



Unlikely tangencies on elliptic surfaces

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Abstract

We study tangencies on an elliptic surface between a curve and the torsion curves. Joint work with D. Ulmer

Elliptic surfaces

An elliptic surface \mathcal{E}/k , where k is an algebraically closed field, is a surface equipped with an elliptic fibration $\pi : \mathcal{E} \rightarrow \mathcal{C}$, where \mathcal{C}/k is a smooth projective algebraic curve, admitting a zero section O .

The general fibre of π is thus an elliptic curve.

When $\mathcal{C} = \mathbb{P}^1$, we have a familiar Weierstrass equation for \mathcal{E} :

$$y^2 = x^3 + a(t)x + b(t)$$

Torsion curves and tangencies

For an integer $n > 0$, we can look at the points of order n on the fibers of π that are elliptic curves. Their closure form a curve $X[n] \subset \mathcal{E}$, the n -torsion curve.

For a fixed curve X on \mathcal{E} , we can compute the intersection number $X \cdot X[n]$, counting intersections with multiplicities.

Theorem

Suppose X is such that $\pi|_X : X \rightarrow \mathcal{C}$ is finite and separable. Then, for n sufficiently large (and prime to the characteristic p , if $p > 0$), the intersections of X and $X[n]$ are transversal.

This was proved by Ulmer and Urzúa when $p = 0$.

Analytic parametrisations over \mathbb{C}

The analytic parametrisation of an elliptic curve over \mathbb{C} is $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, $\Im\tau > 0$. The n torsion points are given by $(a/n) + (b/n)\tau$, for $a, b \in \mathbb{Z}$.

We can also locally parametrise \mathcal{E} by choosing a function τ at a neighbourhood of a point of \mathcal{C} , such that the fibres around that point are given by $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$.

A foliation and tangencies

Consider the foliation \mathcal{F} of $\mathcal{E}(\mathbb{C})$ whose leaves are locally $L_{a,b} = z = a + b\tau$ for $a, b \in \mathbb{R}$. The torsion curves are (unions of) leaves of this foliation but the other leaves are non-algebraic. Ulmer and Urzúa bound the total number of tangencies between a curve and leaves of \mathcal{F} by studying the pullback of the following form to X :

$$\eta = dz - \frac{\Im z}{\Im \tau} d\tau$$

The zeros of η on a curve X correspond to the tangencies of X leaves of \mathcal{F} . Despite η not being algebraic, the number of zeros can be bounded.

Inflexions

We can consider instead the analytic curves on $\mathcal{E}(\mathbb{C})$ given locally by $M_{a,b} = z = a + b\tau$ for $a, b \in \mathbb{C}$. It is a foliation of the tangent bundle $T\mathcal{E}(\mathbb{C})$.

We bound the total number of inflectional contacts between X and the $M_{a,b}$ by studying the pullback of $d^2z/d\tau^2$ to X . This time, this turns out to be an algebraic function on X corresponding to the Manin map.

Manin map

Let K be an extension of $\mathbb{C}(t)$ with $\delta = d/dt$, and let E be given by $E : y^2 = x(x-1)(x-t)$.

The Manin map is a homomorphism $M : E(K) \rightarrow K$ given by:

$$M(P) = \frac{y(P)}{2(x(P)-t)^2} + (1-2t) \left(\frac{x(P)'}{y(P)} \right) \\ + t(1-t) \left(\frac{x(P)'x(P)(x(P)-1)}{2y(P)^3} \left(\frac{x(P)'}{y(P)} \right)' \right)$$

and is, up to a constant, equal to $d^2z(P)/d\tau^2$.

p -descent map

In characteristic $p > 0$, there is a map $\mu : E(K) \rightarrow K$, for any separable extension $K/\mathbb{F}_q(t)$. The tangencies of X to $X[n]$ give rise to zeros of $\mu(P)$, where P is the point corresponding to the multisection X . This gives the main theorem in characteristic $p > 0$. So $\mu(P)$ plays a role similar to that of η in characteristic 0 but, in fact, μ is related to M by $d\mu/dt = M$ up to a constant.

$$\begin{aligned} \mu(P) &= y(P)M(x(P)) + \\ &\wp_A \left(\frac{dx(P)/2y(P)}{\lambda} \right) - \\ &\wp_A \left(-\frac{12x(P)^2 + \left(\frac{d\Delta/\Delta}{\lambda}\right)x(P) + 8a_4}{12y(P)} \right). \end{aligned}$$

Here $\wp_A(z) = z^p - Az$, $M(X) \in K[X]$ has degree $p - 1$ and Δ is the discriminant of E .

Questions?