Cohomology of generalised triangle groups

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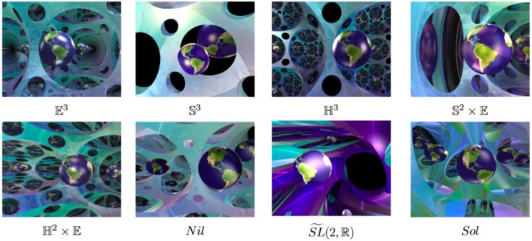
Geometrisation conjecture



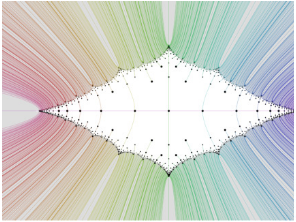
Theorem (Thurston-Hamilton-Perelman)

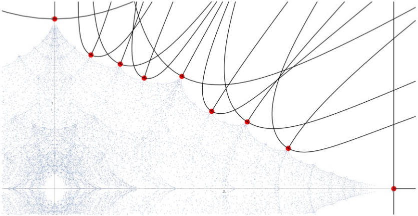
A closed 3-manifold can be decomposed into pieces X/G, where X is a Thurston geometry and G is a discrete subgroup of $\mathrm{Isom}^+(X)$.

Discrete subgroups of $\mathrm{PSL}(2, \mathbb{C})$

- $\mathrm{PSL}(2, \mathbb{C})$ acts on the Riemann sphere by Möbius transformations.
- $\mathrm{PSL}(2, \mathbb{C})$ acts on $\mathbb{H}^3$ by Poincaré extension, in $\mathrm{PSL}(2, \mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$.
- Three-orbifolds are obtained by quotienting hyperbolic 3-space by a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$.

Two parabolic generators



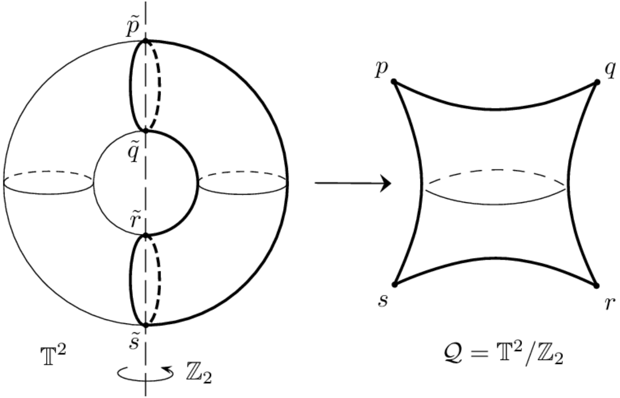


Determined by one complex parameter z,

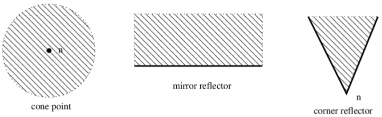
parabolics are $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix}$

Neighbourhoods: Elzenaar-Martin-JS

The pillowcase orbifold



Singular locus of an orbifold

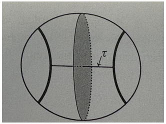


Lemma

Singular points in a two-orbifold are one of:

- *Mirror points, whose local group is $\mathbb{Z}/2\mathbb{Z}$ act by reflection.*
- *Elliptic or cone points of order n, whose local group is $\mathbb{Z}/n\mathbb{Z}$ act by rotations.*
- *Corner reflectors, local group is D_n, reflections in lines meeting at angle $\frac{\pi}{n}$.*

Rational tangles



$\frac{1}{2} = [0, 2]$

$\frac{5}{4} = [1, 4]$

$\frac{21}{16} = [1, 3, 5]$

Theorem (Conway)

Rational tangles are isotopic if and only if they have the same rational number associated to them.

A two-bridge knot or link is the closure of the sum of two rational tangles.

