Smooth representations of Affine Kac-Moody algebras

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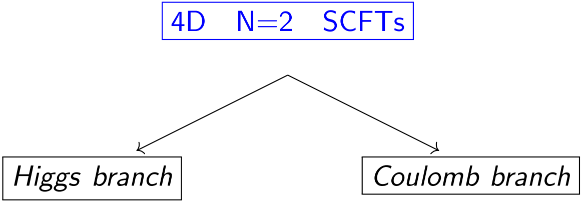
4D N=2 SCFTs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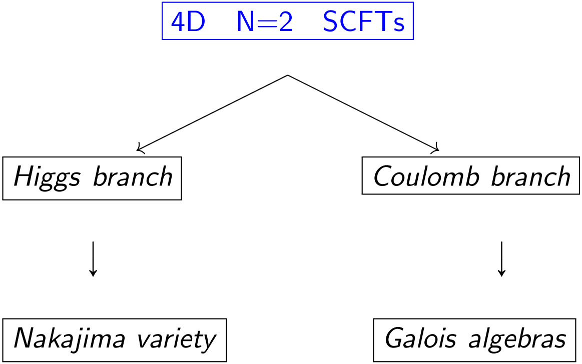
4D N=2 SCFTs
Higgs branch
Coulomb branch
Nakajima variety
Galois algebras

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- $\boxed{\text{2D} \quad \text{N=2} \quad \text{SCFTs}} \longrightarrow \boxed{\text{VOAs}}$

- **Conjecture** (Beem, Rastelli, 2018): If T is in the image $(T \mapsto \mathcal{V})$ then $Higgs(T) = X_{\mathcal{V}}$

- $\mathfrak{g}$ a simple finite-dimensional Lie algebra
- κ a $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$
- Affine Kac–Moody algebra:

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$$

with $[a \otimes f(t), b \otimes g(t)] =$

$$= [a, b] \otimes f(t)g(t) - \kappa(a, b) Res_{t=0}(f(t)dg(t))c,$$

- $\widetilde{\mathfrak{g}} = \widehat{\mathfrak{g}} \oplus \mathbb{C}d$ with $[d, x \otimes t^n] = n(x \otimes t^n)$

Affine vertex algebras

- The universal affine vertex algebra:

$$\mathcal{V}_k(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}\otimes\mathbb{C}[t]+\mathbb{C}c)} \mathbb{C}v,$$

where $\mathfrak{g} \otimes \mathbb{C}[t]v = 0$ and $cv = kv$ (vacuum module)

For $a \in \mathfrak{g}$ and $a_n := a \otimes t^n$: $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$,

$$[a(z), b(w)] = [a, b](w)\delta(z-w) + \kappa(a, b)c\partial_w\delta(z-w)$$

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$R_{\mathcal{V}_k(\mathfrak{g})} \simeq S(\mathfrak{g})$, $X_{\mathcal{V}_k(\mathfrak{g})} = \mathfrak{g}^*$

• V is smooth if for any $v \in V$, $\widehat{\mathfrak{g}}_i v = 0$ for all sufficiently large i, where $\widehat{\mathfrak{g}}_i = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}t^i$, for $i \neq 0$ and $\widehat{\mathfrak{g}}_0 = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}1 \oplus \mathbb{C}c$

Representations of the universal affine vertex algebra $\mathcal{V}_k(\mathfrak{g})$

$\Updownarrow$

Smooth representations of $\widehat{\mathfrak{g}}$ of level k

P1: Describe simple smooth $\widehat{\mathfrak{g}}$-modules

Examples:

- **Category** $\mathcal{O}$;
- **Whittaker modules**;
- **Parabolic induced modules**: Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ be a parabolic subalgebra of $\mathfrak{g}$, $\widehat{\mathfrak{p}} = \widehat{\mathfrak{l}} \oplus \widehat{\mathfrak{u}}$ a parabolic subalgebra of $\widehat{\mathfrak{g}}$, where $\widehat{\mathfrak{l}} = \mathfrak{l} + \widehat{\mathfrak{h}}$, $\widehat{\mathfrak{u}} = \mathfrak{g} \otimes t\mathbb{C}[t] \oplus \mathfrak{u}$, $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c$.

