The Jucys-Murphy method and fusion procedure for the Sergeev superalgebra

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(joint with Alexander Molev and Vera Serganova)

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The *Sergeev superalgebra* $\mathcal{S}_n$ is the graded tensor product of two superalgebras

$$\mathcal{S}_n = \mathbb{C}\mathfrak{S}_n^- \otimes \mathcal{C}l_n,$$

where $\mathbb{C}\mathfrak{S}_n^-$ is the *spin symmetric group algebra* generated by odd elements $t_1, \ldots, t_{n-1}$ subject to the relations

$$t_a^2 = 1, \qquad t_a t_{a+1} t_a = t_{a+1} t_a t_{a+1}, \qquad t_a t_b = -t_b t_a, \qquad |a-b| > 1,$$

while $\mathcal{C}l_n$ is the *Clifford super algebra* generated by odd elements $c_1, \ldots, c_n$ subject to the relations

$$c_a^2 = -1, \qquad c_a c_b = -c_b c_a, \qquad a \neq b.$$

The group algebra $\mathbb{C}\mathfrak{S}_n$ of the symmetric group $\mathfrak{S}_n$ is embedded in $\mathcal{S}_n$ so that the adjacent transpositions $s_a = (a, a+1) \in \mathfrak{S}_n$ are identified with the elements of $\mathcal{S}_n$ by

$$s_a = \frac{1}{\sqrt{2}}\, t_a(c_{a+1} - c_a).$$

This leads to the alternative presentation of $\mathcal{S}_n$ as the semidirect product $\mathbb{C}\mathfrak{S}_n \ltimes \mathcal{C}l_n$ with the relations between elements of the symmetric group

$$s_a c_a = c_{a+1} s_a, \qquad s_a c_{a+1} = c_a s_a, \qquad s_a c_b = c_b s_a, \qquad b \neq a, a+1.$$

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The Sergeev superalgebra was introduced in [Sergeev, 1985]: $\mathcal{S}_n$ plays the role of the symmetric group for a superversion of Schur-Weyl duality (known as *Sergeev duality*). If $V = \mathbb{C}^{k|k}$ is the standard representation of the Lie superalgebra $\mathfrak{q}(k)$, then both $\mathcal{S}_n$ and $\mathfrak{q}(k)$ act on the tensor product $V^{\otimes n}$ and each algebra is the commutant algebra of the other.

The algebra $\mathcal{S}_n$ admits an affinization, $\mathcal{H}^{aff}_{\mathcal{C}l}(n)$, called the *degenerate affine Hecke-Clifford algebra (DAHCA)*: $\mathcal{S}_n$ is a subalgebra of $\mathcal{H}^{aff}_{\mathcal{C}l}(n)$, there also exists a natural surjection $\mathcal{H}^{aff}_{\mathcal{C}l}(n) \to \mathcal{S}_n$, thus the representation theory of $\mathcal{H}^{aff}_{\mathcal{C}l}(n)$ contains that of the Sergeev superalgebra.

The representations of $\mathcal{S}_n$ were studied in the foundational work of [Nazarov, 1997] along with those of DAHCA. $\mathcal{S}_n$ is known to be semisimple and its simple modules are parameterized by strict partitions of n.

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In the same paper Nazarov introduced the *Jucys–Murphy elements* $x_1, \ldots, x_n$ of $\mathcal{S}_n$, they generate a commutative subalgebra of $\mathcal{S}_n$ and act semisimply on each simple module. We use them

1. to construct a complete set of primitive idempotents for $\mathcal{S}_n$;
2. to get the seminormal basis of $\mathcal{S}_n$ and explicit realizations of all its simple modules;
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Follow [Okounkov and Vershik, 1996].

The *Jucys–Murphy elements* of $\mathfrak{S}_n$ are given by

$$x_1 = 0, \qquad x_a = (1, a) + \cdots + (a-1, a), \qquad a = 2, \ldots, n.$$

Simple $\mathfrak{S}_n$-modules are parameterized by partitions $\lambda \vdash n$.
Identify λ with its *Young diagram*; e.g. $\lambda = (6, 3, 1)$ is drawn as

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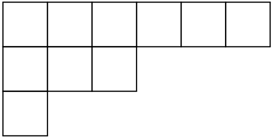

The entries of a *standard* λ*-tableau* increase from left to right in each row and from top to bottom in each column.

