Exploring distant corners of Calabi-Yau moduli spaces

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Outline

Based on...

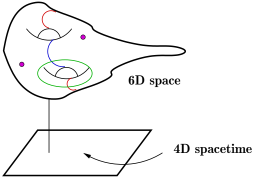
- Collaborators: Andrei Căldăraru, Richard Eager, David Erkinger, Kentaro Hori, Joseph McGovern, Robert Pryor, Mauricio Romo, Emanuel Scheidegger, Thorsten Schimannek, Eric Sharpe
- Papers:
  - McGovern-JK: "A single point as a Calabi-Yau zerofold" [arXiv:2506.16726] (short proceedings article!)
  - McGovern-JK: "Noncommutative resolutions and CICY quotients from a non-abelian GLSM" [arXiv:2504.06147]
  - Pryor-JK: "B-type D-branes in hybrid models" [arXiv:2404.14613]
  - Erkinger-JK: "On genus-0 invariants of Calabi-Yau hybrid models" [arXiv:2210.01226]
  - Scheidegger-Schimannek-JK: "On genus one fibered Calabi-Yau threefolds with 5-sections" [arXiv:2107.056477]
  - ...

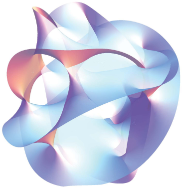
Calabi-Yau compactifications

- We consider type IIA/B string theory on compact Calabi-Yau threefolds.
- Sometimes we add (A-type)/B-type D-branes.
- Perturbative and non-perturbative corrections depend on moduli.

$$\mathcal{M}_{CY} = \mathcal{M}_K \times \mathcal{M}_{CS}$$

- Geometric picture

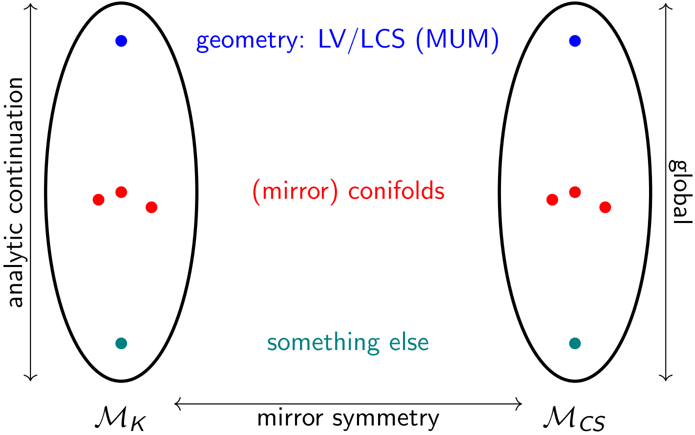




  - Depending on the locus in $\mathcal{M}$, this picture may not be accurate.

Moduli spaces

- Structure of stringy moduli spaces of one-parameter models:



Questions

- Main question: What is the "something else"?
- Related Question: How to characterise string theory on a Calabi-Yau threefold?
  - String theory compactified on a smooth compact Calabi-Yau manifold?
  - A maximal star triangulation of a reflexive lattice polytope in the Kreuzer-Skarke list? [Kreuzer-Skarke 00]
  - A point of maximally unipotent monodromy (MUM) in the associated moduli space?
  - A non-linear sigma model with Calabi-Yau target?
  - A phase of a gauged linear sigma model (GLSM) whose vacuum manifold is a smooth compact Calabi-Yau? [Witten 93]
  - An $\mathcal{N} = (2, 2)$ SCFT with central charge $c = (9, 9)$?
- This talk: It's not necessarily any or all of the above.

