Parametrisation of replete admissible weights

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Joint work with T. Arakawa, I. Blatt and W.-B. Yan.

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Summary: Affine vertex algebra $L_k(\mathfrak{g})$ has nice representation category at $k \in \mathbb{Z}_+$ (WZW model, rational CFT).

For $k = -h^\vee + p/u$ more complicated.

For the Hamiltonian reduction $W(\mathbb{O}_u, p/u)$, representation category again nice (Also rational CFT).

To understand category for general (p, u), suffices to understand cases of u minimal (WZW case) and p minimal (boundary admissible case).

Surprise: many of these boundary admissible algebras appear to be isomorphic. Mysteriously related to 4-dimensional physics.

Let $\mathfrak{g}$ be a simple Lie algebra of rank ℓ, and $k \in \mathbb{R}$.

We consider the affine Kac-Moody algebra

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K, \qquad [at^m, bt^n] = [a, b]t^{m+n} + \delta_{m,-n}\, m\,(a, b)K.$$

We say a representation has "level" k if $K = k\mathrm{Id}$.

There is an affine root system $\widehat{\Delta}$, with simple roots $\alpha_0, \alpha_1, \ldots, \alpha_\ell$. Dominant integral weights are

$$\widehat{P}_+ = \{\widehat{\lambda} \in \widehat{\mathfrak{h}}^* \mid \left\langle \widehat{\lambda}, \alpha_i^\vee \right\rangle \geq 0, i = 0, 1, \ldots, \ell\}.$$

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Can instead describe weights in terms of their level k (which must be in $\mathbb{Z}_+$ for dominant integral) and "finite part": so

$$\widehat{\lambda} = k\Lambda_0 + \lambda, \qquad \text{where } \lambda \in P_+^k$$

There is an associated **vertex algebra** $V^k(\mathfrak{g})$, with closely related representation theory:

$$\{V^k(\mathfrak{g})\text{-modules}\} = \{\text{Smooth } \widehat{\mathfrak{g}}\text{-modules of level } k\}$$

As a vertex algebra, $V^k(\mathfrak{g})$ is simple for generic k. It has an interesting simple quotient $L_k(\mathfrak{g}) = V^k(\mathfrak{g})/N_k$ at special values of k. The representations of $L_k(\mathfrak{g})$ are a distinguished *subset* of smooth $\widehat{\mathfrak{g}}$-modules of level k.

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More precisely

$$L(k\Lambda_0 + (\nu - \rho)) \qquad \text{for } \nu \in P_+^{p,\mathrm{reg}}$$

where

$$P_+^{p,\mathrm{reg}} = \{\nu \in P \mid \langle p\Lambda_0 + \nu, \alpha_i^\vee \rangle > 0, i = 0, 1, \ldots, \ell\}$$

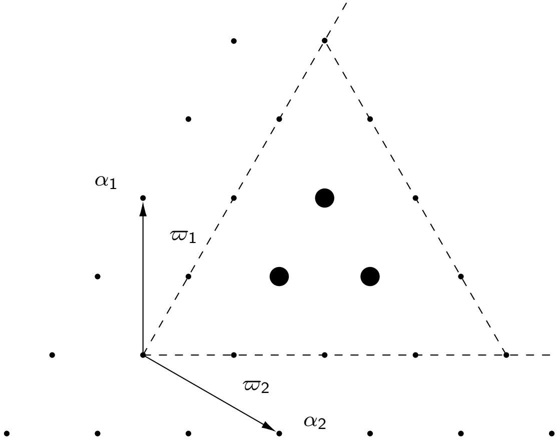
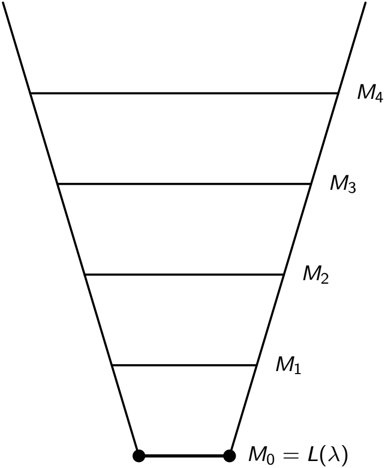
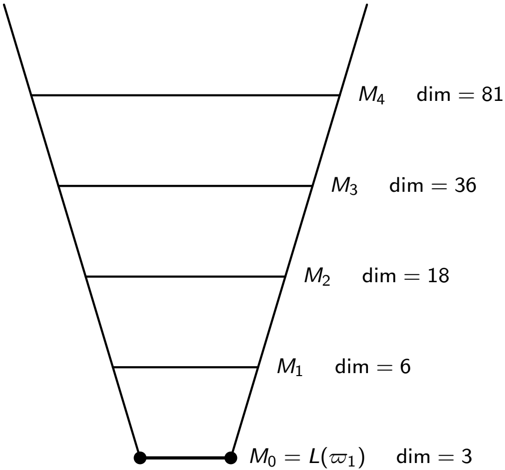


