Universal objects for $\mathcal{W}$-algebras in classical Lie types

Andrew Linshaw

University of Denver

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1. $\mathcal{W}$-algebras

Class of VOAs associated to

1. A simple, finite-dimensional Lie (super)algebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$,
2. A **nilpotent** element f in the even part of $\mathfrak{g}$.

$\mathcal{W}^k(\mathfrak{g}, f)$ the (universal) $\mathcal{W}$-algebra at level k associated to $\mathfrak{g}$ and f via **quantum Drinfeld-Sokolov reduction** (Kac, Roan, Wakimoto, 2003).

$\mathcal{W}_k(\mathfrak{g}, f)$ is the simple quotient.

When $f = 0$, $\mathcal{W}^k(\mathfrak{g}, f)$ is just the affine VOA $V^k(\mathfrak{g})$.

When $\mathfrak{g} = \mathfrak{sl}_2$ and $f \neq 0$,

$$\mathcal{W}^k(\mathfrak{sl}_2, f) \cong \mathrm{Vir}^{c_k}, \qquad c_k = -\frac{(2k+1)(3k+4)}{k+2}.$$

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2. $\mathcal{W}$-algebras

Complete f to a copy of $\mathfrak{sl}_2$ and decompose $\mathfrak{g}$ as an $\mathfrak{sl}_2$-module.

Strong generators of $\mathcal{W}^k(\mathfrak{g}, f)$ correspond to irreducible $\mathfrak{sl}_2$-modules.

Module of dimension d gives rise to a field of weight $\frac{d+1}{2}$.

Let $\mathfrak{g}^\natural$ be the centralizer of this $\mathfrak{sl}_2$ in $\mathfrak{g}$.

$\mathcal{W}^k(\mathfrak{g}, f)$ contains affine subVOA $V^{k'}(\mathfrak{g}^\natural)$.

If $f = f_{\text{prin}}$ is a principal nilpotent, write $\mathcal{W}^k(\mathfrak{g}, f) = \mathcal{W}^k(\mathfrak{g})$.

OPEs of $\mathcal{W}$-algebras are generally **nonlinear**. Aside from low rank examples, unknown except when f is a minimal nilpotent.

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3. Principal $\mathcal{W}$-algebras

Principal $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g})$ is of type $\mathcal{W}(d_1, \dots, d_r)$, where

1. $r = \text{rank}(\mathfrak{g})$,
2. $d_1, \dots, d_r$ are degrees of generators of $\mathbb{C}[\mathfrak{g}]^G$.

This means generators have conformal weights $d_1, \dots, d_r$.

Ex: For $\mathfrak{g} = \mathfrak{sl}_n$, $\mathcal{W}^k(\mathfrak{sl}_n)$ is of type $\mathcal{W}(2, 3, \dots, n)$.

Thm: (Feigin, Frenkel, 1991) Let $\mathfrak{g}$ be a simple Lie algebra. Then

$$\mathcal{W}^k(\mathfrak{g}) \cong \mathcal{W}^{k'}({}^L\mathfrak{g}), \qquad r(k + h^\vee)(k' + {}^L h^\vee) = 1.$$

Here ${}^L\mathfrak{g}$ is **Langlands dual** and r is the **lacing number** of $\mathfrak{g}$.

Analogous to isomorphism $\mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[{}^L\mathfrak{g}]^{{}^L G}$.

Note: For a Lie algebra $\mathfrak{g}$ with dual Coxeter number $h^\vee$, $\psi = k + h^\vee$ is called the **shifted level**.

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4. Coset construction of principal $\mathcal{W}$-algebras

Thm: (Arakawa, Creutzig, L., 2018) Let $\mathfrak{g}$ be simple and simply-laced. We have diagonal embedding

$$V^{k+1}(\mathfrak{g}) \hookrightarrow V^k(\mathfrak{g}) \otimes L_1(\mathfrak{g}), \qquad u \mapsto u \otimes 1 + 1 \otimes u, \qquad u \in \mathfrak{g}.$$

Set

$$\mathcal{C}^k(\mathfrak{g}) = \mathrm{Com}(V^{k+1}(\mathfrak{g}), V^k(\mathfrak{g}) \otimes L_1(\mathfrak{g})).$$

We have an isomorphism

$$C^k(\mathfrak{g}) \cong \mathcal{W}^\ell(\mathfrak{g}), \qquad \ell + h^\vee = \frac{k + h^\vee}{k + 1 + h^\vee}.$$

Conjectured by Bais, Bouwknegt, Surridge, Schoutens (1988).
Case $n = 2$ was proven by Goddard, Kent, Olive (1985).

