The Spin Brauer Category

(joint with A. Savage)

- Deligne
- Wenzel
- Aboumrad

$\mathfrak{gl}_n$:

$$V^{\otimes 3} \cong L_{\ydiagram{3}} \oplus L_{\ydiagram{2,1}}^{\oplus 2} \oplus L_{\ydiagram{1,1,1}}$$

$\#$ summands stable as $n \to \infty$

SO_{2n+1} :

$$S \otimes S \cong \bigoplus_{k=0}^{n} \Lambda^k V$$

$\#$ summands $\to \infty$ as $n \to \infty$

Work / $\mathbb{C}$. V – finite dimensional vectorspace.

$\langle \cdot , \cdot \rangle$ – nondegenerate bilinear form

$N = \dim V \qquad n = \lfloor \frac{N}{2} \rfloor = \operatorname{rank}(\mathfrak{so}(V))$.

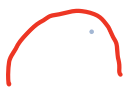
Brauer Category

V

V

$$V \xrightarrow{\mathrm{id}} V$$

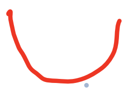
$\mathbb{C}$



$V \otimes V$

$$V \otimes V \xrightarrow{\langle \cdot , \cdot \rangle} \mathbb{C}$$

$V \otimes V$

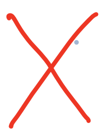


$\mathbb{C}$

$$\mathbb{C} \longrightarrow V \otimes V$$

$$1 \longmapsto \sum_i b_i \otimes b_i^\vee$$

$V \otimes V$

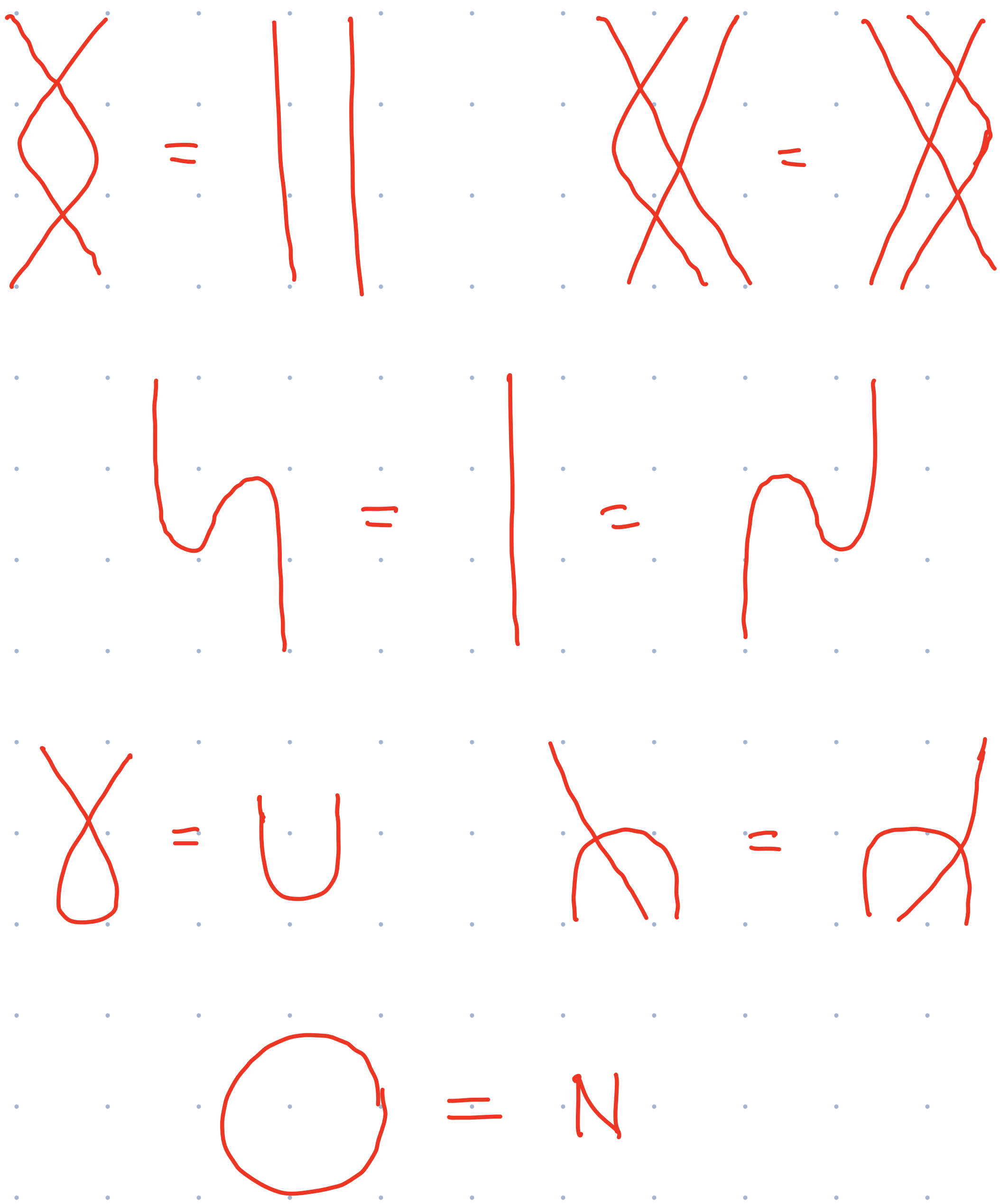


$V \otimes V$

$$V \otimes V \xrightarrow{\text{flip}} V \otimes V$$

$$x \otimes y \longmapsto y \otimes x$$

Relations



$\mathrm{Br}(d)$ = strict $\mathbb{C}$-linear monoidal category with this presentation (replace N by d)

Incarnation Functor $\mathrm{Br}(N) \longrightarrow O(V)\text{-mod}.$

Four groups with the same Lie algebra