More questions

- Assuming we have a well-understood geometric locus somewhere in the moduli space, what can we expect to find when we move to the "other end" of the moduli space?
  - Is it another geometry/non-linear sigma model? (sometimes)
  - If not, what is the correct mathematical and physical description? (usually hard)
  - Do new things happen when we consider CYs which are not complete intersections in toric spaces? (yes)
  - Can we do calculations? (often yes, but how to interpret?)
- What is a universal language that holds everywhere in the moduli space?
  - It should not rely on the existence of a smooth Calabi-Yau geometry, e.g. branes wrapping cycles.
  - Examples: worldsheet CFT, GLSM

Even more questions

- What are the objects we can study?
- More concretely, what is equivalent to the following concepts related to a compactification on a smooth Calabi-Yau?
  - Physical description in terms of a non-linear sigma model.
  - CFT state spaces (chiral rings) and their relation to cohomology classes of the Calabi-Yau.
  - Correlation functions, i.e. Gromov-Witten invariants, and related enumerative invariants (eg. GV invariants).
  - Description of B-type D-branes in terms of the bounded derived category of coherent sheaves on the Calabi-Yau.
- In the remainder of the talk we will report on how to give partial answers to these questions in certain settings.

Disclaimers

- For simplicity, we mostly restrict to one-parameter models, i.e. CYs with either $\dim_{\mathbb{C}}\mathcal{M}_K = 1$ or $\dim_{\mathbb{C}}\mathcal{M}_{CS} = 1$.
  - Many statements generalise to multi-parameter models.
  - We exclude examples where $\dim\mathcal{M}_* = 0$.
- We focus on $\mathcal{M}_K$.
  - This is the type IIA picture.
  - Moduli are (complexified) volume parameters.
  - There are equivalent/mirror statements on $\mathcal{M}_{CS}$.
- We mostly study asymptotic regions of the moduli space, i.e. those where $|\text{moduli}| \gg 0$.
  - This excludes interesting loci in the "middle" of the moduli space such as (mirror) conifold points, attractor points, desert points, etc.
- We only consider B-branes.

How to address these questions?

- Mirror symmetry and topological strings
  - Extremely useful, well-developed standard technology for explicit calculations that is well-suited to reach remote areas of the moduli space. [Huang-Klemm-Quackenbush 06]
  - However, it is not necessarily clear how to interpret what one computes.
- Gauged linear sigma model (GLSM) [Witten 93]
  - Can probe $\mathcal{M}_K$ directly: Kähler parameters $\leftrightarrow$ (FI,θ) couplings
  - Phases: different low-energy effective theories at different values of couplings
  - Exact results via supersymmetric localisation
  - Rest of the talk.

GLSM data

- Symmetries
  - 2D $\mathcal{N} = (2,2)$-SUSY (including $U(1)_{V,A}$ R-symmetry)
  - Gauge symmetry: gauge group G (not necessarily abelian)
- Field content
  - Chiral multiplets: $\Phi^i = (\phi^i, \ldots)$
    - transform in some representation ρ_V of G, $\phi^i \in V$
    - gauge charges Q_i^a: weights of ρ_V
    - CY: $\rho_V : G \to SL(V)$
  - Vector multiplets: $\Sigma_a = (\sigma_a, \ldots)$, $\sigma_a \in \mathfrak{t}_{\mathbb{C}}$
- Parameters
  - FI-theta parameters (ζ^a, θ^a)
    - not renormalised in CY case
    - $t^a = \zeta^a - i\theta^a$: coordinates on $\mathcal{M}_K$
  - gauge couplings
- Superpotential $W(\Phi)$

Phases

- Phases of the GLSM are low-energy effective theories depending on the value of the FI parameter(s).
  - The GLSM probes $\mathcal{M}_K$.
- Higgs vacua (all σ_a zero) are determined by the solutions of the D-term and F-term equations:

$$\mu(\phi) = \zeta \qquad dW(\phi) = 0$$

  - $\mu : V \to \mathfrak{t} \ldots$ moment map
  - Gauge symmetry broken to a, potentially continuous, subgroup.
  - Parameter space is divided into chambers corresponding to different solutions.
  - Obtain low-energy effective theory by analysing the physics near the vacuum.