For instance,

$$\mathcal{T} \quad = \quad \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 4 & 5 & 8 & 10 \\ \hline 3 & 6 & 9 \\ \cline{1-3} 7 \\ \cline{1-1} \end{array}$$

is a standard λ-tableau for $\lambda = (6, 3, 1)$.

The *content* $\sigma_a(\mathcal{T})$ equals $j - i$ if a occupies the box (i, j) in $\mathcal{T}$.

In the example, $\sigma_6(\mathcal{T}) = 0$, $\sigma_7(\mathcal{T}) = -2$ and $\sigma_8(\mathcal{T}) = 4$.

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For $\lambda \vdash n$ the module V^λ over $\mathfrak{S}_n$ is afforded by the vector space

$$V^\lambda = \bigoplus_{\mathrm{sh}(\mathcal{T})=\lambda} \mathbb{C}\, v_{\mathcal{T}}$$

with the basis vectors $v_{\mathcal{T}}$ labelled by the standard λ-tableaux $\mathcal{T}$.

The generators $s_a = (a, a+1)$ and the JM elements x_a act by

$$s_a\, v_{\mathcal{T}} = \frac{1}{\sigma_{a+1}(\mathcal{T}) - \sigma_a(\mathcal{T})}\, v_{\mathcal{T}} + \mathcal{X}_a(\mathcal{T})\, v_{s_a\mathcal{T}},$$

$$x_a\, v_{\mathcal{T}} = \sigma_a(\mathcal{T})\, v_{\mathcal{T}},$$

where $v_{s_a\mathcal{T}} := 0$ if the tableau $s_a\mathcal{T}$ is not standard, and

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Primitive idempotents

If $\mathcal{U}$ is a standard λ-tableau, let $\mathcal{V}$ be obtained from $\mathcal{U}$ by deleting the box α occupied by n. Let c be the content of α.

Define by induction,

$$e_{\mathcal{U}} = e_{\mathcal{V}} \frac{(x_n - a_1) \dots (x_n - a_l)}{(c - a_1) \dots (c - a_l)},$$

where $a_1, \dots, a_l$ are the contents of all addable boxes of $\mathrm{sh}(\mathcal{V})$ except for α; and $e_{\boxed{1}} = 1$ for the one-box tableau.

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Theorem (Jucys 1971, Murphy 1981)

The elements $e_{\mathcal{U}}$ are primitive idempotents in $\mathbb{C}\mathfrak{S}_n$. They are pairwise orthogonal and form a decomposition of the identity:

$$e_{\mathcal{U}}\, e_{\mathcal{V}} = \delta_{\mathcal{U}\mathcal{V}}\, e_{\mathcal{V}}, \qquad 1 = \sum_{\lambda \vdash n} \sum_{\mathrm{sh}(\mathcal{U})=\lambda} e_{\mathcal{U}}.$$

Moreover,

$$x_a\, e_{\mathcal{U}} = e_{\mathcal{U}}\, x_a = \sigma_a(\mathcal{U})\, e_{\mathcal{U}}.$$

Proof. Observe that the $\mathbb{C}\mathfrak{S}_n$-module $V = \bigoplus_{\lambda \vdash n} V^{\lambda}$ is faithful. All relations follow from the action: $e_{\mathcal{U}}\, v_{\mathcal{T}} = \delta_{\mathcal{U}\mathcal{T}}\, v_{\mathcal{T}}$.

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Intertwiners

Follow [Cherednik, 1986] to introduce the *intertwining elements*

$$\phi_a = s_a(x_a - x_{a+1}) + 1.$$

They satisfy the braid relations

$$\phi_a\phi_{a+1}\phi_a = \phi_{a+1}\phi_a\phi_{a+1}, \qquad \phi_a\phi_b = \phi_b\phi_a, \qquad |a-b| > 1,$$

and have the property

$$\phi_a^2 = 1 - (x_a - x_{a+1})^2.$$

For each $w \in \mathfrak{S}_n$ there are well-defined elements

$$\phi_w = \phi_{a_1} \dots \phi_{a_r} \qquad \text{and} \qquad \phi_w^* = \phi_{w^{-1}} = \phi_{a_r} \dots \phi_{a_1},$$

where $w = s_{a_1} \dots s_{a_r}$ is a reduced decomposition.