Figure: The set $P_+^{p,\mathrm{reg}}$ of regular weights of shifted level $p = 4$ for $\mathfrak{g} = \mathfrak{sl}_3$



Each $\widehat{\mathfrak{g}}$-module $L_k(\lambda)$ has a $\mathbb{Z}_+$-grading with lowest piece being the finite $\mathfrak{g}$-module $L(\lambda)$

The Weyl character formula tells us the dimensions of graded pieces.



Example: $\mathfrak{g} = \mathfrak{sl}_3$ at level $k = 1$ and $\lambda = \varpi_1$

Characters of the irreducible modules

$$\widetilde{\chi}_M(\tau) = \sum_{n=0}^{\infty} \dim(M_n) q^n \qquad \text{where } q = e^{2\pi i \tau}$$

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...with S-matrix entries given by some explicit cyclotomic sums

$$S_{\nu,\nu'} = \frac{i^{|\Delta_+|}}{|P/p\,Q|^{1/2}} \sum_{w \in W} \varepsilon(w)\, e^{-\frac{2\pi i}{p}(w(\nu),\nu')}$$

(Kac-Peterson 1984)

There is also a theory of tensor products of vertex algebra modules (Kazhdan-Lusztig, Huang-Lepowsky), and in the present case one actually obtains a **modular tensor category (MTC)**, with S-matrix stipulated by the modular behaviour of characters.

In particular the Verlinde formula holds:

$$M^{\nu_1} \dot{\otimes} M^{\nu_2} \cong \bigoplus_{\nu_3} N_{\nu_1,\nu_2}^{\nu_3} M^{\nu_3} \qquad N_{\nu_1,\nu_2}^{\nu_3} = \sum_{\nu'} \frac{S_{\nu_1,\nu'} S_{\nu_2,\nu'} \overline{S}_{\nu_3,\nu'}}{S_{\rho,\nu'}}$$

The coefficients $N \in \mathbb{Z}_{\geq 0}$ are called the "fusion rules".

Example: $\mathfrak{g} = \mathfrak{sl}_3$, $k = 1$.

$$S = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\alpha & -1+\alpha \\ 1 & -1+\alpha & -\alpha \end{bmatrix}$$

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General pattern: fusion rules coincide with those of quantum groups at roots of unity.

Principal admissible levels

$$k = -h^\vee + \frac{p}{u}, \qquad (p,u) = (r^\vee, u) = 1, \qquad u \geq 1, \quad p \geq h^\vee.$$

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We say $\widehat{\lambda}$ is principal admissible if it is dominant integral with respect to $\Pi^\vee(\widehat{\lambda})$.

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Does $L_k(\mathfrak{g})$ have an MTC of representations, though (for $k \notin \mathbb{Z}_+$)? No.

Every admissible level $k = -h^\vee + p/u$ determines a special **nilpotent orbit** $\mathbb{O}_u \subset \mathfrak{g}$ characterised by

$$\overline{\mathbb{O}}_u = \{x \in \mathfrak{g} \mid \mathrm{ad}(x)^{2u} = 0\}.$$

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In general a nilpotent element $f \in \mathfrak{g}$ determines an $\mathfrak{sl}_2$-triple $\{e, h, f\} \subset \mathfrak{g}$ and a grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ by eigenvalues of $\mathrm{ad}(h)/2$.