Thm: (Creutzig, L., 2021) We have an isomorphism

$$\mathrm{Com}(V^k(\mathfrak{sp}_{2n}), V^k(\mathfrak{osp}_{1|2n})) \cong \mathcal{W}^\ell(\mathfrak{so}_{2n+1}), \quad \ell + h^\vee_{\mathfrak{so}_{2n+1}} = \frac{k + h^\vee_{\mathfrak{osp}_{1|2n}}}{k + h^\vee_{\mathfrak{sp}_{2n}}}.$$

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5. Gaiotto-Rapčák conjectures

Generalization of both Feigin-Frenkel duality and coset realization.

For $n \geq 1$, consider $\mathfrak{sl}_{n+m}$ with **hook-type** partition $(n, 1^m)$.

$\mathfrak{sl}_{n+m}$ has shifted level $\psi = k + n + m$, and we define

$$\mathcal{W}^{\psi}(n, m) := \mathcal{W}^{\psi-n-m}(\mathfrak{sl}_{n+m}, f_{n,1^m}).$$

Similarly, for $n \geq 1$, consider $\mathfrak{sl}_{n|m}$ with the partition $f_{n|1^m}$.

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6. **Gaiotto-Rapčák conjectures**

For $n = m$, use instead $\mathcal{V}^{\psi}(n,n) := \mathcal{W}^{\psi}(\mathfrak{psl}_{n|n}, f_{n|1^n})$.

For $n = 0$, need a different definition:

$$\mathcal{W}^{\psi}(0,m) = V^{\psi-m}(\mathfrak{sl}_m) \otimes \mathcal{S}(m),$$
$$\mathcal{V}^{\psi}(0,m) = V^{-\psi-m}(\mathfrak{sl}_m) \otimes \mathcal{E}(m).$$

Here $\mathcal{S}(m)$ and $\mathcal{E}(m)$ are rank m $\beta\gamma$-**system** and bc-**system**.

Analogous to **Weyl algebra** and **Clifford algebra**, respectively.

They have actions of $L_{-1}(\mathfrak{gl}_m)$ and $L_1(\mathfrak{gl}_m)$, respectively,

$\mathcal{W}^{\psi}(n,m)$ and $\mathcal{V}^{\psi}(n,m)$ have **affine subVOAs** $V^{\psi-m-1}(\mathfrak{gl}_m)$ and $V^{-\psi-m+1}(\mathfrak{gl}_m)$, respectively.

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$$\mathcal{W}^{\psi}(0,m) = V^{\psi-m}(\mathfrak{sl}_m) \otimes \mathcal{S}(m),$$
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Here $\mathcal{S}(m)$ and $\mathcal{E}(m)$ are rank m $\beta\gamma$-**system** and bc-**system**.

Analogous to **Weyl algebra** and **Clifford algebra**, respectively.

They have actions of $L_{-1}(\mathfrak{gl}_m)$ and $L_1(\mathfrak{gl}_m)$, respectively,

$\mathcal{W}^{\psi}(n,m)$ and $\mathcal{V}^{\psi}(n,m)$ have **affine subVOAs** $V^{\psi-m-1}(\mathfrak{gl}_m)$ and $V^{-\psi-m+1}(\mathfrak{gl}_m)$, respectively.

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Consider the cosets

$$\mathcal{C}^{\psi}(n,m) = \mathrm{Com}(V^{\psi-m-1}(\mathfrak{gl}_m), \mathcal{W}^{\psi}(n,m)),$$

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$$\mathcal{D}^{\psi}(n,n) = \mathrm{Com}(V^{-\psi-n+1}(\mathfrak{sl}_n), \mathcal{V}^{\psi}(n,n))^{U(1)}.$$

Cases $\mathcal{C}^{\psi}(0,m)$ and $\mathcal{D}^{\psi}(0,m)$ are called **GKO cosets**.