  For a simple $\mathfrak{l}$-module N set

  $$\mathbb{M}_{k,\mathfrak{p}}(N) = U(\widehat{\mathfrak{g}}) \otimes_{U(\widehat{\mathfrak{p}})} N,$$

  with $\widehat{\mathfrak{u}}N = 0$ and c acts by $k \neq -h^{\vee}$ (non-critical level). It has a unique simple quotient $\mathbb{L}_{k,\mathfrak{p}}(N)$. Both $\mathbb{M}_{k,\mathfrak{p}}(N)$ and $\mathbb{L}_{k,\mathfrak{p}}(N)$ are smooth (positive energy modules)

Mixed modules

V.Futorny, X.Guo, Y.Xue, K.Zhao, Advances in Math., 481, 2025

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- Let $\Sigma = \{(N_\alpha, M_\alpha),\ N_\alpha + M_\alpha \geqslant 0,\ \alpha \in \pi\}$,

S_Σ the Lie subalgebra of $\widehat{\mathfrak{g}}$ generated by c and the elements $(x(n) = x \otimes t^n)$

$$h_\alpha(n_{\alpha,0}),\ e_\alpha(n_\alpha),\ f_\alpha(m_\alpha),$$

where $n_{\alpha,0}, n_\alpha, m_\alpha \in, n_{\alpha,0} \geqslant 0, n_\alpha > N_\alpha, m_\alpha > M_\alpha,\ \alpha \in \pi$. Consider a Lie algebra homomorphism $\varphi_\Sigma : S_\Sigma \longrightarrow \mathbb{C}$ and smooth $\widehat{h}$-weight modules

$$\widehat{M}(\varphi_\Sigma) = \mathrm{Ind}_{S_\Sigma}^{\widehat{\mathfrak{g}}} \mathbb{C}v,$$

where S_Σ acts on v by φ_Σ.

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- If $N_\alpha = n$ and $M_\alpha = m$ for all α, write $\widehat{M}(\varphi_{n,m})$ and we can assume $n = -1$

Takiff realization

Consider

$$T_m(\mathfrak{g}) = \mathfrak{g} \otimes (\mathbb{C}[t]/t^{m+1}\mathbb{C}[t]),$$

with

$$T_m(\mathfrak{g}) = T_m(\mathfrak{g})_+ \oplus T_m(\mathfrak{g})_0 \oplus T_m(\mathfrak{g})_-,$$

$$T_m(\mathfrak{g})_\pm = \mathfrak{n}_\pm \otimes (\mathbb{C}[t]/t^{m+1}\mathbb{C}[t]), \quad T_m(\mathfrak{g})_0 = \mathfrak{h} \otimes (\mathbb{C}[t]/t^{m+1}\mathbb{C}[t])$$

Let $\psi : T_m(\mathfrak{g})_0 \longrightarrow \mathbb{C}$ and $V(\psi)$ is the induced $T_m(\mathfrak{g})$-module from $\mathbb{C}$ with a trivial action of $T_m(\mathfrak{g})_+$.

View $V(\psi)$ as a $\mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}c$-module, on which c acts by $\theta \in \mathbb{C}$ and define a $\widehat{\mathfrak{g}}$-module

$$M(\psi, \theta) = \mathrm{Ind}^{\widehat{\mathfrak{g}}}_{\mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}c} V(\psi)$$

- $M(\psi, \theta) \cong \widehat{M}(\varphi)$, for some homomorphism $\varphi : S_{-1,m} \longrightarrow \mathbb{C}$ such that $\varphi(c) = \theta$.