The intertwiners act in V^λ by

$$\phi_a v_{\mathcal{T}} = \Big(\sigma_a(\mathcal{T}) - \sigma_{a+1}(\mathcal{T})\Big) \mathcal{X}_a(\mathcal{T})\, v_{s_a \mathcal{T}}.$$

Hence,

$$\phi_{d(\mathcal{T})} v_{\mathcal{R}^\lambda} = b_{\mathcal{T}} v_{\mathcal{T}}, \qquad b_{\mathcal{T}} \neq 0,$$

where $\mathcal{R}^\lambda$ is the *row-tableau* of shape λ and $\mathcal{T} = d(\mathcal{T})\mathcal{R}^\lambda$.

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Murphy's basis

For standard λ-tableaux $\mathcal{T}$ and $\mathcal{U}$ set

$$\zeta_{\mathcal{T}\mathcal{U}} = \phi_{d(\mathcal{T})}\, e_{\mathcal{R}^\lambda}\, \phi^*_{d(\mathcal{U})}.$$

Using the properties

$$\phi_a\, e_{\mathcal{U}} = e_{s_a\mathcal{U}}\, \phi_a,$$

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Theorem (Murphy 1992)

As λ runs over the Young diagrams with n boxes, the elements $\zeta_{\mathcal{T}\mathcal{U}}$ associated with standard λ-tableaux $\mathcal{T}$ and $\mathcal{U}$ form a basis of the group algebra $\mathbb{C}\mathfrak{S}_n$.

Corollary. We have the direct sum decomposition

$$\mathbb{C}\mathfrak{S}_n = \bigoplus_{\lambda \vdash n} \bigoplus_{\mathrm{sh}(\mathcal{U})=\lambda} \mathbb{C}\mathfrak{S}_n e_{\mathcal{U}},$$

where every left ideal $\mathbb{C}\mathfrak{S}_n e_{\mathcal{U}}$ is a simple $\mathfrak{S}_n$-module associated with the partition $\lambda = \mathrm{sh}(\mathcal{U})$.

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Corollary. Fix a standard λ-tableau $\mathcal{U}$.
The mapping

$$\zeta_{\mathcal{T}\mathcal{U}} \mapsto b_{\mathcal{T}} v_{\mathcal{T}}, \qquad \mathrm{sh}(\mathcal{T}) = \lambda,$$

defines an $\mathfrak{S}_n$-module isomorphism

$$\mathbb{C}\mathfrak{S}_n e_{\mathcal{U}} \to V^{\lambda}.$$

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Fusion procedure

Take n variables $u_1, \ldots, u_n$ and consider the rational function with values in $\mathbb{C}\mathfrak{S}_n$ defined by

$$\Phi(u_1, \ldots, u_n) = \prod_{1 \leqslant a < b \leqslant n} \Big(1 - \frac{(a, b)}{u_a - u_b}\Big),$$

where the product is taken in the lexicographical order on the set of pairs (a, b).

Given a standard λ-tableau $\mathcal{U}$, set $\sigma_a = \sigma_a(\mathcal{U})$ for $a = 1, \ldots, n$.

Theorem (Jucys 1966; Molev 2008)

The consecutive evaluations are well-defined and we have

$$\Phi(u_1, \ldots, u_n)\big|_{u_1=\sigma_1}\big|_{u_2=\sigma_2} \cdots \big|_{u_n=\sigma_n} = \frac{n!}{f_\lambda}\, e_{\mathcal{U}}.$$

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Introduce analogs of the transpositions in the Sergeev superalgebra

$$t_{ab} = (-1)^{b-a-1}\, t_{b-1} \dots t_{a+1} t_a t_{a+1} \dots t_{b-1}, \quad a < b, \quad t_{ba} = -t_{ab}.$$

The even Jucys–Murphy elements are

$$x_1 = 0, \quad x_a = \sqrt{2}\,(t_{1a} + \dots + t_{a-1,a})\, c_a, \quad a = 2, \dots, n.$$

Note that the x_a pairwise commute.

The irreducible representations of $\mathcal{S}_n$ and $\mathbb{C}\mathfrak{S}_n^-$ are parameterized by strict partitions $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of n ($\lambda \Vdash n$) with $\lambda_1 > \dots > \lambda_\ell > 0$ and $\lambda_1 + \dots + \lambda_\ell = n$; $\ell(\lambda) = \ell$ the length of λ. An example of a (*shifted*) λ-*tableau*

1	2	4	5	8	10
	3	6	9		
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is a standard λ-tableau for $\lambda = (6, 3, 1)$ with the diagonal entries $1, 3$ and 7.

