How Killing fields yield to tractors - a tale of algebra exposed

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Work with Simon Goodwin, Thomas Leistner and Jonathan Kress,

Links: e.g.

G. + Leistner, . . *prolongation* . . *Killing* . , Ann. Mat. Pura Appl. (2019)

G. + Macbeth, *Detecting Einstein geodesics* . . Diff. Geom. Appl (2014)

G. *Laplacian operators.* . ., Math. Ann. 2006

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Symmetries and hidden symmetries

Recall that on a (psuedo-) Riemannian manifold (M, g) a vector field $k \in \mathfrak{X}$ is an *infintesimal symmetry* or **Killing vector** field if

$$\mathcal{L}_k g = 0.$$

If we fix g, it is an overdetermined linear PDE on $k_a := g_{ab} k^b$:

$$\nabla_a k_b + \nabla_b k_a = 0.$$

A higher rank $k \in \Gamma(\odot^\ell T^* M)$ is a **Killing tensor** (or "hidden symmetry") if

$$symmetric - part(\nabla k) = 0 \quad \text{i.e.} \quad \nabla_{(a_1} k_{a_2 \cdots a_{\ell+1})} = 0$$

Both provide **first integrals** for **geodesics** i.e. curves $\gamma : \mathbb{R} \to M$

$$\nabla_u u = 0, \quad \text{where} \quad u := \dot{\gamma}, \ \nabla = \text{Levi-Civita}(g),$$

as

$$k(u, \cdots, u) = k_{a_2 \cdots a_{\ell+1}} u^{a_2} \cdots u^{a_{\ell+1}}$$

is **necessarily constant** along any geodesic: $u \cdot (k(u, \cdots, u)) = 0$.

Killing tensors i.e. $k \in \Gamma(\odot^\ell T^*M)$ such that

$$\nabla_{(a_1} k_{a_2 \cdots a_{\ell+1})} = 0$$

have many other related applications: If a linear differential operator $L : C^\infty \to C^\infty$ is a **symmetry of** the Laplacian Δ^g, i.e. $[L, \Delta^d] = 0$ (so preserves its spectrum) then its leading symbol is a Killing tensor $L = k^{a_1 \cdots a_\ell} \nabla_{a_1} \cdots \nabla_{a_\ell} + lots$.

Families of Killing tensors are used for the construction of **separation of variables** coordinates (for the Laplacian and other natural operators) and in **superintegrable systems** – where there are enough (compatible) Killing tensors so that the Hamilton-Jacobi equations can be solved algebraically.

Classical mechanics formulation

For a Riemannian manifold (including Euclidean space $\mathbb{E}^n$) the geodesics – the trajectories – are determined by a Hamiltonian of the form

$$\mathcal{H} = g_q(p, p), \qquad q \in M, \ p \in T^*M$$

The resulting Hamilton equations are just the geodesic equations. Hard to treat directly so an idea is seek first integrals i.e. functions k on phase space s.t.

$$\{\mathcal{H}, k\} = 0, \quad \text{where } \{\cdot, \cdot\} \text{ is Poisson bracket on } T^*M$$

If k takes the form $k = k_q(p, \cdots, p)$, monomial in p at each point q then k_q is a Killing tensor.

E.g. If we have $2n - 1$ such k that are functionally independent – superintegrable – then the unparametrised paths are determined algebraically (and, in suitable settings, bounded orbits are closed).

Potentials

If $\mathcal{H} = g_q(p, p)$ is easy (e.g. g is Euclidean or hyperbolic metric) then one can ramp things up by adding a *potential*

$$\mathcal{H}(p, q) = g_q(p, p) + V(q). \qquad (1)$$

E.g. The Kepler system where g is the Euclidean metric and $V = k/r$, where r radial distance. [(1) quantises to a Schrödinger operator $\Delta^g + V$ on functions, Kepler $\rightsquigarrow$ Hydrogen atom]

Problem: First integrals for (1) ? It's no longer in the form $g_q(p, p)$ so trajectories are not just geodesics. One can attack this directly. Or:

Enter conformal geometry: In its simplest form, the **Stäckel transform** of the Hamiltonian (1) is

$$\tilde{\mathcal{H}} = \tilde{g}_q(p, p), \qquad \tilde{g} = V \cdot g \quad \text{conformally related to } g$$

Constants of motion for $\mathcal{H}$ can be mapped to constants of motion for $\tilde{\mathcal{H}}$, in particular, this mapping preserves independence and therefore superintegrability.