Two bridge link complements

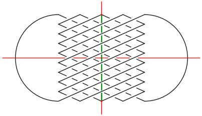


Figure: $K(4/9)$

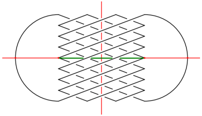


Figure: $K(3/8)$

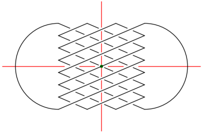


Figure: $K(3/7)$

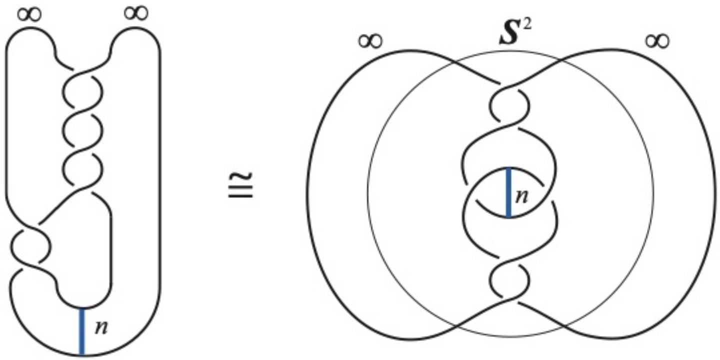
- $M(r) = \mathbb{S}^3 - K(r)$ where $r = s/t$ is hyperbolic iff $s \not\equiv \pm 1 \mod t$ (torus knot).
- Klein 4-group preserves pillow diagrams.
- If t is even we get a link, if t is odd a knot.

Fundamental groups for 2-bridge link complements

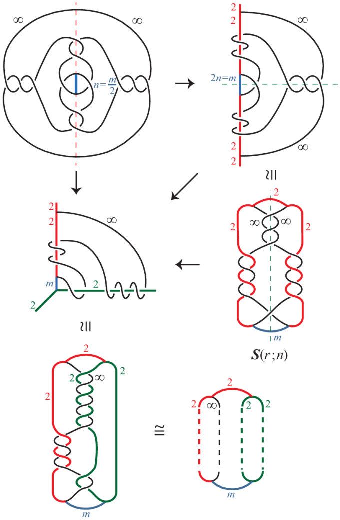
- Deformation retraction:
  $M(r) \to \mathbb{S} \cup D_\infty \cup D_r$, so
  $\pi_1(M(r)) = \pi_1(\mathbb{S} \cup D_\infty \cup D_r)$.
- $\mathbb{S} \cup D_\infty \simeq V_2$, $\pi_1(\mathbb{S} \cup D_\infty) = \pi_1\mathbb{S}/\langle\langle b^2 \rangle\rangle$ is freely generated by f and g.
- Let w_r be the image of $\Omega_r^2 \in \pi_1(\mathbb{S})$ in $\langle f, g \rangle$ under this quotient. Then

$$G(r) = \pi_1 M(r) = \langle f, g \mid w_r \rangle .$$

Even parabolic Heckoid orbifold



Odd parabolic Heckoid orbifold



Agol's conjecture

Theorem (Sakuma et al.)

A non-free Kleinian group Γ is generated by two non-commuting parabolic elements if and only if one of the following holds.

- Γ *is conjugate to the hyperbolic 2-bridge link group,* $G(r) = \pi_1(M(r))$*, for* $r = \frac{s}{t} \in \mathbb{Q}$*, where* $s \not\equiv \pm 1 \mod t$.
- Γ *is conjugate to the Heckoid group,* $G(r; n) = \pi_1(S(r, n))$*, for some* $r \in \mathbb{Q}$ *and some* $n \in \frac{1}{2}\mathbb{Z}$ *with* $n \geq 3/2$.

Drilling preserves 2-generation

Lemma

Let $\Sigma_\Gamma \subset \mathbb{S}^3$ *be the singular locus of* $\mathcal{O}_\Gamma$. *Let* $\Gamma = \pi_1^{orb}(\mathcal{O}_\Gamma) = \langle a_1, \cdots, a_n \rangle$ *where* a_1 *has finite order* $p \geq 2$. *Then* $\widehat{\Gamma}$, *the uniformizing group of the orbifold obtained by drilling the geodesic for* a_1 *is generated by* $\hat{a}_1$ *and* $a_2, \cdots, a_n$, *where* $\hat{a}_1$ *is the Wirtinger generator corresponding to* a_1.

Drilling preserves hyperbolicity

We rely on an extension of Thurston's hyperbolisation theorem, verifying these conditions takes quite some work.

Theorem (Boileau, Leeb, Porti)

Let $\mathcal{O}$ be a compact orientable connected Haken 3-orbifold. If $\mathcal{O}$ is topologically atoroidal and not Seifert fibred, nor Euclidean, then $\mathcal{O}$ is hyperbolic.

Orbifolds by two torsion elements

Theorem (Chesebro-Martin-JS)

Suppose that Γ is a 2-elliptic generator Kleinian group. Then Γ is the uniformizing group for a hyperbolic orbifold $M(r)_p$, $M(r)_{(p,q)}$, or $S(r;n)_{(p,q)}$.