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Simple modules in $\mathcal{S}_n$

The respective dimensions of the simple $\mathcal{S}_n$-module U^λ and the simple $\mathbb{C}\mathfrak{S}_n^-$-module V^λ are given by

$$\dim U^\lambda = 2^{n-\lfloor\frac{\ell(\lambda)}{2}\rfloor} g_\lambda \qquad \text{and} \qquad \dim V^\lambda = 2^{\lceil\frac{n-\ell(\lambda)}{2}\rceil} g_\lambda,$$

where g_λ is the number of standard λ-tableaux, which is found by the *Schur formula* [1911]

$$g_\lambda = \frac{n!}{\lambda_1!\ldots\lambda_\ell!} \prod_{1\leqslant i<j\leqslant\ell} \frac{\lambda_i-\lambda_j}{\lambda_i+\lambda_j}.$$

The dimension count in the decomposition yields the Schur identity analogous to $\mathbb{C}\mathfrak{S}_n$ case:

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Simple modules in $\mathcal{S}_n$

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We need to extend the set of tableaux by allowing any non-diagonal entry to occur with a bar on it [Sagan 1987]:

1	2	$\bar{4}$	$\bar{5}$	8	$\overline{10}$
	3	$\bar{6}$	9		
		7			

.

Given a strict partition $\lambda \Vdash n$, the number of the corresponding standard barred tableaux equals $2^{n-\ell(\lambda)} g_\lambda$.

Given a standard barred tableaux $\mathcal{U}$, introduce the *signed content* $\kappa_{\mathrm{a}}(\mathcal{U})$ of any barred or unbarred entry a of $\mathcal{U}$ by the formula

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Idempotents in $\mathcal{S}_n$

For any standard barred tableau $\mathcal{U}$ set $e_{\boxed{1}} = 1$ for the one-box tableau, and

$$e_{\mathcal{U}} = e_{\mathcal{V}} \frac{(x_n - b_1)\dots(x_n - b_p)}{(\kappa - b_1)\dots(\kappa - b_p)},$$

where $\mathcal{V}$ is the barred tableau obtained from $\mathcal{U}$ by removing the box α occupied by n (resp. $\bar{n}$). The shape of $\mathcal{V}$ denote by μ, then $b_1, \dots, b_p$ are the signed contents in all addable boxes of μ (barred and unbarred), except for the entry n (resp. $\bar{n}$), while κ is the signed content of the entry n (resp. $\bar{n}$).

Theorem

$$e_{\mathcal{U}}\, e_{\mathcal{V}} = \delta_{\mathcal{U}\mathcal{V}}\, e_{\mathcal{V}}, \qquad 1 = \sum_{\lambda \Vdash n} \sum_{\mathrm{sh}(\mathcal{U})=\lambda} e_{\mathcal{U}}.$$

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$$\widehat{U} = \bigoplus_{\lambda \Vdash n} \widehat{U}^\lambda \quad \widehat{U}^\lambda = \bigoplus_{\mathrm{sh}(\mathcal{T})=\lambda} \mathcal{C}l_n \, v_{\mathcal{T}}$$

with the basis vectors $v_{\mathcal{T}}$ labelled by standard λ-tableaux $\mathcal{T}$.

The generators of $\mathcal{S}_n$ act by

$$s_a \, v_{\mathcal{T}} = \left(\frac{1}{\kappa_{a+1}(\mathcal{T}) - \kappa_a(\mathcal{T})} + \frac{c_a c_{a+1}}{\kappa_{a+1}(\mathcal{T}) + \kappa_a(\mathcal{T})} \right) v_{\mathcal{T}} + \mathcal{Y}_a(\mathcal{T}) \, v_{s_a \mathcal{T}},$$

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Example

For $n = 3$ we have six barred tableaux $\mathcal{U}$:

1	2	3

1	2	$\bar{3}$

1	$\bar{2}$	3

1	$\bar{2}$	$\bar{3}$

1	2
	3

1	$\bar{2}$
	3

The respective elements $e_{\mathcal{U}} \in \mathcal{S}_3$ are

$$\frac{\sqrt{2}+x_2}{2\sqrt{2}} \cdot \frac{x_3(x_3+\sqrt{6})}{12}, \quad \frac{\sqrt{2}+x_2}{2\sqrt{2}} \cdot \frac{x_3(x_3-\sqrt{6})}{12}, \quad \frac{\sqrt{2}-x_2}{2\sqrt{2}} \cdot \frac{x_3(x_3+\sqrt{6})}{12},$$

$$\frac{\sqrt{2}-x_2}{2\sqrt{2}} \cdot \frac{x_3(x_3-\sqrt{6})}{12}, \quad \frac{\sqrt{2}+x_2}{2\sqrt{2}} \cdot \frac{6-x_3^2}{6}, \quad \frac{\sqrt{2}-x_2}{2\sqrt{2}} \cdot \frac{6-x_3^2}{6}.$$

Note that $x_2^2 = 2$ and $x_3^3 = 6x_3$ so that all these elements are idempotents in $\mathcal{S}_3$.