Thm: (Creutzig-L., 2020) Let $n \geq m$ be non-negative integers. We have isomorphisms of 1-parameter VOAs

$$\mathcal{D}^{\psi}(n,m) \cong \mathcal{C}^{\psi^{-1}}(n-m,m) \cong \mathcal{D}^{\psi'}(m,n), \qquad \frac{1}{\psi} + \frac{1}{\psi'} = 1.$$

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8. Idea of proof

Exists a 2-parameter VOA $\mathcal{W}_\infty$ which classifies VOAs of type $\mathcal{W}(2,3,\dots,N)$ for some N.

$\mathcal{W}_\infty$ of type $\mathcal{W}(2,3,\dots)$, defined over polynomial ring $\mathbb{C}[c,\lambda]$.

Conjectured to exist by many authors: Bakas, Kiritsis (1991), Yu, Wu (1992), Gaberdiel, Gopakumar (2011), and constructed rigorously by L. (2017).

Given a prime ideal $I \subseteq \mathbb{C}[c,\lambda]$, $\mathcal{W}^I_\infty = \mathcal{W}_\infty / I \cdot \mathcal{W}_\infty$, is a VOA over $R = \mathbb{C}[c,\lambda]/I$.

$\mathcal{W}^I_\infty$ simple for a generic ideal I. Otherwise, $\mathcal{W}_{\infty,I}$ denotes simple quotient.

All simple, one-parameter VOAs of type $\mathcal{W}(2,3,\dots,N)$ are of this form, including $\mathcal{C}^\psi(n,m)$ and $\mathcal{D}^\psi(n,m)$.

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9. General nilpotents

Goal: Construct similar universal 2-parameter VOAs for **all** nilpotents, not only hook-type.

Let $P = (n_0^{m_0}, n_1^{m_1}, \dots, n_t^{m_t})$ be a partition of $N = \sum_{i=0}^{t} n_i m_i$ consisting of m_i parts of size n_i, where $n_0 > n_1 > \cdots > n_t \geq 2$.

Let $M = \{m_0, \dots, m_t\}$ is the set of *multiplicities*.

Let $S = \{d_1, \dots, d_t\}$ is the set of *height differences* $d_{i+1} = n_i - n_{i+1}$.

If $t = 0$, $S = \emptyset$, and $M = \{m\}$, $P = (n^m)$ is called **rectangular**.

Observation: Let $f_P \in \mathfrak{sl}_N$ be the nilpotent corresponding to P.

Then $\mathcal{W}^k(\mathfrak{sl}_N, f_P)$ has many features that only depend on S and M, and are independent of choice of n_t.

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10. General nilpotents

1. For all $n_t \geq 2$,
$$\mathfrak{sl}_N^\natural \cong \mathfrak{t} \oplus \Big(\bigoplus_{i=0}^{t} \mathfrak{sl}_{m_i}\Big),$$
where $\mathfrak{t}$ is an abelian Lie algebra of dimension t.
2. Up to any fixed conformal weight, strong generating type of $\mathcal{W}^k(\mathfrak{sl}_N, f_P)$ is independent of n_t, for n_t sufficiently large.
3. Action of $\mathfrak{sl}_N^\natural$ on the generating fields of higher weight, is independent of n_t for n_t sufficiently large.

Ex: For $t = 0$, $S = \emptyset$ and $M = \{m\}$, $P = (n^m)$, and $\mathfrak{sl}_{nm}^\natural = \mathfrak{sl}_m$.

Then $\mathcal{W}^k(\mathfrak{sl}_{nm}, f_{n^m})$ has generating type
$$\mathcal{W}(1^{m^2-1}, 2^{m^2}, 3^{m^2}, \dots, n^{m^2}).$$

For $d \geq 2$, m^2 fields in weight d transform under $\mathfrak{sl}_m$ as trivial plus adjoint module.

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11. **General nilpotents**

Ex: $S = \{2\}$ and $M = \{1, 3\}$, so $P = (n+2, n^3)$ and $N = 4n+2$.

Then $\mathfrak{sl}^{\natural}_{4n+2} \cong \mathfrak{gl}_3$.