Let $\hat{\Gamma}$ be the uniformising group for the drilled orbifold. Then $\hat{\Gamma} = \langle f_\infty, g_\infty \rangle$ where f_∞ and g_∞ are parabolics. By Agol's conjecture $\mathbb{H}^3/\hat{\Gamma}$ is a parabolic Heckoid orbifold.

Groups of type *FL*

Definition

G is of *type FL* if $\mathbb{Z} \in \mathbb{Z}G$-Mod admits

$$0 \longrightarrow F_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

with $F_i \in \mathbb{Z}G$-Mod f.g. and free for $1 \leq i \leq n$.
The cohomological dimension
$\mathrm{cd}(G) := \mathrm{pd}_{\mathbb{Z}G}(\mathbb{Z})$ is the infimum over the length of all free resolutions of $\mathbb{Z} \in \mathbb{Z}G$-Mod.

Groups of type VFL

Definition

A group Γ is *virtually type* **FL** (**VFL**) if every finite index torsion-free subgroup $\Gamma' \leq \Gamma$ is type **FL**. In this case we define the *virtual cohomological dimension* of Γ to be $\mathrm{vcd}(\Gamma) := \mathrm{cd}(\Gamma')$.

This is well-defined thanks to J.P. Serre.

Cocompact Kleinian groups I

Lemma

A finitely generated Kleinian group Γ *is cocompact if and only if* $\mathrm{vcd}(\Gamma) = 3$*.*

Proof.

Let $M = \mathbb{H}^3/\Gamma'$ where Γ' is a f.i. torsion-free subgroup (Selberg's lemma). If Γ is cocompact, then M is a closed orientable 3-manifold. By Poincaré duality $H^3(M, \mathbb{Z}) \cong H_0(M, \mathbb{Z}) \cong \mathbb{Z}$ thus $\mathrm{cd}(\Gamma') = 3$. □

Cocompact Kleinian groups II

Proof.

If Γ is not cocompact, then by Scott's core theorem there exists a compact, connected 3-manifold $N \subset M$ such that $N \hookrightarrow M$ is a homotopy equivalence. Since M is not compact $\partial N \neq \emptyset$, and N is aspherical with incompressible boundary. Hence Γ' is a duality group of dimension 2 (Bieri-Eckmann) and thus $\mathrm{cd}(\Gamma') \leq 2$. $\square$

Sean's machine

We study the central extension of groups

$$1 \longrightarrow \langle c \rangle \longrightarrow G \longrightarrow \Gamma \longrightarrow 1$$

where
$G = G_w(l, m, n) = \langle f, g, w \mid f^l = g^m = w^n \rangle$,
for $w = w(f, g)$, $\langle c = f^l \rangle \cong \mathbb{Z}$, and $\Gamma \cong G/\langle c \rangle$
has the presentation

$$\Gamma = \Gamma_w(l, m, n) = \langle f, g, w \mid f^l = g^m = w^n = 1 \rangle.$$

$\mathrm{cd}(G) \leq 3$

Lemma

G is type **FL** *with* $\mathrm{cd}(G) = 3$ *and* $\chi(G) = 0$.

Let Γ' be of type **FL** with $\mathrm{cd}(\Gamma') \leq 2$.

$$1 \longrightarrow \langle c \rangle \longrightarrow G' \longrightarrow \Gamma' \longrightarrow 1,$$

Since S^1 is a $K(\mathbb{Z}, 1)$ space $\mathrm{cd}(\mathbb{Z}) = 1$ and $\chi(\mathbb{Z}) = \chi(S^1) = 0$. Hence $\chi(G) = \chi(G') = 0$. By Feldman $\mathrm{cd}(G') \leq 3$. Since $[G : G'] < \infty$, we obtain $\mathrm{cd}(G) \leq 3$ by Eckmann-Shapiro.

Fox derivatives

A *derivation* is a map $D : G \to A$ such that $D(u \cdot v) = D(u) + u \cdot D(v)$ (A: $\mathbb{Z}G$-module).