There are two orthogonal primitive idempotents in $\mathcal{C}l_2$:

$$\frac{1 + i c_1 c_2}{2} \qquad \text{and} \qquad \frac{1 - i c_1 c_2}{2}.$$

Since

$$\mathcal{C}l_{2m} \cong (\mathcal{C}l_2)^{\otimes m} \qquad \text{and} \qquad \mathcal{C}l_{2m+1} \cong (\mathcal{C}l_2)^{\otimes m} \otimes \mathcal{C}l_1,$$

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Let $\mathcal{U}$ be a standard barred λ-tableau with diagonal entries $d_1 < \cdots < d_\ell$, then $e_{\mathcal{U}}$ commutes with $c_{d_1}, \ldots, c_{d_\ell}$.

Consider the subalgebra $\mathcal{C}l_\ell^{\mathcal{U}}$ of $\mathcal{C}l_n$ generated by $c_{d_1}, \ldots, c_{d_\ell}$.

Let $\delta = (\delta_1, \ldots, \delta_m)$ be an m-tuple with $\delta_a \in \{1, -1\}$, where m is defined by $\ell = 2m$ or $\ell = 2m + 1$.

$$\mathcal{E}_\delta^{\mathcal{U}} = \prod_{a=1}^{m} \frac{1 + i\delta_a c_{d_{2a-1}} c_{d_{2a}}}{2} \in \mathcal{C}l_\ell^{\mathcal{U}} \tag{1}$$

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Given δ, denote by U^λ_δ the $\mathcal{S}_n$-submodule of $\widehat{U}^\lambda$, generated by the vectors $\mathcal{E}^{\mathcal{U}}_\delta v_{\mathcal{U}}$, where $\mathcal{U}$ runs over all standard barred tableaux of shape λ.

Theorem

We have the direct sum decomposition of $\mathcal{S}_n$-modules

$$\widehat{U}^\lambda = \bigoplus_\delta U^\lambda_\delta,$$

summed over all m-tuples δ. Moreover, each $\mathcal{S}_n$-module U^λ_δ is irreducible and isomorphic to U^λ.

Given δ, denote by U_δ^λ the $\mathcal{S}_n$-submodule of $\widehat{U}^\lambda$, generated by the vectors $\mathcal{E}_\delta^{\mathcal{U}} v_{\mathcal{U}}$, where $\mathcal{U}$ runs over all standard barred tableaux of shape λ.

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$$\widehat{U}^\lambda = \bigoplus_\delta U_\delta^\lambda,$$

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Proof. We verify that the submodule U_δ^λ is spanned over $\mathbb{C}$ by the vectors of the form

$$c_{d_{j_1}} \dots c_{d_{j_s}} \mathcal{E}_\delta^{\mathcal{U}} v_{\mathcal{U}}, \qquad s \geqslant 0, \tag{2}$$

where $1 \leqslant j_1 < \dots < j_s \leqslant \ell(\lambda)$, all j_a are odd, and $\mathcal{U}$ runs over the standard barred tableaux $\mathcal{U}$ of shape λ.

Corollary. Let $\delta = (\delta_1, \dots, \delta_m)$ be an m-tuple with $\delta_a \in \{1, -1\}$. The vectors (2) form a $\mathbb{C}$-basis of the simple submodule U_δ^λ of $\widehat{U}^\lambda$.

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Nazarov's intertwiners

$$\phi_a = s_a(x_a^2 - x_{a+1}^2) + x_a + x_{a+1} - c_a c_{a+1}(x_a - x_{a+1}).$$

They satisfy the braid relations

$$\phi_a\phi_{a+1}\phi_a = \phi_{a+1}\phi_a\phi_{a+1}, \qquad \phi_a\phi_b = \phi_b\phi_a, \qquad |a-b| > 1,$$

and have the properties

$$\phi_a^2 = 2(x_a^2 + x_{a+1}^2) - (x_a^2 - x_{a+1}^2)^2$$

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Murphy-type basis in $\mathcal{S}_n$

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Let $\mathcal{R}^\lambda$ denote the *row-tableau* of shape λ.