Conformal Killing tensors

In terms of Killing tensors finding constants of motion for $\mathcal{H} = g_q(p,p) + V(q)$ corresponds exactly to the question of finding Killing tensors for g that are also, after a possible trace adjustment, also Killing tensors for the conformally related $\tilde{g}$.

A weaker condition that is conformally invariant is the **conformal Killing equation CKE:**

$$\nabla_{(a_1} k_{a_2 \cdots a_{\ell+1})_0} = 0 \quad \text{i.e.} \quad \textbf{\textit{trace}} - \textbf{\textit{free}} - symmetric - part(\nabla k) = 0,$$

call the (trace-free symmetric) solution $k_{a_1 \cdots a_\ell}$ a conf. Killing tensor.

So the problem of first integrals for $\mathcal{H}$ boils down to finding solutions to the CKE and then whether it is possible to add the trace parts so that it becomes Killing in the different conformally related metrics g and $\tilde{g}$.

Hence some questions:

Q1: When is a conformal Killing tensor on (M, g) actually Killing on the conformally related $(M, \widehat{g} = e^{2\omega}g)$?

For rank 1, i.e. conformal (Killing) vector (fields), this is a classical question and linked to interesting global dynamics questions. A conformal vector field is *essential* if $\nexists \, \omega \in C^\infty$.

For rank $\ell \geq 2$ there is evidently the interesting and natural variant:
Q2: When is a conformal Killing tensor on (M, g) the trace-free part of a Killing tensor on the conformally related $(M, \widehat{g} = e^{2\omega}g)$?

Our first focus is Q2 with $\ell = 2$.

Many related questions that boil down to Q2: E.g. our second focus:
Q3: If $k_1, \cdots, k_s$ is a family of Killing tensors on (M, g) – **when can they each be trace adjusted** so as to be **simultaneously Killing** for the conformally related $(M, \widehat{g} = e^{2\omega}g)$?

Prolongation and tractors

The conf Killing equation $\nabla_{(a}k_{bc)_0} = 0$ is an overdetermined finite type linear PDE. So we should study it by prolongation. Let's do an easier "fundamental eqn" instead:

The metric $\bar{g} := u^{-2}g$ is **Einstein** i.e. $\text{Ric}^{\bar{g}} = \lambda\bar{g}$ iff $\exists\ u > 0$ s.t.

$$\text{trace-free}(\nabla_a\nabla_b u + P_{ab}u) = 0. \qquad \text{(AE) -- almost Einstein eqn}$$

where $\text{Ric} = (n-2)P + gJ$, $J := trace^g(P)$. We drop $u > 0$ and study. Best to replace u by conformal density of weight 1 $\sigma \in \Gamma(\mathcal{E}[1])$, where $\mathcal{E}[2n] \cong (\Lambda^n(TM))^2$. Then (AE) is conformally invariant. AE is overdetermined so study by prolongation. It is equivalent to the closed first order system:

$$\nabla_a\sigma - \mu_a = 0$$
$$\nabla_a\mu_b + P_{ab}\sigma + \rho\boldsymbol{g}_{ab} = 0$$
$$\nabla_a\rho - P_{ab}\mu^b = 0$$

$\boldsymbol{g}$ is the conformal metric, $\otimes^n\boldsymbol{g} : (\Lambda^n(TM))^2 \xrightarrow{\cong} \mathcal{E}[2n]$.