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We consider the **Hamiltonian reduction**. Resolve $L_k(\mathfrak{g})$ by tensoring with a free fermion vertex algebra F, built from $\mathfrak{g}_{>0}$, with some differential d. The cohomology is called a W-**algebra**

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If we choose $f \in \mathbb{O} = \mathbb{O}_u$ then $W(\mathbb{O}, p/u)$ is the "right size", and has an MTC of representations.

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- Irreducible $W(\mathbb{O}, p/u)$-modules can be obtained as Hamiltonian reductions of admissible $L(\lambda)$,
- Some $L(\lambda)$ go to 0. The others, we call **replete** (more precisely, those satisfying $\mathrm{Var}(\mathrm{Ann} L(\lambda)) = \overline{\mathbb{O}}_u$),
- If λ, λ' are related under the "dot"-action of the Weyl group W, then $L(\lambda), L(\lambda')$ have the the same reduction.

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For technical reasons, it is nice if every replete weight have a $W \circ (-)$-representative which is $\mathfrak{g}_0$-integrable.

We denote

$$W(+) = \{w \in W \mid w(\Delta_{0,+}) \subset \Delta_+\} \quad \text{where } \Delta_0 = \Delta(\mathfrak{g}_0).$$

Then we have

Theorem (Arakawa, vE)

Suppose every replete weight has a $\mathfrak{g}_0$-integrable representative. Then the MTC of $W(\mathbb{O}_u, h^\vee/u)$-modules has S-matrix

$$S_{\eta,\eta'} = C \sum_{w \in W(+)} \varepsilon(w)\, e^{-\frac{2\pi i h^\vee}{u}(\eta, w(\eta'))} \prod_{\alpha \in \Delta_{0,+}} \frac{(w(\alpha), \xi)}{(\alpha, \xi)}.$$

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Proof: An application of l'Hopital's rule.

Remark: A similar but more complicated formula holds for $p > h^\vee$ too. The simplification occurs because in the $p = h^\vee$ case we have $P^{p,\mathrm{reg}} = \{\rho\}$.

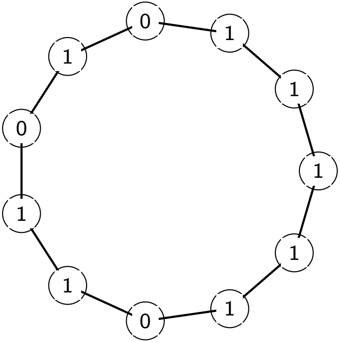
Example: $W(\mathfrak{sl}_{11}[8,3], 11/8)$.

In general in type A the orbit $\mathbb{O}_u \subset \mathfrak{sl}_n$ corresponds to the partition $[u^m, s]$ where $n = um + s$ and $0 \leq s < u$.

In this case $\Delta_{0,+} = \{\alpha_{1,2}, \alpha_{3,4}, \alpha_{5,6}\}$. See this using the combinatorics of "pyramids":

1	3	5					
2	4	6	7	8	9	10	11

So, for $\lambda = w(\rho - (h^\vee/u)\eta) - \rho$ to be **replete**, we need $\eta \in P^8_+$ to be integrable with respect to exactly 3 mutually orthogonal simple roots.



At left: the Dynkin diagram of $\widehat{\mathfrak{sl}}_{11}$, with a labeling of $\eta \in P_{+}^{8}$ corresponding to a replete weight.

There is a simultaneous $\mathbb{Z}/11$-action on the triples (η, w, ν), preserving

$$\lambda = w(\nu - (p/u)\eta) - \rho.$$

The action on η is by rotating the affine Dynkin diagram.

In all there are seven orbits of weights $\eta \in P_+^8$. Namely

$$\begin{array}{ll} (1,1,1,1,1,1,0,1,0,1 \mid 0), & (1,1,1,1,1,0,1,1,0,1 \mid 0), \\ (1,1,1,1,0,1,1,1,0,1 \mid 0), & (1,1,1,0,1,1,1,1,0,1 \mid 0), \\ (1,1,0,1,1,1,1,1,0,1 \mid 0), & (1,1,1,1,0,1,1,0,1,1 \mid 0), \\ (1,1,1,0,1,1,1,0,1,1 \mid 0). & \end{array}$$

And hence $W(\mathfrak{sl}_{11}[8,3], 11/8)$ has 7 irreducible modules.