Coulomb branch

- At phase boundaries, a Coulomb branch can emerge.
- Located in FI-theta-space determined by the critical loci of

$$\widetilde{W}_{eff} = -\langle t, \sigma \rangle - \sum_{i=1}^{\dim V} \langle Q_i, \sigma \rangle \left(\log \langle Q_i, \sigma \rangle - 1 \right) + i\pi \sum_{\alpha > 0} \langle \alpha, \sigma \rangle$$

  - $\alpha > 0$ . . . positive roots of G
  - $\langle \cdot, \cdot \rangle$ . . . $\mathfrak{t}_{\mathbb{C}}^* \times \mathfrak{t}_{\mathbb{C}} \to \mathbb{C}$
- This determines the singular loci in $\mathcal{M}_K$
- More generally:
  - Higher rank G: mixed branches
  - Non-abelian G: strong coupling phenomena

GLSM B-branes

- D-branes (B-type) in GLSMs are G-invariant matrix factorisations of the GLSM potential with R-charge 1

[Herbst-Hori-Page 08][Honda-Okuda,Hori-Romo 13]

- Data:
  - $\mathbb{Z}_2$-graded module (Chan-Paton space): $M = M^0 \oplus M^1$
  - Matrix Factorisation: $Q \in End^1(M)$ with

$$Q^2(\phi) = W(\phi) \cdot \mathrm{id}_M$$

  - Compatible with gauge and R-symmetry:
    - $\rho : G \to GL(M)$ such that $\rho(g)^{-1} Q(g\phi)\rho(g) = Q(\phi)$, $g \in G$.
    - $r_* : u(1)_V \to gl(M)$ such that $\lambda^{\mathbf{r}_*} Q(\lambda^R \phi)\lambda^{-\mathbf{r}_*} = \lambda Q(\phi)$, $\lambda \in U(1)_V$.
- Universal structure: The category of GLSM B-branes reduces to the "correct" category of B-branes in the respective phase.

[Herbst-Hori-Page 08]

GLSM partition functions

- SUSY localisation: For certain curved backgrounds, the path integral can be solved exactly.
- The SUSY partition functions compute fully quantum corrected objects:
  - Sphere partition function Z_{S^2}: Kähler potential $K(t,\bar{t})$ on $\mathcal{M}_K$
    [Benini-Cremonesi 12][Doroud-Gomis-LeFloch-Lee 12]
    [Jockers-Kumar-Lapan-Morrison-Romo 12][Gomis-Lee 12]
    [Gerchkovitz-Gomis-Komargodski 14][Gomis-Hsin-Komargodski-Schwimmer-Seiberg-Theisen 15]
  - Torus partition function Z_{T^2}: Elliptic genus
    [Benini-Eager-Hori-Tachikawa 13]
  - Hemisphere partition function Z_{D^2}: Central charge of a B-brane [Hosono 00][Sugishita-Terashima, Honda-Okuda, Hori-Romo 13]
  - Annulus partition function: Open Witten index of two B-branes [Hori-Romo 13]
- These objects can be evaluated in any phase of the GLSM.

Universal structure

- Claim: The partition functions evaluated in phases have a universal structure.
- Example: Z_{D^2} [Romo-Scheidegger-JK '20][Pryor-JK '24]

$$Z_{D^2}^{phases}(\mathcal{B}) = \langle \mathrm{ch}(\mathfrak{B}), \Gamma \cdot I \rangle.$$

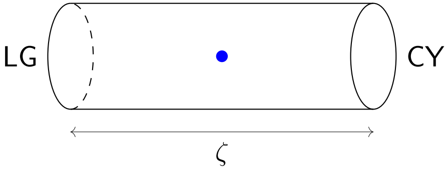
  - $\langle \cdot, \cdot \rangle$ … pairing on the state space ($(a,c) - ring$)
  - Γ … Gamma class
  - I … I-function (mirror periods)
  - $\mathrm{ch}(\mathfrak{B})$ … Chern character of the IR image $\mathfrak{B}$ of $\mathcal{B}$
- These objects can be defined without reference to geometry.
- Assuming this structure holds, use this to compute genus-0 invariants, brane charges, integral bases of periods, "swampland distances", etc. in exotic phases. [Erkinger-JK 19,20,22]

Quintic GLSM

- The quintic is a $G = U(1)$ GLSM with scalars $\phi = (p, x_1, \ldots, x_5)$ with gauge charges $(-5, 1, \ldots, 1)$, σ and FI-theta parameter $t = \zeta - i\theta$, and superpotential $W = pG_5(x)$.