The tractor connection and D-operator

The above system can be collected into a linear connection – in fact the conformally invariant **tractor bundle** $\mathcal{T} = \mathcal{E}[1] \oplus T^*M[1] \oplus \mathcal{E}[-1]$ with invariant $X : \mathcal{E}[-1] \to \mathcal{T}$, and **connection** $\nabla^{\mathcal{T}}$. Given $\overline{g} \in \boldsymbol{c}$ this is given by

$$\mathcal{T} \overset{\overline{g}}{=} \mathcal{E}[1] \oplus T^*M[1] \oplus \mathcal{E}[-1], \qquad \mathcal{E}[1] := (\Lambda^n TM)^{\frac{2}{2n}}$$

$$\nabla^{\mathcal{T}}_a(\sigma, \mu_b, \rho) = (\nabla_a \sigma - \mu_a, \ \nabla \mu_b + P_{ab}\sigma + \boldsymbol{g}_{ab}\rho, \ \nabla_a \rho - P_{ab}\mu^b),$$

and $\nabla^{\mathcal{T}}$ preserves a conformally invariant **tractor metric** h

$$\mathcal{T} \ni V = (\sigma, \mu_b, \rho) \mapsto 2\sigma\rho + \mu_b\mu^b = h(V, V).$$

There is also a 2nd order conformally invariant **Thomas operator** :

$$\Gamma(\mathcal{E}[w]) \ni f \mapsto D_A f \overset{\overline{g}}{=} \begin{pmatrix} wf \\ \nabla_a f \\ -(n + 2w - 2)^{-1}(\Delta f + w\mathsf{J}f) \end{pmatrix}$$

where J is $\text{trace}^{\overline{g}}(P_{ab})$, so a number times $\text{Sc}(\overline{g})$.

The scale tractor

If $I_A \overset{g}{=} (\rho, \mu_a, \sigma)$ is a parallel tractor then $\mu_a = \nabla_a \sigma$, and $\rho = -\frac{1}{n+1}(\Delta \sigma + J\sigma)$. This gives the first statement of:

Proposition

I parallel implies $I_A = D_A \sigma$. So $I \neq 0 \Rightarrow \sigma$ is nonvanishing on an open dense set $M_{\sigma \neq 0}$. On $M_{\sigma \neq 0}$, $\bar{g} = \sigma^{-2} \mathbf{g}$ is Einstein. Conversely if $\bar{g} = \sigma^{-2} \mathbf{g}$ is Einstein then $I := D\sigma$ is parallel.

Drop parallel. The **scale tractor** $I := D\sigma$:

Now if $k^a \in \Gamma(TM)$ then its half-prolongation

$$K_A := \begin{pmatrix} 0 \\ k^a \\ -\frac{1}{n} \nabla_a k^a \end{pmatrix} \quad \in \Gamma(\mathcal{T})[1]$$

is an invariant tractor and $\boxed{I_A K^A = 0 \text{ iff } \nabla \cdot k = 0.}$ As, in the scale $g = \sigma^{-2} \mathbf{g}$, $I_A = (\rho \ 0 \ \sigma)$.

Full prolongation of conformal K vectors

Theorem (G. Math. Ann 2006)

$k \in \Gamma(TM)$ is a conformal Killing vector iff

$$D_B K_C = \mathbb{K}_{BC} := D_{[A}K_{B]} \qquad \textcolor{red}{half}$$

or equivalently

$$\nabla_a \mathbb{K}_{BC} = Curvature \cdot \mathbb{K}_{BC} \qquad \textcolor{red}{full}$$

Proof.

First part: Calculate – or think about conformally flat case first.
Second part: **Conf flat case:** $\Rightarrow$ Using first part

$$D_A D_B K_C = D_A \mathbb{K}_{BC} = D_{(A}D_{B)}K_C \stackrel{\cdot}{=} 0.$$

$\Leftarrow$: top slot of $\nabla_a \mathbb{K}_{BC} = 0$ is $\nabla_{(a}k_{b)_0} = 0$. □

Corollary: a conformal Killing vector is Killing for an Einstein metric $\sigma^{-2}\boldsymbol{g}$ iff

$$I^A \mathbb{K}_{AB} = 0 \qquad I_A = \frac{1}{n}D_A \sigma$$

Purely algebraic condition in c flat case !! Corank($\mathbb{K}$) $\rightsquigarrow$ upper bound on number of Einstein scales . ..