Theorem (F.–Guo–Xue– Zhao)

Let $\mathfrak{g} = \mathfrak{sl}_2$, $n, m \in \mathbb{Z}$, $n + m \geqslant 0$ *and* φ *is a Lie algebra homomorphism from* $S_{n,m}$ *to* $\mathbb{C}$*. The* $\hat{\mathfrak{g}}$*-module* $\widehat{M}(\varphi)$ *is irreducible if and only if* $\varphi(h(n + m + 1)) \neq 0$ *and* $\varphi(c) \neq -2$.

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Remark:

"Full" Whittaker modules ($n = -1, m = 0$) for $\widehat{\mathfrak{sl}}_2$ were studied by Adamovich–Lu–Zhao 2016

Twisted localization

There is a family of functors parametrized by a real root α and a complex x which preserve the smoothness:

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Let $f_\alpha \in \widehat{\mathfrak{g}}_{-\alpha} \setminus \{0\}$, $F_\alpha = \{f_\alpha^k \mid k \in \mathbb{Z}_{\geqslant 0}\}$, $D_\alpha U(\widehat{\mathfrak{g}})$ the localization of $U(\widehat{\mathfrak{g}})$ relative to F_α. For a $\widehat{\mathfrak{g}}$-module M define:

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- For $x \in \mathbb{C}$ consider the automorphism θ^x of $D_\alpha U(\widehat{\mathfrak{g}})$:

$$\theta^x(u) := \sum_{i \geqslant 0} \binom{x}{i} \mathrm{ad}(f_\alpha)^i(u)\, f_\alpha^{-i}$$

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For a set $S = \{\alpha_1, \dots, \alpha_k\}$ of commuting real roots and $\mathbf{x} = \{x_1, \dots, x_n\} \in \mathbb{C}^n$ define $D_S^{\mathbf{x}} M := D_{\alpha_1}^{x_1} \dots D_{\alpha_n}^{x_n} M$.

Harish-Chandra modules

• A $\tilde{\mathfrak{g}}$-module V is weight if a Cartan subalgebra $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ is diagonalizable on V.

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Example: If $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u} \subset \mathfrak{g}$ and N is a Harish-Chandra $\mathfrak{l} + \tilde{\mathfrak{h}}$-module then $\mathbb{M}_{k,\mathfrak{p}}(N)$ and $\mathbb{L}_{k,\mathfrak{p}}(N)$ are Harish-Chandra $\tilde{\mathfrak{g}}$-modules (in particular, the Verma $\tilde{\mathfrak{g}}$-module $\mathbb{M}(\lambda) = \mathbb{M}_{k,\mathfrak{h}}(\mathbb{C})$ and its simple quotient $\mathbb{L}(\lambda) = \mathbb{L}_{k,\mathfrak{h}}(\mathbb{C})$)

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Theorem (F–Tsylke, 2001)

Let V be a simple Harish-Chandra $\widetilde{\mathfrak{g}}$-module with a nonzero action of c. Then $V \simeq L_{k,\mathfrak{p}}(N)$, for some parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}$ and a simple $\mathfrak{l}$-module $N \simeq \otimes D_{S_i}^{\mu_i} L(\lambda_i)$ for some sets of roots S_i's, $\mu_i \in \mathfrak{h}^*$ and simple λ_i-highest weight $\mathfrak{l}_i$-modules $L(\lambda_i)$, where $\mathfrak{l}_i$'s are simple components of $\mathfrak{l}$ of type A and C.

Based on two steps:
1) Fernando-F parabolic reduction to cuspidal $\mathfrak{g}$-modules
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Corollary: Simple Harish-Chandra $\mathcal{V}_k(\mathfrak{g})$-modules are obtained from $\mathbb{L}(\lambda)$ by twisted localization up to a spectral flow

Question: Is it possible to classify simple smooth weight $\widetilde{\mathfrak{g}}$-modules?

