Periodic elements in Artin groups and stability conditions

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§0. OVERVIEW

We'll use topology of hyperplane arrangements to study algebraic properties of Artin groups.

GOALS: Let $A(\Gamma)$ be an Artin group (of spherical type).

In main part of talk I'll explain the following result:

THEOREM 1 (HLY, '25)*

$\alpha \in A(\Gamma)$ is periodic $\iff$ α has a fixed point in $\widetilde{V_\Gamma^{reg}}$

periodic: $\exists k$ s.t. $\alpha^k \in Z(A)$

$\widetilde{V_\Gamma^{reg}}$: the universal cover of the complexified hyperplane complement corresponding to the Coxeter arrangement

* more precise statement below.

- A key input to our work is the realisation of $\widetilde{V^{reg}_{\mathbb{C}}}$ as the space $\mathrm{Stab}(\mathcal{T})$ of Bridgeland stability conditions on a 2-CY category $\mathcal{T}$.

- One of the main motivations for our work is an analogy, from work of Kontsevich and collaborators, which relates:

$$\mathrm{Aut}(\mathcal{T}) \circlearrowright \mathrm{Stab}(\mathcal{T}) \quad \leftrightsquigarrow \quad \mathrm{MCG}(S) \circlearrowright \mathrm{Teich}(S)$$

- In second part of talk I'll explain WIP which generalises Theorem 1 to larger classes of elements.

§1. BACKGROUND

- Γ = **Coxeter graph** = graph w/ edges labelled by $\mathbb{Z}_{\geq 3} \cup \{\infty\}$

- $W = W(\Gamma)$ = **Coxeter group**

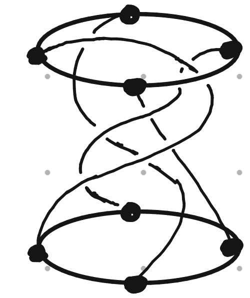
$$= \langle s_i \mid i \in \Gamma,\ s_i^2 = 1,\ \overbrace{s_i s_j \cdots}^{m_{ij}} = \overbrace{s_j s_i \cdots}^{m_{ij}} \rangle$$

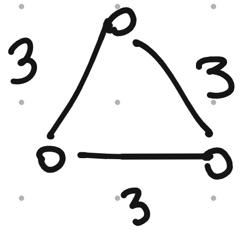
where $i \overset{m_{ij}}{\text{———}} j$ in Γ, $m_{ij} = \infty$ means no relation

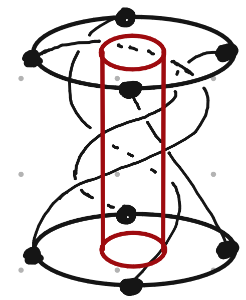
and $i \not\text{——} j$ means $s_i s_j = s_j s_i$.

- $A = A(\Gamma)$ = **Artin (-Tits) group**

$$= \langle \sigma_i \mid i \in \Gamma,\ \sigma_i \sigma_j \cdots = \sigma_j \sigma_i \cdots \rangle$$

EX: • $\Gamma = \circ \overset{3}{\text{———}} \circ \overset{3}{\text{———}} \circ$. $W(\Gamma) \cong S_4$, $A(\Gamma) \cong B_4 \ni$ 

• $\Gamma =$ 

. $W(\Gamma) \cong \widetilde{S}_4$, $A(\Gamma) \cong \widetilde{B}_4 \ni$ 

DEF: • The reflection representation of W is given by

$V_\Gamma' := \bigoplus_{i \in \Gamma} \mathbb{R}\alpha_i$, $s_i \cdot v = v - (\alpha_i, v)\alpha_i$ where

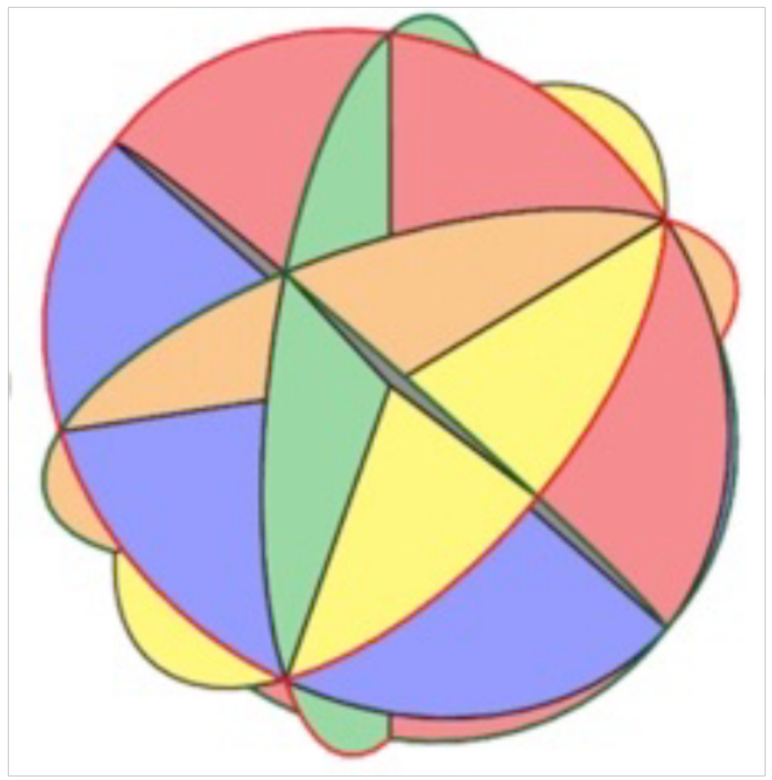
$$(\alpha_i, \alpha_j) := \begin{cases} 2 & \text{if } i = j, \\ -2\cos\left(\frac{\pi}{m_{ij}}\right) & \text{if } i \neq j. \end{cases}$$