[Witten 93]

- Phases
  - $\zeta \gg 0$: $\{x \in \mathbb{P}^4 | G_5(x) = 0\}$ NLSM
  - $\zeta \ll 0$: $\{x \in \mathbb{C}^5/\mathbb{Z}_5, W^{LG} = \langle p \rangle G_5(x)\}$ Landau-Ginzburg orbifold
- Coulomb branch: $e^{-t} = -5^{-5}$
- Moduli space:



Landau-Ginzburg orbifolds

- Low-energy effective theory and state spaces
  - 2D $\mathcal{N} = (2, 2)$ Landau-Ginzburg orbifold
  - (Un-)Twisted sectors [Vafa 89][Intriligator-Vafa 90]
- Genus-0 correlators
  - FJRW theory [Fan-Jarvis-Ruan 07][Chiodo-Ruan 08][Chiodo-Iritiani-Ruan 12]
  - Computable via the Z_{D^2}/Z_{S^2} [Romo-Scheidegger-JK 20]
- B-branes
  - Matrix factorisations of W^{LG}

    [Kontsevich][Kapustin-Li 02][Brunner-Herbst-Lerche-Scheuner 03][Walcher 04][. . .]
  - D-brane transport and monodromies via the GLSM (algebraic not numeric!) [Herbst-Hori-Page 08]
  - Central charges and integral bases of (mirror) periods via Z_{D^2}

    [Romo-Scheidegger-JK 20]
- Classification [Kreuzer-Skarke 92]

Good Hybrids I

- Good Hybrid models
  - Landau-Ginzburg orbifolds fibered over a geometric base.

    [Witten 93][Bertolini-Melnikov-Plesser 13]
  - Generalises LG models and NLSMs.
- Examples
  - Small volume phases of 3 of the 14 hypergeometic Calabi-Yau threefolds ("K-points"), e.g. hybrid phase of $\mathbb{P}^5[3,3]$ is a $\mathbb{Z}_3$ Landau-Ginzburg orbifold fibered over $\mathbb{P}^1$
  - Phases of GLSMs associated to well-known CY hypersurfaces in toric spaces, e.g. $\mathbb{P}^4(11222)[8]$, $\mathbb{P}^4(11169)[18]$
  - Phases of non-abelian GLSMs

[Hori-JK 13,16][Guo-Romo 21][Lee-Lian-Romo 23][Lee-Lian-Romo-Santilli 25][...]

Good Hybrids II

- Low-energy effective theory and state spaces
  - 2D $\mathcal{N} = (2,2)$ hybrid model (NLSM with potential+"good")
  - State spaces have been constructed.

  [Bertolini-Melnikov-Plesser 13][Clader 13]
- Genus-0 correlators
  - Hybrid FJRW theory [Clader 13][...]
  - Computable via the Z_{D^2}/Z_{S^2} [Erkinger-JK 20,22]
- B-branes
  - Hybrid B-branes [Herbst-Lazaroiu 04][Lin-Pomerleano 11][Pryor-JK 24]
  - D-brane transport, monodromies, and D-brane charges via the GLSM [Pryor-JK 24]
- No classification yet.

Bad hybrids/Pseudo-hybrids

- Often, hybrid models are "not good".
  - Solutions of GLSM vacuum equations are not unique and lead to different possible symmetry breaking patterns.
  - R-charges of the IR CFT not well-defined.
  - Singular CFT with extra massless fields. [Aspinwall-Plesser 09]
  - Non-regular: GLSM gauge sector does not decouple.

    [Hori-Tong 06][Guo-Romo-Smith 25][McGovern-JK 25]
  - No known formulation of the low-energy effective theory.
  - Generic phase of a GLSM?
- Examples
  - Small volume phases 6 out of the 14 hypergeometric Calabi-Yau threefolds (3 "C-points", 3 "F-points"), e.g. $\mathbb{P}^6[2,2,3]$
- GLSM partition functions and mirror symmetry often give sensible results.
  - E.g. test of the refined swampland distance conjecture.