Let β be a set of non-diagonal boxes of λ. Denote by $\mathcal{R}^{\lambda,\beta}$ the tableau obtained from $\mathcal{R}^\lambda$ by adding the bars to all entries occupying the boxes of β.

For any two standard unbarred tableaux $\mathcal{T}$ and $\mathcal{U}$ of shape λ set

$$\zeta^{\beta}_{\mathcal{T}\mathcal{U}} = \phi_{d(\mathcal{T})}\, e_{\mathcal{R}^{\lambda,\beta}}\, \phi^*_{d(\mathcal{U})},$$

where $d(\mathcal{T}) \in \mathfrak{S}_n$ is such that $\mathcal{T} = d(\mathcal{T})\mathcal{R}^\lambda$.

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Theorem

As λ runs over shifted Young diagrams with n boxes, the elements $\zeta^{\beta}_{\mathcal{T}\mathcal{U}}$ associated with standard unbarred tableaux $\mathcal{T}$ and $\mathcal{U}$ of shape λ and sets β of non-diagonal boxes of λ form a basis of the Sergeev superalgebra $\mathcal{S}_n$ over $\mathcal{C}l_n$.

Proof. The number of elements is

$$\sum_{\lambda \Vdash n} 2^{n-\ell(\lambda)}\, g_\lambda^2 = n!,$$

and so coincides with the rank of $\mathcal{S}_n$ as a $\mathcal{C}l_n$-module. It is enough to show that the elements are linearly independent over $\mathcal{C}l_n$ by acting in $\widehat{U}$.

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Seminormal form for $\mathcal{S}_n$

Recall the $\mathcal{S}_n$-module

$$\widehat{U}^\lambda = \bigoplus_{\mathrm{sh}(\mathcal{T})=\lambda} \mathcal{C}l_n v_\mathcal{T}$$

with the basis vectors $v_\mathcal{T}$ labelled by standard λ-tableaux $\mathcal{T}$.

Given $\mathcal{T}$, let $d_1 < \cdots < d_\ell$ be the diagonal entries.

Theorem

The simple module U^λ over $\mathcal{S}_n$ is the quotient of $\widehat{U}^\lambda$ by the relations

$$c_{d_{2a}} v_\mathcal{T} = i\, c_{d_{2a-1}} v_\mathcal{T}, \qquad a = 1, \ldots, \left\lfloor \frac{\ell(\lambda)}{2} \right\rfloor.$$

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Fusion procedure

Take n variables $u_1, \dots, u_n$ and consider the rational function with values in $\mathbb{C}\mathfrak{S}_n$ defined by [Nazarov 1997],

$$\Phi(u_1, \dots, u_n) = \prod_{1 \leqslant a < b \leqslant n} \left(1 - \frac{(a,b)}{u_a - u_b} + \frac{(a,b)\, c_a c_b}{u_a + v_b}\right),$$

where the product is taken in the lexicographical order on the set of pairs (a, b).

Let $\mathcal{U}$ be a standard barred λ-tableau. For every $a = 1, \dots, n$ set $\kappa_a = \kappa_{\mathrm{a}}(\mathcal{U})$ if $\mathrm{a} = a$ or $\mathrm{a} = \bar{a}$ is the entry of $\mathcal{U}$.

Fusion procedure

Take n variables $u_1, \dots, u_n$ and consider the rational function with values in $\mathbb{C}\mathfrak{S}_n$ defined by [Nazarov 1997],

$$\Phi(u_1, \dots, u_n) = \prod_{1 \leqslant a < b \leqslant n} \left(1 - \frac{(a,b)}{u_a - u_b} + \frac{(a,b)\, c_a c_b}{u_a + v_b}\right),$$

where the product is taken in the lexicographical order on the set of pairs (a, b).

Let $\mathcal{U}$ be a standard barred λ-tableau. For every $a = 1, \dots, n$ set $\kappa_a = \kappa_{\mathrm{a}}(\mathcal{U})$ if $\mathrm{a} = a$ or $\mathrm{a} = \bar{a}$ is the entry of $\mathcal{U}$.

Theorem

The consecutive evaluations are well-defined and we have

$$\Phi(u_1, \ldots, u_n)\big|_{u_1=\kappa_1}\big|_{u_2=\kappa_2} \ldots \big|_{u_n=\kappa_n} = \frac{n!}{g_\lambda}\, e_{\mathcal{U}}.$$

Proof. This is essentially equivalent to the JM formula for $e_{\mathcal{U}}$.

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