Lemma: (Half-prolongation)

$$k_{ab} \mapsto K_{AB} = \begin{pmatrix} 0 & 0 & 0 \\ \rho_a & k_{ab} & 0 \\ \rho & \rho_b & 0 \end{pmatrix} \qquad \begin{array}{l} \rho_a = \frac{2}{n+2}\nabla^b k_{ab} \\ \rho = \frac{1}{n+1}\left(\frac{1}{n+2}\nabla^c\nabla^d k_{cd} + P^{cd}k_{cd}\right) \end{array}$$

is an isomorphism between sections of $\mathcal{E}_{(ab)_0}[4]$ and sections of $\mathcal{E}_{(AB)_0}[2]$ that satisfy the equations

$$X^A K_{AB} = 0 \quad \text{and} \quad \mathbb{D}^A K_{AB} = 0. \tag{2}$$

Theorem

$\Gamma(S_0^2 T^*M)[4] \ni k_{ab}$ is conf Killing iff

$$D_{(A}K_{BC)} = X_{(A}F_{BC)} \overset{conf.flat}{=} 0.$$

For a conf. Killing tensor k_{bc}, we say a scale $\sigma \in \Gamma(\mathcal{E}[1])$ is a strong Killing scale (SKS) if it is Killing for $g = \sigma^{-2}\boldsymbol{g}$, i.e. $\nabla^b k_{bc} = 0$.

By inspection of the half-prolongation above we have:
Proposition: 1. A scale $\sigma \in \Gamma(\mathcal{E}[1])$ is a SKS if and only if $I^A K_{AB} = X_B F$ for some $F \in \mathcal{E}[1]$.
2. σ is an Einstein SKS if and only if $I^A K_{AB} = 0$ and I_A is parallel for the conformal tractor connection.

We say that $\sigma \rightsquigarrow I = D\sigma$ is a Killing scale (KS) if k_{bc} is the trace-free part of a Killing tensor for $g = \sigma^{-2}\boldsymbol{g}$.
Proposition: Given a conformal Killing tensor k_{bc}, σ is a KS iff $\exists$ $\lambda \in \Gamma(\mathcal{E}[2])$ s.t.

$$I^A K_{AB} = \frac{\sigma}{2} D_B \lambda - \lambda I_B + X_B F \ . \tag{3}$$

If so $\overline{k}_{bc} = k_{bc} + \lambda g$ is the Killing tensor.

Proof: The display is the same as $\rho_a = \frac{2}{n+2}\nabla^b k_{ab} = -\nabla_a \lambda$ being a gradient $\longleftrightarrow$ Bertrand-Darboux $(\star)$

The full prolongation – forever Young!

Recall Thm above $\nabla_{(a}k_{bc)_0} \Leftrightarrow D_{(A}K_{BC)} = 0$ if c-flat.

Corollary: [cf e.g. BGG Machinery Čap et al] On a conformally flat manifolds, there is a one-to-one correspondence:

conformal Killing tensors $k_{bc} \leftrightarrow \mathbb{K}_{ABCD}$ parallel tractors

where the $\mathbb{K}_{ABCD}$ have Weyl tensor symmetries. Say $\mathbb{K} \in \mathbb{W}[0]$

It's the proof here that's important:

Proof: $\Rightarrow$: $D_B D_C K_{DE}$ is symmetric on BC and DE, but $D_{(C}K_{DE)} = 0$ (and $D_B D_C K_{DE}$ is trace-free) whence

$$\mathbb{K}_{BCDE} := D_B D_C K_{DE} \in \overline{\mathbb{W}}[0] = \boxed{}_0 \qquad \text{alt. Weyl space.}$$

$D_A D_B D_C K_{DE}$ is symmetric on ABC and DE but $D_{(C}K_{DE)} = 0$. Thus $D_A D_B D_C K_{DE} = 0$ whence

$$\mathbb{K}_{BCDE} \qquad \text{is parallel}$$

$\Leftarrow$: top slot of $\nabla_a D_B D_C K_{DE} = 0$ is $\nabla_{(a}k_{bc)_0} = 0$.