$\mathcal{W}^k(\mathfrak{sl}_{4n+2,}, f_P)$ has generating type

$$\mathcal{W}(1^9, 2^{16}, 3^{16}, \dots, n^{16}, (n+1)^7, n+2).$$

Under action of $\mathfrak{gl}_3$:

1. 16 fields in each weight $2 \leq d \leq n$ transform as sum of the trivial, standard, dual standard, and adjoint modules.
2. 7 fields in weight $n+1$ transform as sum of trivial, standard, and dual standard modules.
3. Field in weight $n+2$ transforms as trivial module.

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12. General nilpotents

Conj: Let $S = \{d_1, \dots, d_t\}$, $M = \{m_0, \dots, m_t\}$ be as above. There exists a unique 2-parameter VOA $\mathcal{W}_{\infty}^{A,S,M}$ such that:

1. $\mathcal{W}_{\infty}^{A,S,M}$ is defined over a finite localization of the polynomial ring in two variables.
2. $\mathcal{W}_{\infty}^{A,S,M}$ is freely generated of the appropriate type determined by S and M.
3. $\mathcal{W}_{\infty}^{A,S,M}$ admits all the $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{sl}_N, f_P)$ as 1-parameter quotients.
4. $\mathcal{W}_{\infty}^{A,S,M}$ is weakly generated by the fields in weight at most 3.
5. $\mathcal{W}_{\infty}^{A,S,M}$ is an extension of $\mathcal{H}(|M|-1)$ tensored with $|M|$ commuting copies of $\mathcal{W}_{\infty}$, where $|M| = \sum_{i=0}^{t} m_i$.
6. $\mathcal{W}_{\infty}^{A,S,M}$ serves as a **classifying object** for vertex algebras satisfying (1)-(4); all are quotients of $\mathcal{W}_{\infty}^{A,S,M}$.

13. Some examples

Ex: $\mathcal{W}_\infty^{A,\emptyset,\{1\}} \cong \mathcal{W}_\infty$, generating type $\mathcal{W}(2,3,4,\dots)$.

Ex: $\mathcal{W}_\infty^{A,\emptyset,\{m\}} \otimes \mathcal{H}$ is the algebra $\mathcal{W}_\infty^{(K)}$ for $K = m$ constructed recently by Gaiotto, Rapčák, and Zhou (2023).

$\mathcal{W}_\infty^{A,\emptyset,\{m\}}$ has generating type $\mathcal{W}(1^{m^2-1}, 2^{m^2}, 3^{m^2}, \dots)$.

Note: Uniqueness of $\mathcal{W}_\infty^{A,\emptyset,\{m\}}$ as a 2-parameter VOA was **not** proven.

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14. More quotients of $\mathcal{W}_\infty^{A,S,M}$

Use same notation P for partition $(n_0^{m_0}, n_1^{m_1}, \dots, n_t^{m_t}, 1^s)$ of $N+s$.

Let f_P denote the corresponding nilpotent in either $\mathfrak{sl}_{N+s}$ of $\mathfrak{sl}_{N|s}$.

Both $\mathfrak{sl}^\natural_{N+s}$ and $\mathfrak{sl}^\natural_{N|s}$ are $\mathfrak{t} \oplus (\bigoplus_{i=0}^t \mathfrak{sl}_{m_i}) \oplus \mathfrak{gl}_s$.

$\mathcal{W}^k(\mathfrak{sl}_{N+s}, f_P)$ and $\mathcal{W}^k(\mathfrak{sl}_{N|s}, f_P)$ have affine subVOA of type $V^{k'}(\mathfrak{gl}_s)$, $V^{k''}(\mathfrak{gl}_s)$ or some shifted levels k', k''.

Thm: The cosets

$$\mathcal{C}^k_{S,M}(n_t, s) = \mathrm{Com}(V^{k'}(\mathfrak{gl}_s), \mathcal{W}^k(\mathfrak{sl}_{N+s}, f_P)),$$
$$\mathcal{D}^k_{S,M}(n_t, s) = \mathrm{Com}(V^{k''}(\mathfrak{gl}_s), \mathcal{W}^k(\mathfrak{sl}_{N|s}, f_P)),$$

have same generating type as $\mathcal{W}_\infty^{A,S,M}$.

Analogous to Gaiotto-Rapčák algebras $\mathcal{C}^\psi(n,m)$ and $\mathcal{D}^\psi(n,m)$ for $n \geq 2$.