[Blumenhagen-Kläwer-Schlechter-Wolf 18][Erkinger-JK 19]

Further geometric phases

- Sometimes, Calabi-Yau moduli spaces/phases of GLSMs have more than one point of maximally unipotent monodromy.
- Many cases:
  - There is another smooth Calabi-Yau geometry.
  - The two Calabi-Yaus are topologically distinct but birational, i.e. related by flop transitions.
  - Toric settings: different maximal star triangulations
- However, this is not always the case. We can have two MUM points, but:
  - Calabi-Yaus may be non-birational.
  - Calabi-Yaus may be non-commutative resolutions of singular spaces.
  - Geometry may arise through non-perturbative effects.
  - Non-regular models. [McGovern-JK 25]

Rødland model I

- Consider a GLSM with $G = U(2)$ and field content

[Rødland 98][Hori-Tong 06]

	$p^{1,\dots,7}$	$x_{1,\dots,7}$	FI
$U(2)$	$\det^{-1}$	$\square$	ζ

- Superpotential

$$W = \sum_{ijk=1}^{1} \sum_{ab=1}^{2} A_k^{ij} p^k x_i^a \epsilon_{ab} x_j^b = \sum_{ij=1}^{7} A^{ij}(p)[x_i x_j]$$

- Vacuum equations

$$-|p|^2 \mathrm{id}_{2\times 2} + xx^\dagger = \zeta \mathrm{id}_{2\times 2}, \qquad dW = 0$$

Rødland model II

- $\zeta \gg 0$-phase
  - $p = 0$, x must have maximal rank, G broken completely

$$X = \{[x_i x_j] \in G(2,7) | A_k^{ij}[x_i x_j] = 0\}$$

- $\zeta \ll 0$-phase
  - Vacuum: $p \in \mathbb{P}^6$, $x = 0$, G broken to $SU(2)$ – strongly coupled
  - Fluctuations of x create a potential $W = A^{ij}(\langle p \rangle)[x_i x_j]$, giving mass to the x-fields.
  - Non-trivial IR physics where the rank of the mass matrix drops:

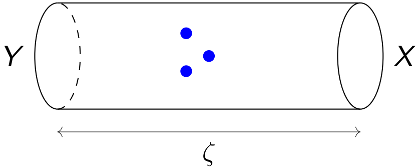
$$Y = \{p \in \mathbb{P}^6 | \mathrm{rk} A(p) = 4\}$$

- Coulomb branch:

$$e^{-t} = \frac{1}{(1+\omega^k)^7}, \quad \omega = e^{\frac{2\pi i}{7}}, \quad k = 1, 2, 3,$$

Rødland model III

- Moduli space



- Some facts:
  - X and Y are not birational.
  - X is a complete intersection in a non-toric variety.
  - Y is a non-complete intersection (determinantal variety).
  - Y is a strongly coupled phase due to the unbroken $SU(2)$.
  - Y is not a solution of the classical vacuum equations. Rather it is generated by non-perturbative effects.
  - B-branes and brane transport are interesting.

[Donovan-Segal 12][EHKR asymptoting completion]

Non-commutative resolutions

- Another source of MUM points are non-commutative resolutions Calabi-Yaus with nodal singularities.
  - Let $\hat{X}$ be a non-commutative (non-Kähler) resolution of the geometry with order N torsion: $\mathrm{Tors} H_2(\hat{X}, \mathbb{Z}) \simeq \mathbb{Z}_N$.
  - The singularities are resolved by turning on a discrete B-field.

[Vafa-Witten 94]

- This modifies the topological string partition function, leading to the definition of torsion-refined Gopakumar-Vafa invariants.

[Schimannek 21][Katz-Klemm-Schimannek-Sharpe 22]

- Many new examples of Calabi-Yaus.

[Katz-Schimannek 23][Lee-Lian-Romo 23][Schimannek 25][McGovern-JK 25]

- Interesting properties of D-brane categories expected.