Einstein SKS if conformally flat

From the above we have at once:

Theorem

Let (M, g) be conformally flat and k_{ab} is a conformal Killing tensor with associated K_{AB} and $\mathbb{K}_{ABCD} = D_A D_C K_{BD}$. Then an Einstein SKS σ is equivalent to

$$I_A \text{ is parallel and satisfies } I^C \mathbb{K}_{ABCD} = 0.$$

Proof.

$\Rightarrow$: Ein SKS means $I^B K_{BD} = 0$ and I parallel. Latter means $[D, I] = 0$.

$\Leftarrow$ $I^C \mathbb{K}_{ABCD} = 0$ means $X^A X^B I^C \mathbb{K}_{ABCD} = 0 = 2I^C K_{CD}$. $\qquad\square$

Corollary: On conformally flat spaces, for a conformal Killing tensor k_{ab} the dimension of the space of Einstein SKS is bounded by the dimension of the nullity of K_{ABCD} and is locally equal to this dimension. In particular there are conformal Killing tensors that admit no Einstein SKS.

How can we incorporate the trace adjustment etc?? First hint:

Theorem: On conformally flat manifolds there is a one-to-one correspondence between:

1. Sections r_{ab} of $\mathcal{E}_{(ab)}$ that are Killing in the Einstein scale $\sigma = X^A I_A$. and

2. Parallel tractors R_{ABCD} with Riemann tensor symmetries ⊞ and $I^A R_{ABCD} = 0$.

Proof: When I is parallel then $I^\perp \subset \mathcal{T}$ is the projective tractor bundle with connection induced by $\nabla^{\mathcal{T}}$ [G.+Macbeth 2014]. Then from [G.+Leistner 2019] (or BGG theory)

$$r_{ab} \overset{1-1}{\leftrightarrow} R_{ABCD}.$$

A punchline

Theorem

On a conformally flat manifold, let k_{ab} be a conformal Killing tensor with full conformal prolongation $\mathbb{W}_{ABCD}$, where $\mathbb{W}_{ABCD}$ has alg. Weyl tensor symmetries. Then k_{ab} is, after trace adjustment, Killing for the Einstein scale $\sigma = X^A I_A$ if and only if

$$0 = I^A \mathbb{W}_{AB[CD} I_{E]}.$$

Proof.

Case $I^2 \neq 0$ Sketch. $\Leftarrow$: $0 = I^A \mathbb{W}_{AB[CD} I_{E]}$ with I and $\mathbb{W}$ parallel implies that

$R_{ABCD} := I^2 \mathbb{W}_{ABCD} + (h \wedge P)_{ABCD},$ is parallel and orthogonal to I, where

$$P_{BD} := \frac{J}{n} \left(I^2 h_{BD} - 2 I_B I_D \right) - I^A I^C \mathbb{W}_{ABCD}.$$

Then $R_{ABCD} \leftrightarrow r_{ab}$ Killing for the scale $\sigma = X^A I_A$ by prev slide.

□

On conformally flat simply connected M.

Summary:

- Einstein metrics g correspond to parallel tractors I_A.
- conformal Killing tensors k_{bc} correspond to parallel tractors

$$\mathbb{W} \in \boxed{}_0 \qquad \mathbb{W} = \mathrm{prol}(k).$$

k is the trace-free part of a Killing tensor for g iff

$$\boxed{0 = I^A \mathbb{W}_{AB[CD}I_{E]}.} \qquad (\star)$$

Given $\mathbb{W}_{ABCD}$ we can solve for the variety of Is that solve $(\star)$.

Given I we can solve for the $\mathbb{W}$ that solve $(\star)$.

Superintegrability !?

Can we use this to construct superintergrable systems? Especially new and maximal ?

Yes, yes, and yes but more subtle!