Theorem (Fox 1953)

Let F be free group on $\{x_1, \ldots, x_k\}$. For each x_i there exists a unique derivation $\partial_i : F \to \mathbb{Z}F$ defined by $\partial_i\left(x_j\right) = \delta_{ij}$. For each $y \in F$ there is a fundamental formula

$$y - 1 = \sum_{i=1}^{k} \partial_i\,(y)\,(x_i - 1).$$

$\mathrm{cd}(G) \geq 3$

Consider the start of a free resolution for $\mathbb{Z} \in \mathbb{Z}G$-Mod (Lyndon)

$$0 \longrightarrow \ker(d_2) \xrightarrow{\iota} \mathbb{Z}G^3 \xrightarrow{d_2} \mathbb{Z}G^3 \xrightarrow{d_1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

$(\partial_f(w)(f-1),\ \partial_g(w)(g-1),\ c-1) \in \mathbb{Z}G^3 \in \ker(d_2)$. Hence $\ker(d_2) \neq 0$, so $\mathrm{cd}(G) \geq 3$ by Schanuel, thus $\mathrm{cd}(G) = 3$.

Resolution for trivial module

Theorem

The trivial module $\mathbb{Z} \in \mathbb{Z}G\text{-}\mathrm{Mod}$ *has a finite type free resolution of the form*

$$0 \longrightarrow \mathbb{Z}G \xrightarrow{d_3} \mathbb{Z}G^3 \xrightarrow{d_2} \mathbb{Z}G^3 \xrightarrow{d_1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

with differentials given by

$$d_1 = \begin{bmatrix} 1 - f \\ 1 - g \\ 1 - w \end{bmatrix}, \quad d_2 = \begin{bmatrix} -\mathbf{N}_f^{l-1} & 0 & \mathbf{N}_w^{n-1} \\ 0 & -\mathbf{N}_g^{m-1} & \mathbf{N}_w^{n-1} \\ \partial_f(w) & \partial_g(w) & -1 \end{bmatrix},$$

$$d_3 = \begin{bmatrix} \partial_f(w)(f-1) & \partial_g(w)(g-1) & c-1 \end{bmatrix}.$$

A 2-periodic resolution

Theorem The trivial module $\mathbb{Z} \in \mathbb{Z}\Gamma\text{-Mod}$ has a 2-periodic resolution in degrees ≥ 2 of the form

$$\cdots \xrightarrow{\partial_3} \mathbb{Z}\Gamma^4 \xrightarrow{\partial_4} \mathbb{Z}\Gamma^4 \xrightarrow{\partial_3} \mathbb{Z}\Gamma^4 \xrightarrow{\partial_2} \mathbb{Z}\Gamma^3 \xrightarrow{\partial_1} \mathbb{Z}\Gamma \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

with the differentials given by

$$\partial_1 = \begin{bmatrix} 1-f \\ 1-g \\ 1-w \end{bmatrix}, \quad \partial_3 = \begin{bmatrix} 1-f & 1-f & 0 & 1-f \\ 1-g & 0 & 1-g & 1-g \\ 1-w & 0 & 0 & 1-w \\ 0 & \partial_f(w)(f-1) & \partial_g(w)(g-1) & c-1 \end{bmatrix}$$

$$\partial_2 = \begin{bmatrix} -\partial_f(w) & -\partial_g(w) & -\mathbf{N}_w^{n-1} \\ -\mathbf{N}_f^{l-1} & 0 & \mathbf{N}_w^{n-1} \\ 0 & -\mathbf{N}_g^{m-1} & \mathbf{N}_w^{n-1} \\ \partial_f(w) & \partial_g(w) & -1 \end{bmatrix}, \quad \partial_4 = \begin{bmatrix} -\partial_f(w) & -\partial_g(w) & -\mathbf{N}_w^{n-1} & 0 \\ -\mathbf{N}_f^{l-1} & 0 & \mathbf{N}_w^{n-1} & 0 \\ 0 & -\mathbf{N}_g^{m-1} & \mathbf{N}_w^{n-1} & 0 \\ \partial_f(w) & \partial_g(w) & -1 & 1 \end{bmatrix}.$$

Cohomology of Heckoid groups

Augmentation map and Smith normal form yield

Corollary

$$H_i(\Gamma, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \textit{ or } 2, \\ \mathbb{Z}/l \oplus \mathbb{Z}/m & i = 1, \\ \mathbb{Z}/l \oplus \mathbb{Z}/m \oplus \mathbb{Z}/n & i \equiv 1 \pmod 2,\ i > 1, \\ 0 & i \equiv 0 \pmod 2,\ i > 2, \end{cases}$$

$$H^i(\Gamma, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, \\ \mathbb{Z}/l \oplus \mathbb{Z}/m \oplus \mathbb{Z} & i = 2, \\ 0 & i \equiv 1 \pmod 2, \\ \mathbb{Z}/l \oplus \mathbb{Z}/m \oplus \mathbb{Z}/n & i \equiv 0 \pmod 2,\ i > 2. \end{cases}$$