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Wrong conjecture:
Any such modules is isomorphic to $D_S^{\mu}\mathbb{L}(\lambda)$ for some S, $\lambda, \mu \in \widetilde{\mathfrak{h}}^*$

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Let $\lambda \in \widetilde{\mathfrak{h}}^*$, $\mathbb{M}(\lambda)$ a simple Verma module with highest weight λ. Take (almost) any nonzero homogeneous *ad*-nilpotent $u \in U(\widetilde{\mathfrak{g}})$ and consider $D_u^x\mathbb{M}(\lambda)$ with $x \in \mathbb{C}$. Then generically $D_u^x\mathbb{M}(\lambda)$ is a simple smooth weight module.

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Theorem (F–H.Gao–Křižka, 2025)

Let $\alpha \in \widehat{\Delta}^{\mathrm{re}}$*. Assume that the Verma module* $\mathbb{M}(\lambda)$ *is simple. Then*

- *If* M *is a smooth* $\widetilde{\mathfrak{g}}$*-module then* $T_\alpha(M)$ *is a smooth module.*
- *Simplicity and finite length of* $T_\alpha(\mathbb{M}(\lambda))$
- $T_\alpha(\mathbb{M}(\lambda))$ *is a Harish-Chandra module if and only if* α *is a simple root.*
- *If* $\mathrm{supp}(\alpha) \neq \widehat{\pi}$ *then the support of* $T_\alpha(\mathbb{M}_{k,\mathfrak{g}}(\lambda))$ *is bounded by a hyperplane.*

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• Twisting functor can be used to construct new simple modules beyond the Harish-Chandra category

- Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ be a parabolic subalgebra of $\mathfrak{g}$
- $\mathbb{F}_\lambda$ the finite-dimensional simple $\mathfrak{p}$-module of highest weight λ
- $\alpha \in \widehat{\Delta}^{\mathrm{re}}$
- $\Gamma_\alpha \subset U(\tilde{\mathfrak{g}})$ generated by the Cartan subalgebra and the Casimir element of the $\mathfrak{sl}_2$-subalgebra based on α

$$W_{\mathfrak{p}}(\lambda, \alpha) = T_\alpha(\mathbb{M}_{k,\mathfrak{p}}(\mathbb{F}_\lambda))$$

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Theorem (F.–Křižka, 2023)

The module $W_{\mathfrak{p}}(\lambda, \alpha)$ has finite Γ_α-multiplicities and finite Γ-multiplicities for any commutative subalgebra Γ of $U(\widetilde{\mathfrak{g}})$ containing Γ_α

Simple affine vertex algebra: $\mathcal{L}_k(\mathfrak{g}) = \mathbb{L}_{k,\mathfrak{g}}(\mathbb{C})$

• By Gorelik, Kac, 2006, $\mathbb{M}_{k,\mathfrak{g}}(\mathbb{C})$ is simple unless k is admissible:

$$k + h^\vee = \frac{p}{q} \quad \text{with } p, q \in \mathbb{N},\ (p,q) = 1,\ p \geqslant \begin{cases} h^\vee & \text{if } (r^\vee, q) = 1 \\ h & \text{if } (r^\vee, q) = r^\vee \end{cases}$$

• Maximal submodule is generated by one singular vector, e.g. formula for a singular vector of $\mathcal{V}_{-\frac{14}{3}}(D_4)$ has 500 pages and > 600 monomials (Adamovich, Vukovara, 2025)

• All simple Harish-Chandra $\mathcal{L}_k(\mathfrak{g})$-modules have form $\mathbb{L}_{k,\mathfrak{p}}(N)$, the question is which ones. Such modules are called **Relaxed highest weight modules** following Semikhatov

- Simple highest weight $\mathcal{L}_k(\mathfrak{g})$-modules: Kac–Wakimoto, 1989 and Arakawa, 2016

  e.g. $\mathcal{L}_{-2}(D_4)$ has 5 simple modules in $\mathcal{O}$, while $\mathcal{L}_{-\frac{14}{3}}(D_4)$ has 405 such modules

- For sl_2: Adamovic–Milas 1995
- For sl_3: Arakawa–F.–Ramirez 2017
- General algorithm: Kawasetsu–Ridout 2019

Problem 2: How to construct explicitly relaxed highest weight representations?

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Ongoing . . . . . .

Thank you!