To get maximal superintegrability we need,
Non-Einstein scales, $\tilde{I} = \hat{D}\tilde{\sigma}$.
Then the equation becomes

$$3\tilde{I}^A X^B Z_c^C Z_d^D X^E \mathbb{W}_{AB[CD}\tilde{I}_{E]} = \tilde{\sigma} X^B Z_c^C Z_d^D X^E \mathbb{W}_{AB[CD}\hat{D}_{E]}\tilde{I}^A. \quad (4)$$

In general this is hard! But:

Non-Einstein scales?

Idea: Examine scales that "lie between" Einstein scales I, J. That is metrics
$$\tilde{g} = f^{-2}(s)g$$
where g is Einstein and $s^{-2}g$ is Einstein. (cf Bertrand)
Then the scale tractor $\tilde{I} = I^f$ satisfies

$$\tilde{I}_A = (f(s) - sf'(s))I_A + f'(s)J_A \mod X_A$$

E.g. I is a Killing scale $I^A \mathbb{W}_{AB[CD}I_{E]} = 0$ and J any other Einstein scale then (4) reduces to the form

$$X^B Z_c^C Z_d^D X^E \left(h(s)(J^A R_{AB[CD}I_{E]}) + \ell(s)(J^A R_{AB[CD}J_{E]}) \right) = 0$$

where R in $I^\perp$ is s.t. $\mathbb{W}$ is the trace-free part of R, and
$h(s) = (3(f(s) - sf'(s))f'(s) + sf(s)f''(s))$ and
$\ell(s) = (3f'(s)^2 - f(s)f''(s))$,

Taking into account also the need for functional independence we have this: For any Einstein scales I, J and any potential $V(s)$ (with s interpolating between the scales) we get $2n - 3$ constants from $\mathbb{W}$s (effectively killing vectors) in $I^\perp \cap J^\perp$, 1 constant from $\mathcal{H}$ itself, and then we need 1 more to have maximal superintergrability! This corresponds to finding a suitable quadratic Killing tensor that is, after trace adjustment, Killing for two scales (and is a new one).

For a given $\mathbb{W}$, if I is a Killing scale, but J (Einstein) is not then $\tilde{g} = f^{-2}(s)g$ is a Killing scale if, with
$h(s) = (3(f(s) - sf'(s))f'(s) + sf(s)f''(s))$ and
$\ell(s) = (3f'(s)^2 - f(s)f''(s))$, either

$$(1) \quad \ell(s) = 0 \quad \text{or} \quad (2) \quad h(s) = s\ell(s) \quad I, J \text{ null cases}$$

E.g. with I Euclidean, the Harmonic oscillator has $g(s) = 0$ and Kepler $h(s) = s\ell(s)$, where $s = -r^2/2$.

Key points:

1. It is easy to solve these ODEs.

2. For each solution it's straightforward to find the R which gives the extra constant of motion – in this generality: i.e. I and J Einstein scales and background conformally flat.

3. In fact:

Thm: Given two Einstein scales $\mathcal{H} = g^{-1} + V(s)$ is superintegrable iff one of the ODEs holds $\rightsquigarrow$ exactly two superintegrable $V(s)$, for each pair I, J.

This generalises a classical Theorem of Bertand which states that in Euclidean 3 spaces there are are just two radially symmetric superintegral potentials.

Enough already,

Thank you for Listening.

Abstract

How Killing fields yield to tractors - a tale of algebra exposed
This is story about a beautiful interaction of mechanics, geometry, and algebra. The motion of particles or objects is a classical and fundamental problem in physics. Quantisation yields related problems for partial differential operators on function spaces. A class of these problems corresponds to finding "shortest paths" or geodesics on geometric manifolds, namely Riemannian geometries. It has long been known that conformal geometry plays a role in understanding these topics, but previously little has been done to caste and treat the problems using the central tools of conformal geometry such as the tractor connection.
Objects called Killing tensors are important as they yield first integrals for the geodesic equation, symmetries for the Laplacian, and have a host of other applications. If there are enough of these (the so-called superintegrable setting) then trajectories can be determined algebraically. Conformal Killing tensors satisfy a weaker (conformally invariant) equation. It turns out to be important to