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15. More quotients of $\mathcal{W}_{\infty}^{A,S,M}$: GKO cosets

Both $\mathcal{C}^k_{S,M}(n_t, s)$ and $\mathcal{D}^k_{S,M}(n_t, s)$ also make sense and have the correct generating type if we set $n_t = 1$.

These are analogous to $\mathcal{C}^{\psi}(1, m)$ and $\mathcal{D}^{\psi}(1, m)$, respectively.

What about setting $n_t = 0$?

This has the effect of replacing P with smaller partition

$$P' = (n_0^{m_0}, \dots, n_{t-1}^{m_{t-1}})$$

of $N' = \sum_{i=0}^{t-1} m_i n_i$, where $d_{i+1} = n_i - n_{n+1}$ for $i = 0, \dots, t-2$.

Let $f_{P'} \in \mathfrak{sl}_{N'}$ be the corresponding nilpotent.

For all $s \geq 1$, we use the same notation for this nilpotent in $\mathfrak{sl}_{N'+s}$.

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16. More quotients of $\mathcal{W}_\infty^{A,S,M}$: GKO cosets

Consider the tensor products

$$\mathcal{W}^k(\mathfrak{sl}_{N'+s}, f_{P'}) \otimes \mathcal{S}(m_t s), \qquad \mathcal{W}^k(\mathfrak{sl}_{N'+s}, f_{P'}) \otimes \mathcal{E}(m_t s),$$

where $\mathcal{S}(m_t s)$ and $\mathcal{E}(m_t s)$ are the $\beta\gamma$-system and bc-system of rank $m_t s$, respectively.

Note that

1. $\mathcal{W}^k(\mathfrak{sl}_{N'+s}, f_{P'})$ has an affine subVOA $V^{k'}(\mathfrak{gl}_s)$ for some k',
2. $\mathcal{S}(m_t s)$ has an action of $V^{-m_t}(\mathfrak{gl}_s) \otimes V^{-s}(\mathfrak{sl}_{m_t})$,
3. $\mathcal{E}(m_t s)$ has an action of $V^{m_t}(\mathfrak{gl}_s) \otimes V^{s}(\mathfrak{sl}_{m_t})$.

We therefore have diagonal embeddings

$$V^{k'-m_t}(\mathfrak{gl}_s) \hookrightarrow \mathcal{W}^k(\mathfrak{sl}_{N'+s}, f_{P'}) \otimes \mathcal{S}(m_t s),$$
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Consider the following diagonal cosets:

$$\mathcal{C}^k_{S,M}(0,s) := \mathrm{Com}(V^{k'-m_t}(\mathfrak{gl}_s), \mathcal{W}^k(\mathfrak{sl}_{N'+s}, f_{P'}) \otimes \mathcal{S}(m_t s)),$$
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Thm: These also have same generating type as $\mathcal{W}_\infty^{A,S,M}$.

Analogues of GKO cosets $\mathcal{C}^\psi(0,m)$ and $\mathcal{D}^\psi(0,m)$, respectively.

Note: $\mathcal{W}_\infty^{A,S,M}$ has an affine subVOA of type $V^k(\mathfrak{sl}_{m_t})$, and $\mathcal{D}^k_{S,M}(0,s)$ has affine subVOA $L_s(\mathfrak{sl}_{m_t})$.

Ex: For $S = \emptyset$ and $M = \{m\}$, so $P = (n^m)$, we have $N' = 0$ so P' is empty. Then

$$\mathcal{C}^k_{\emptyset,\{m\}}(0,s) = \mathrm{Com}(V^{k-m}(\mathfrak{gl}_s), V^k(\mathfrak{sl}_s) \otimes \mathcal{S}(ms)),$$
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Analogues of GKO cosets $\mathcal{C}^{\psi}(0,m)$ and $\mathcal{D}^{\psi}(0,m)$, respectively.

Note: $\mathcal{W}_{\infty}^{A,S,M}$ has an affine subVOA of type $V^k(\mathfrak{sl}_{m_t})$, and $\mathcal{D}^k_{S,M}(0,s)$ has affine subVOA $L_s(\mathfrak{sl}_{m_t})$.