$$
\begin{tikzcd}
O(V) \arrow[r, phantom, "\overset{\text{index } 2}{\supseteq}"] & SO(V) \quad \text{(connected)} \\
Pin(V) \arrow[u, two heads, "\text{double cover}"'] \arrow[r, phantom, "\overset{\text{index } 2}{\supseteq}"] & Spin(V) \arrow[u, two heads, "\text{double cover}"'] \quad \left(\begin{matrix}\text{simply} \\ \text{connected}\end{matrix}\right) \; (N \gg 0)
\end{tikzcd}
$$

Let $G(V) = \begin{cases} Spin(V) & \text{if } N \text{ is odd} \\ Pin(V) & \text{if } N \text{ is even} \end{cases}$

$$Cl(V) = T(V) \Big/ \Big\langle vw + wv = 2\langle v, w\rangle \Big\rangle$$

$$\cong \begin{cases} \mathrm{Mat}_{2^n}(\mathbb{C}), & N \text{ odd} \\ \mathrm{Mat}_{2^n}(\mathbb{C}) \times \mathrm{Mat}_{2^n}(\mathbb{C}), & N \text{ even} \end{cases}$$

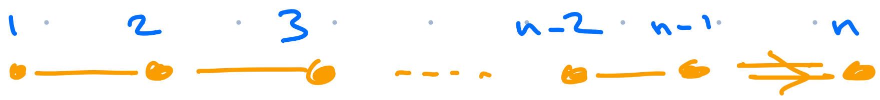
$$GPin(V) = \left\{ g \in Cl(V)^\times \;\middle|\; \begin{array}{l} gVg^{-1} = V \\ g \text{ homogenous} \end{array} \right\}$$

$$Pin(V) = \ker\left(\text{spinor norm}: GPin \to \mathbb{G}_m\right)$$

$$Spin(V) = Pin(V) \cap Cl^{\text{even}}$$

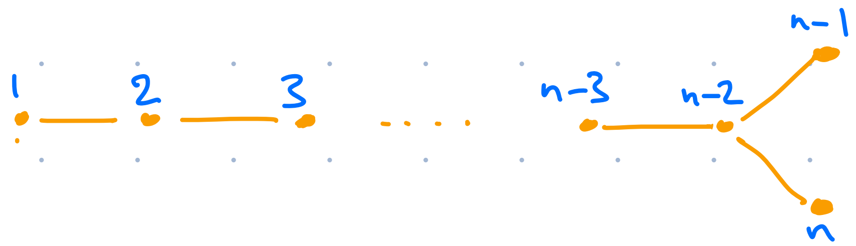
The Spin Module

Type B (N odd):



$$S = L(\omega_n)$$

Type D (N even):



$$S = L(\omega_{n-1}) \oplus L(\omega_n)$$

- S is an irreducible $G(V)$-module
- S is a tensor generator of $G(V)$-mod.

Spin Brauer Category.

Generating objects S, V.