• $\Phi = \{ w \cdot \alpha_i \mid i \in \Gamma, w \in W \} \subset V_\Gamma'$ roots.

- $W \circlearrowright V_r := (V_r')^*$ contragradient representation.

- $\alpha \in \Phi$, $H_\alpha = \{ f \in V_r \mid f(\alpha) = 0 \}$ hyperplane.

- The Coxeter arrangement is $(V_r, \mathcal{H} = \{ H_\alpha \mid \alpha \in \Phi \})$.



The A_3 Coxeter arrangement from [Dermenjian et al, '22]

- The complexified hyperplane complement is

$$V^{reg} = V_{\Gamma}^{reg} := V_{\Gamma} \otimes_{\mathbb{R}} \mathbb{C} \setminus \bigcup_{\alpha \in \Phi} H_{\alpha} \otimes_{\mathbb{R}} \mathbb{C}$$

- $W \circlearrowright V^{reg} \rightsquigarrow \kappa : V^{reg} \to V^{reg}/W$ ← normal cover

THEOREM :

1. [Brieskorn, '71] $A \cong \pi_1(V^{reg}/W, pt)$

2. [Deligne, '72] V^{reg}/W is a $K(\pi, 1)$, i.e. all higher homotopy groups are trivial.

§2 Periodic Elements

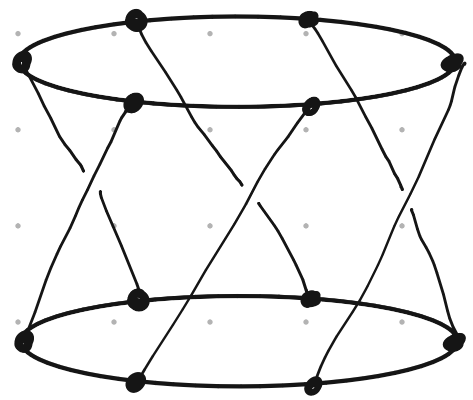
- Let $\widetilde{V}^{reg}$ be the universal cover of V^{reg}/W.
- $A \circlearrowright \widetilde{V}^{reg}$ by deck transformations.
- $\mathbb{C}^\times \circlearrowright V^{reg}$ by scaling. It lifts to $\mathbb{C} \circlearrowright \widetilde{V}^{reg}$ which commutes with the A-action
- Set:

$$\mathbb{P}(\widetilde{V}^{reg}) := \widetilde{V}^{reg} / \mathbb{C}$$

Note that A acts on $\mathbb{P}(\widetilde{V}^{reg})$.

- Recall that $\alpha \in A$ is periodic if $\exists k$ s.t. $\alpha^k \in Z(A)$.

THEOREM: [Brouwer (1919), Kerékjártó (1919), Eilenberg (1934)] Any periodic elt in B_n is conjugate to a power of



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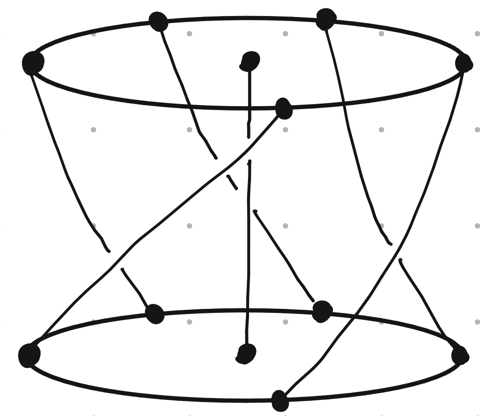


THEOREM 1 (HLY, '25) Suppose $|W| < \infty$ and let $\alpha \in A$. Then:

$$\alpha \text{ periodic} \iff \alpha \text{ has a fixed point in } \mathbb{P}\left(\widetilde{V_\Gamma^{reg}}\right).$$

Compare to Thurston's Theorem:

$\alpha \in MCG(S)$ is periodic $\iff$ α has a fixed pt in Teich(S)

INGREDIENTS OF PROOF: In $\Rightarrow$ direction we use:

Thm ("Bessis' Springer theory for Artin groups", 2015)

1. Let $\alpha \in A$ be periodic. Then α is conjugate to a power of a root of θ = full-twist. ($Z(A) = \langle \theta \rangle \cong \mathbb{Z}$).