Ex: For $S = \emptyset$ and $M = \{m\}$, so $P = (n^m)$, we have $N' = 0$ so P' is empty. Then

$$\mathcal{C}^k_{\emptyset,\{m\}}(0,s) = \mathrm{Com}(V^{k-m}(\mathfrak{gl}_s), V^k(\mathfrak{sl}_s) \otimes \mathcal{S}(ms)),$$
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17. More quotients of $\mathcal{W}_\infty^{A,S,M}$: GKO cosets

Consider the following diagonal cosets:

$$\mathcal{C}^k_{S,M}(0,s) := \mathrm{Com}(V^{k'-m_t}(\mathfrak{gl}_s), \mathcal{W}^k(\mathfrak{sl}_{N'+s}, f_{P'}) \otimes \mathcal{S}(m_t s)),$$
$$\mathcal{D}^k_{S,M}(0,s) := \mathrm{Com}(V^{k'+m_t}(\mathfrak{gl}_s), \mathcal{W}^k(\mathfrak{sl}_{N'+s}, f_{P'}) \otimes \mathcal{E}(m_t s)).$$

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Ex: For $S=\emptyset$ and $M=\{m\}$, so $P=(n^m)$, we have $N'=0$ so P' is empty. Then

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18. More quotients of $\mathcal{W}_\infty^{A,S,M}$: GKO cosets

Ex: $S = \{2\}$ and $M = \{1,3\}$, so $P = (n+2, n^3)$ and $N = 4n+2$.

Then $\mathfrak{sl}_{4n+2}^\natural \cong \mathfrak{gl}_3$.

Setting $n = 0$, we get $P' = (2)$ and $N' = 2$. Then

$$\mathcal{W}^k(\mathfrak{sl}_{N'+s}, f_{P'}) := \mathcal{W}^k(\mathfrak{sl}_{s+2}, f_{\min}).$$

This has affine subVOA $V^{k'}(\mathfrak{gl}_s)$ for $k' = k+1$.

In this case,

$$\mathcal{C}^k_{\{2\},\{1,3\}}(0,s) = \mathrm{Com}(V^{k'-3}(\mathfrak{gl}_s), \mathcal{W}^k(\mathfrak{sl}_{2+s}, f_{\min}) \otimes \mathcal{S}(3s)),$$
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19. Intersections of truncation curves

Idea: Regard $\mathcal{W}_{\infty}^{A,S,M}$ as a **moduli space** of VOAs with appropriate strong generating type.

Parameter space $X = \mathbb{C}^2 \setminus \bigcup_i C_i$ where $\bigcup_i C_i$ is the union of finitely many curves.

Each 1-parameter quotient $\mathcal{C}^k_{S,M}(n_t, s)$ or $\mathcal{D}^k_{S,M}(n_t, s)$ corresponds to an ideal $I_{n_t,s}$ or $J_{n_t,s}$ in ring of parameters.

Curves $V(I_{n_t,s})$ and $V(J_{n_t,s})$ are called **truncation curves** in X.

At intersection point on two of these curves, we expect an isomorphism between simple quotients of corresponding VOAs.

A subtlety: even though $\mathcal{C}^k_{S,M}(n_t, s)$ or $\mathcal{D}^k_{S,M}(n_t, s)$ are quotients of $\mathcal{W}_{\infty}^{A,S,M}$ for generic k, this can fail at a particular level.

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20. Rationality of GKO cosets

GKO cosets play a special role because they often have points where simple quotient is **strongly rational**.

Via intersections of truncation curves, this provides a way to prove new rationality results.

Ex: For $S = \emptyset$ and $M = \{1\}$, so $\mathcal{W}_{\infty}^{A,\emptyset,\{1\}} = \mathcal{W}_{\infty}$, recall GKO coset

$$\mathcal{D}^k_{\emptyset,\{1\}}(0,s) = \text{Com}(V^{k+1}(\mathfrak{gl}_s), V^k(\mathfrak{sl}_s) \otimes \mathcal{E}(s)).$$

When k is admissible for $\mathfrak{sl}_s$, simple quotient

$$\mathcal{D}_{k,\emptyset,\{1\}}(0,s) = \text{Com}(L_{k+1}(\mathfrak{gl}_s), L_k(\mathfrak{sl}_s) \otimes \mathcal{E}(s)),$$

which is always strongly rational.

