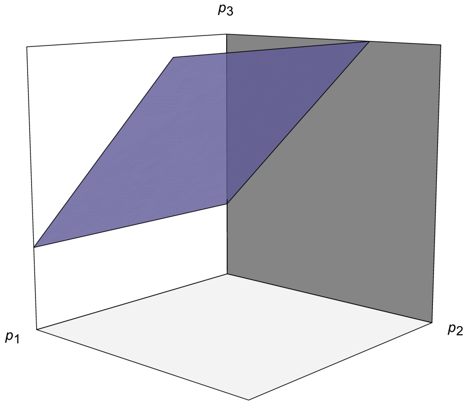


Figure: $|z_2|^2 - |z_3|^2 = c, \quad c > 0$

Open GW theory

for symplectic cut we have:

$$W(\mathbf{t}, c, \epsilon)$$

open GW theory (counting of holomorphic disks)

for $\mathbb{C}^3$ examples we have

$$W \sim Li_2(e^c) + ...$$

$\mathbb{C}^3$ example

$$\mathcal{F}^D = \Gamma(\epsilon_1)\Gamma(\epsilon_2)\Gamma(\epsilon_3)$$

$$\mathcal{H}^D = \Gamma(\epsilon_1)\Gamma(\epsilon_2+\epsilon_3)\frac{e^{\epsilon_3 c}}{(1+e^c)^{\epsilon_2+\epsilon_3}}$$

$$\mathcal{F}^D(t,\epsilon) = \int_{-\infty}^{+\infty} dc\ \mathcal{H}^D(t,c,\epsilon)$$

$$\partial_c W = \log(1+e^c) + ...$$

Open GW theory

Braverman's construction for symplectic cut

cut of $\mathbb{C}^2$: look at $|z_1|^2 + |z_2|^2 + |z_3|^2 - |z_4|^2 = t$

CY3 fold with CY symplectic cut $\rightarrow$ CY4

extended PF system in higher dimensional manifolds

K-theoretical version

if we mix two pertubations we get rather cute formula

$$Z(T, q_i, \mathfrak{q}) = \sum_{Q_i^a n^i = T^a} \prod_{i=1}^{N} \frac{q_i^{n_i}}{(\mathfrak{q}, \mathfrak{q})_{n_i}}$$

where $q_i = e^{\hbar \epsilon_i}$, $\mathfrak{q} = e^{\lambda \hbar}$

But extracting GW invariants is complicated

Summary

Equivariance helps but not always!!!

Equivariant parameters improve the analytical behaviour, but the relation to enumerative geometry is unclear

End

Thanks for your attention!