 $= \mathrm{id}_S$  $= \mathrm{id}_V$

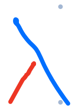
Generating morphisms

 $S \otimes S \to \mathbb{1}$  $V \otimes V \to \mathbb{1}$

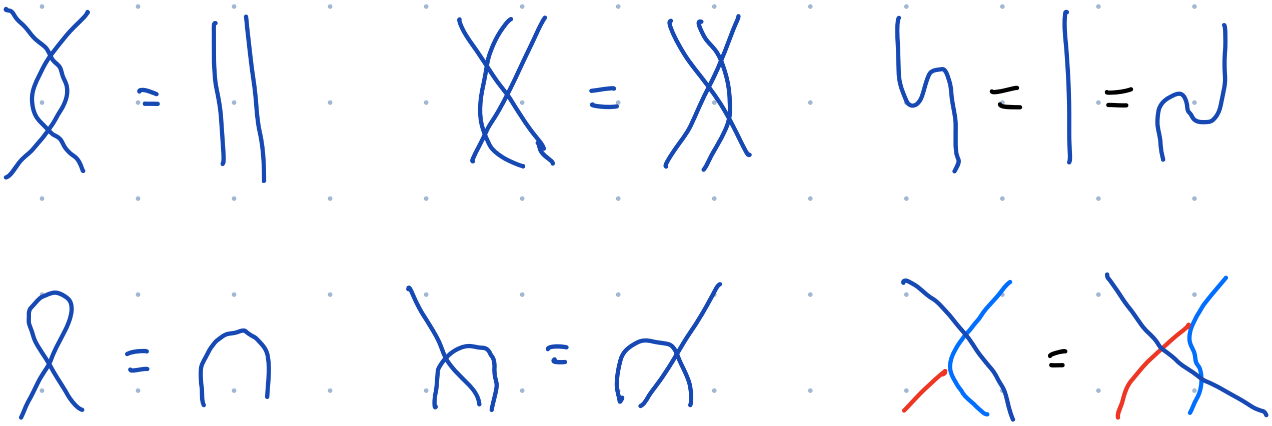
 $\mathbb{1} \to S \otimes S$  $\mathbb{1} \to V \otimes V$

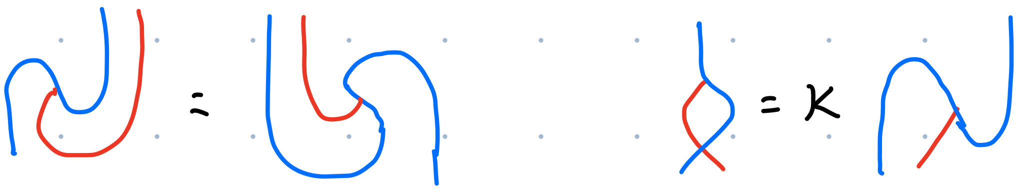
 $S \otimes S \to S \otimes S$  $V \otimes V \to V \otimes V$

 $V \otimes S \to S \otimes V$  $S \otimes V \to V \otimes S$

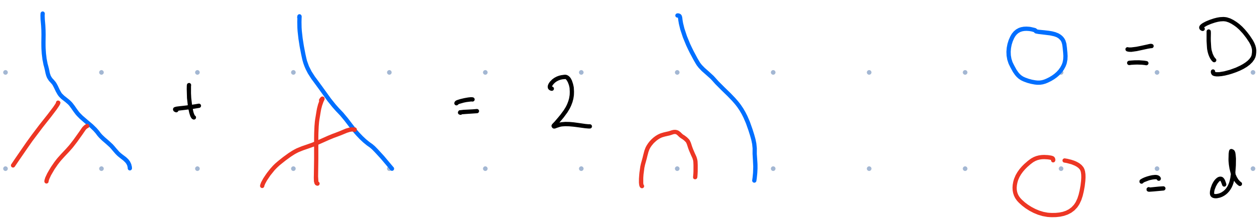
 $V \otimes S \to S$

Relations in SB(d, D)

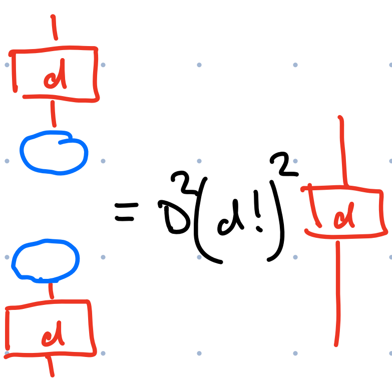




$K = 1$ unless $d > 0$ and $d \equiv 3 \pmod 4$, when $K = -1$.



If d is an odd positive integer, add the extra relation



$$\left(\boxed{r} = \sum_{\sigma \in S_r} (\mathrm{sgn}\,\sigma)\, \sigma \right)$$

$$\boxed{\begin{array}{c} SB(d, D) \\ d \in \mathbb{C},\ D \in \mathbb{C} \end{array}}$$

Type D interpolating category, except when d is an odd positive integer.