Ex: For $S = \emptyset$ and $M = \{m\}$, so $P = (n^m)$, recall GKO coset

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$$\mathcal{D}_{k,\emptyset,\{m\}}(0,s) = \mathrm{Com}(L_{k+m}(\mathfrak{gl}_s), L_k(\mathfrak{sl}_s) \otimes \mathcal{E}(ms)),$$

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$$\mathrm{Com}(L_{k+i}(\mathfrak{gl}_s), L_{k+i-1}(\mathfrak{sl}_s) \otimes \mathcal{E}(s)), \qquad i = 1, \dots, m,$$

hence is strongly rational.

Ex: $S = \{2\}$ and $M = \{1,3\}$, so $P = (n+2, n^3)$ and $N = 4n+2$. Recall $\mathcal{W}_{\infty}^{A,\{2\},\{1,3\}}$ has type $\mathcal{W}(1^9, 2^{16}, 3^{16}, \dots)$.

Recall GKO coset

$$\mathcal{D}^k_{\{2\},\{1,3\}}(0,s) = \mathrm{Com}(V^{k'+3}(\mathfrak{gl}_s), \mathcal{W}^k(\mathfrak{sl}_{2+s}, f_{\min}) \otimes \mathcal{E}(3s)).$$

Thm: Simple quotient $\mathcal{D}_{k,\{2\},\{1,3\}}(0,s)$ strongly rational when $k = -(s+2) + \frac{m+s+2}{2}$, $m \in \mathbb{N}_{\geq 2}$, $m+s$ is odd.

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22. **Orthosymplectic types**

All $\mathcal{W}$-algebras of types B, C, or D are expected to be governed by universal 2-parameter VOAs $\mathcal{W}_{\infty}^{C,S,M}$ and $\mathcal{W}_{\infty}^{BD,S,M}$.

Here $S = \{d_1, \dots, d_t\}$ and $M = \{m_0, \dots, m_r\}$ are as above, but there are some restrictions on parities of elements $m_i \in M$.

1-parameter quotients include 8 families of $\mathcal{W}$-algebras and their cosets, and 4 families of GKO cosets.

Ex: $\mathcal{W}_{\infty}^{BD,\emptyset,\{1\}} = \mathcal{W}_{\infty}^{\mathrm{ev}}$. Generating type $\mathcal{W}(2,4,6,\dots)$ constructed by Kanade-L. (2019). Used to prove trialities of orthosymplectic Y-algebras (Creutzig, L. 2022).

Ex: $\mathcal{W}_{\infty}^{C,\emptyset,\{2\}}$ has generating type $\mathcal{W}(1^3, 2, 3^3, 4, \dots)$, affine subVOA is of type $\mathfrak{sp}_2$. (Creutzig, Kovalchuk, L. 2024).

Ex: $\mathcal{W}_{\infty}^{BD,\emptyset,\{2\}}$ has generating type $\mathcal{W}(1, 2^3, 3, 4^3, \dots)$, affine subVOA is of type $\mathfrak{so}_2$. (Creutzig, Kovalchuk, L. 2025).

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All $\mathcal{W}$-algebras of types B, C, or D are expected to be governed by universal 2-parameter VOAs $\mathcal{W}_\infty^{C,S,M}$ and $\mathcal{W}_\infty^{BD,S,M}$.

Here $S = \{d_1, \dots, d_t\}$ and $M = \{m_0, \dots, m_r\}$ are as above, but there are some restrictions on parities of elements $m_i \in M$.

1-parameter quotients include 8 families of $\mathcal{W}$-algebras and their cosets, and 4 families of GKO cosets.

Ex: $\mathcal{W}_\infty^{BD,\emptyset,\{1\}} = \mathcal{W}_\infty^{\mathrm{ev}}$. Generating type $\mathcal{W}(2,4,6,\dots)$ constructed by Kanade-L. (2019). Used to prove trialities of orthosymplectic Y-algebras (Creutzig, L. 2022).

Ex: $\mathcal{W}_\infty^{C,\emptyset,\{2\}}$ has generating type $\mathcal{W}(1^3, 2, 3^3, 4, \dots)$, affine subVOA is of type $\mathfrak{sp}_2$. (Creutzig, Kovalchuk, L. 2024).

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