NCR Example I

- Consider a GLSM with $G = U(1)$ and field content

[Căldăraru-Distler-Hellerman-Pantev-Sharpe 07]

	$p^{1,\dots,4}$	$x_{1,\dots,8}$	FI
$U(1)$	-2	1	ζ

- Superpotential

$$W = \sum_{i=1}^{4} p^i G_{2,i}(x)$$

- Vacuum equations

$$-2|p^2| + |x|^2 = \zeta, \qquad dW = 0$$

NCR example II

- $\zeta \gg 0$-phase
  - $\mathbb{P}^7[2,2,2,2]$

$$X = \{x_i \in \mathbb{P}^7 | G_{2,i}(x) = 0\}$$

  - This is one of the 14 hypergeometric models.
- $\zeta \ll 0$-phase
  - Interpretation 1
    - Good hybrid: $\mathbb{Z}_2$ LG orbifold fibered over $\mathbb{P}^3$
    - Hybrid invariants definable/computable. [Clader 13][Erkinger-JK 22]
  - Interpretation 2
    - Non-commutative resolution of a nodal octic.
    - $\mathbb{Z}_2$-refined GV invariants computable.

[Katz-Klemm-Schimannek-Sharpe 22]

- Duality between the two interpretations should lead to interesting properties of the associated D-brane categories.

[Guo-Romo 21][Lee-Lian-Romo 23][Lee-Lian-Romo-Santilli 25]

A new non-abelian model I

- We found a GLSM with $G = U(1)^3 \rtimes \mathbb{Z}_3$ that has two MUM points. [McGovern-JK 25]
- Chiral matter

ϕ	p^1, p^2, p^3	x_1, x_2, x_3	y_1, y_2, y_3	z_1, z_2, z_3	FI
$U(1)_1$	-1	1	0	0	ζ
$U(1)_2$	-1	0	1	0	ζ
$U(1)_3$	-1	0	0	1	ζ

$$\mathbb{Z}_3 : \qquad x_i \to y_i \to z_i \to x_i$$

- Superpotential

$$W = \sum_{ijkl=1}^{3} S_l^{ijk} p^l x_i y_j z_k = \sum_{ijk=1}^{3} S^{ijk}(p) x_i y_j z_k$$

A new non-abelian model II

- $\zeta \gg 0$

$$X_{\zeta\gg 0} = \{(x_i, y_i, z_i) \in (\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)/\mathbb{Z}_3 | S_l^{ijk} x_i y_j z_k = 0\}$$

  - This is a non-simply-connected Calabi-Yau threefold with

$$h^{11} = 1, \qquad H^3 = 30, \qquad c_2 \cdot H = 36, \qquad \chi = -30$$

[Candelas-Dale-Lutken-Schimmrigk 87][Candelas-Davies 08][Constantin-Gray-Lukas 16][...]

- $\zeta \ll 0$
  - The phase is non-regular, i.e. no clear separation between gauge and matter sector. [Hori-Tong 06]
  - Low-energy effective description unclear.
  - Conjecture (via mirror symmetry/topological strings): the "geometry" is a non-commutative resolution of a $\mathbb{P}^5_{111223}[4,6]$ with 63 nodes and a $\mathbb{Z}_3$-values B-field.

Summary

- Exotic regions in Calabi-Yau moduli spaces are ubiquitous.
- Some of them, like hybrids or non-perturbative geometric phases, can be understood in terms of low-energy effective theories.
- Many cannot (yet), but GLSM partition functions and topological strings/mirror symmtry still work.
- MUM points do not necessarily relate to smooth geometries.
- It pays off to consider Calabi-Yaus in non-toric spaces.
- GLSM and CFT perspectives indicate that there should be a universal framework to study any limiting regions of the moduli space.

Open questions

- How to characterise a generic limiting region in a Calabi-Yau moduli space?
  - Can a generic phase of a GLSM be described in terms of a "nice" low energy effective theory?

    [Hori-Tong 06][Hori 11][Hori-JK 13][McGovern-JK 25]
  - Does the IR CFT have a "nice" (free-field) realisation?
  - If not, how much can we learn about these regions and what are the appropriate mathematical and physical tools?
  - Do all swampland conjectures hold in exotic/singular regions of the moduli space?
  - Is there new physics/mathematics in distant regions of the moduli space?
- Interesting connections to mathematics:
  - Homological projective duality
  - Noncommutative algebraic geometry
  - Minimal model program