2. For any d, $\sqrt[d]{\theta}$ is a conjugacy class.

3. $\sqrt[d]{\theta} \neq \emptyset \iff d$ is regular for W in sense of Springer,
i.e. $\exists w \in W, v \in V^{reg}$ s.t.
$$w \cdot v = \zeta v \quad , \quad \zeta = e^{\frac{2\pi i}{d}}$$

In $\Leftarrow$ direction we use:

<u>Thm</u> (Khovanov–Seidel, 2002): There is a 2-CY category $\mathcal{T} = \mathcal{T}(\Gamma)$ s.t. $A \circlearrowright \mathcal{T}$, the action is faithful, and it categorifies the Burau representation when $\Gamma = A_n$.

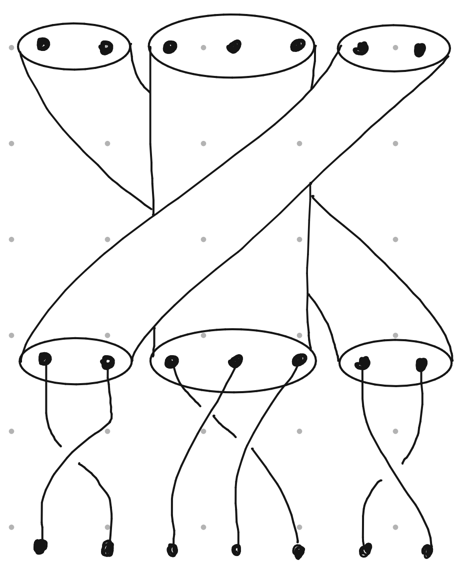
<u>Thm</u> (Bridgeland, 2009): $\mathbb{P}(\widetilde{V}^{reg}) \cong \mathrm{Stab}(\mathcal{T})$

- moduli space of "Bridgeland stability conditions"
- naturally a complex manifold
- each point in $\mathrm{Stab}(\mathcal{T})$ provides a mass function on $\mathrm{Ob}(\mathcal{T})$.

§3. Entropy 0 elements

- Periodic elements are a (small) subset of "entropy 0" elements, and we'd like to generalise Theorem 1 to these elements.

Ex: Entropy 0 braids in B_n are built from periodic braids:



- For general Γ, entropy of $\alpha \in A$ is defined as the growth rate of $m_\tau(\alpha^n(G))$ where $\tau \in Stab(T)$, $G \in T$ pro-generator.

[Dimitrov, Haiden, Katzarkov, Kontsevich]

CONJECTURE 2: Suppose $|W| < \infty$. Then

$\alpha \in A$ has entropy 0 $\iff$ α fixes a multi-scale stability condition in $\mathbb{P}(\mathrm{Stab}(\mathcal{T}))$.

$\exists\, \phi \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_r = \mathcal{T}$ s.t.

(i) $\alpha(\mathcal{T}(\mathcal{J}_i)) = \mathcal{T}(\mathcal{J}_i)$, and

(ii) α fixes a pt in $\mathbb{P}\left(\mathrm{Stab}\left(\mathcal{T}(\mathcal{J}_i)/\mathcal{T}(\mathcal{J}_{i-1})\right)\right)$

- Multi-scale stability conditions were defined by Barbieri-Möller-So in their study of compactifications of $\mathrm{Stab}(\mathcal{T})$.
- Compare to Thurston's Theorem:

$\alpha \in MCG(S)$ entropy 0 $\Rightarrow$ α has a fixed pt in $\partial\left(\overline{\mathrm{Teich}(S)}\right)$

Conjecturally,

$$\mathrm{Stab}\left(T(J_i) / T(J_{i-1}) \right) \cong \left(\begin{array}{l} \text{complexified hyp. comp.} \\ \text{of restriction of } J_i\text{-arrangement} \\ \text{to } J_{i-1} \end{array} \right)$$

Theorem 3 (Garnier-Heng-Licata-Y. '25). The fundamental group of a quotient of this complexified hyperplane complement is iso. to

$$\mathrm{Norm}_{A(J_i)}(A(J_{i-1})) / A(J_{i-1})$$

To prove Conj 2 we need to develop Bessis' Springer theory for this group.

Proof-of-concept: we can prove all of this in type A.