Theorems:

Theorem: Let $d = N$, $D = (-1)^{\binom{n}{2}+nN} 2^n$. Then there is a monoidal functor

$$F : SB(d, D) \longrightarrow G(V)\text{-mod}$$

(the incarnation functor)

- F is full.
- Kar F is essentially surjective.

Theorem:

$$\mathrm{End}_{SB(d,D)}(\mathbb{1}) \cong \mathbb{C}.$$

What about positive characteristic? (char $k \neq 2$)

$$\mathrm{Rep}\, G(V) \supseteq \mathrm{Tilt}\, G(V)$$

tilting modules: have a Weyl flag and a dual Weyl flag.

Theorem (Donkin): $\mathrm{Tilt}\, G(V)$ is closed under tensor products.

Our incarnation functor lands in $\mathrm{Tilt}\, G(V)$.

Conjecture: $F : SB(d, D) \to \mathrm{Tilt}\, G(V)$.

F is full and

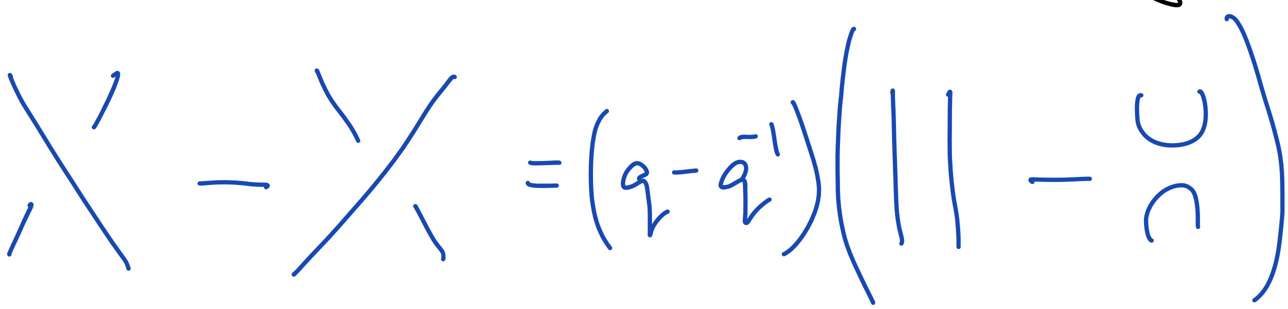
$\mathrm{Kar}\, F$ is essentially surjective

Remark: $\dim \mathrm{End}(S^{\otimes m})$ is independent of characteristic.

What about the quantum case?

- Now crossings come in two flavours

Skein relation for vector–vector crossings



No Skein relation for spin–spin crossings

(so can't simplify a closed diagram with a non-trivial spin link to a scalar)

For generic q:

- Never need the extra relation
- Have an incarnation functor to $\mathrm{Rep}(U_q(N))$ which is full and whose Karoubian envelope is essentially surjective.

Affine Spin Brauer category.

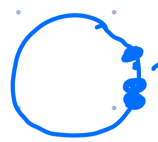
If C is monoidal then there is $C \longrightarrow \mathrm{End}(C)$ where $X \in \mathrm{ob}\, C$ gets sent to the functor $X \otimes -$.

So we have

$$SB(d, D) \longrightarrow \mathrm{End}\left(G(V)\text{-mod}\right).$$

This extends to an action of an affine spin Brauer category by considering additional generators

(action of $\Delta(C) - 1 \otimes C$ $\quad (C = \text{Casimir})$)

Theorem: The elements  r dots $\quad 2 \leq r \leq 2n$, r even,

generate $Z(\mathfrak{so}(V))^{G(V)} \subset \text{centre}\left(\mathfrak{so}(V)\text{-mod}\right)$

under the incarnation functor.

cf: the functions $f_r(X) = \mathrm{tr}\left(\pi(X)^r\right)$ $\left(\pi : \mathfrak{so}(V) \to \mathrm{End}\, S\right)$

generate $\mathbb{C}[\mathfrak{so}(V)]^{G(V)}$.

Have you seen this symmetric function?

$$W_r(x_1, x_2, \ldots, x_n) = \frac{1}{2^n} \sum_{\zeta_1, \ldots, \zeta_n \in \{\pm 1\}} \left(\sum_{i=1}^{n} \zeta_i \sqrt{x_i} \right)^{2r}.$$

Conjecture: W_r is Schur-positive