Next example: $W(\mathfrak{sl}_{19}[8,8,3], 19/8)$.

The pyramid associated with this nilpotent orbit is:

1	4	7					
2	5	8	10	12	14	16	18
3	6	9	11	13	15	17	19

The weights in $\Delta_{0,+}$ are given by columns, i.e., $\alpha_{1,2}$, $\alpha_{2,3}$, etc. In particular $\mathfrak{g}_0$ contains 3 copies of $\mathfrak{sl}_3$ and 5 copies of $\mathfrak{sl}_2$.

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The possible weights $\eta \in P_+^8$ are those whose coefficients exhibit 3 blocks of $(0,0)$ and 5 "blocks" of (0). Like the following

$$(1,0,1,0,1,0,1,0,1,0,1,0,0,1,0,0,1,0,\mid 0).$$

There are 7 classes of replete weights in total. But the point is they are in obvious bijection with the previous case: just shorten/lengthen by 1 the lengths of the blocks of zeros.

Proposition (Arakawa, Blatt, vE, Yan)

Fix u, s coprime and write $n = um + s$ for varying $m \in \mathbb{Z}_+$. The irreducible $W(\mathfrak{sl}_n[u^m, s], n/u)$-modules are in bijection with each other, and are

$$\frac{1}{u} \sum_{d \mid \gcd(s, u-s)} \varphi(d) \binom{u/d}{s/d}$$

in number, where $\varphi(n) = \sum_{d|n} d$ is the Euler totient function.

How the $m = 0$ case fits into this parametrisation needs a bit of interpretation. Continuing our example: $W(\mathfrak{sl}_3[3], 3/8)$.

Since $u > h^\vee$ the orbit $\mathbb{O}_u$ is the principal nilpotent orbit, and in this case $\Delta_{0,+} = \emptyset$. So the condition of being replete means $\eta \in P_+^{8,\mathrm{reg}}$. These weights are

$$\begin{aligned} &(6, 1, \mid 1), \quad (5, 2, \mid 1), \\ &(4, 3, \mid 1), \quad (3, 4, \mid 1), \\ &(2, 5, \mid 1), \quad (4, 2, \mid 2), \\ &(3, 3, \mid 2). \end{aligned}$$

These weights are indeed in natural bijection with the other cases.

In the $m = 0$ case (i.e., the principal case), our formula for the S-matrix reduces to that given by [Frenkel-Kac-Wakimoto] in 1993.

In particular, in the principal nilpotent case, fusion rules are given by Kronecker products of fusion rules of quantum groups at roots of unity.

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So could it be that the bijections between irreducible modules lift to the level of modular data?

Theorem (Arakawa, Blatt, vE, Yan)

The bijections between sets of irreducible $W(\mathfrak{sl}_n[u^m, s], n/u)$-modules induce

- *Equalities of q-characters (in particular equalities of T-matrices),*
- *Equalities of S-matrices.*

In a sense, our theorem determines all fusion rules of exceptional W-algebras in type A.

Indeed our formula for the S-matrix (stated above only for $p = h^\vee$), has the feature that fusion rules of $W(\mathbb{O}_u, p/u)$, roughly speaking, “factorise” into a Kronecker product

$$(\text{Fusion rules of } L_{p-h^\vee}(\mathfrak{g})) \times (\text{Fusion rules of } W(\mathbb{O}_u, h^\vee/u)).$$

The theorem tells us

$$(\text{Fusion rules of } W(\mathbb{O}_u, h^\vee/u)) = (\text{Fusion rules of } W(\mathfrak{sl}_s[\text{prin}], s/u)),$$

which in turn is quantum group fusion rules, by Frenkel-Kac-Wakimoto.

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$$S_{\lambda,\lambda'} = C \sum_{w \in W^f} \varepsilon(w) e^{-2\pi i \frac{n}{u}(\eta, w(\eta'))}.$$

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The projections of the weights η, η' to an $\mathfrak{sl}_u$ components can be shown to lie in the orbit of its Weyl vector. So these contribute constant factors, and the S-matrix ends up reducing to that of the $\mathfrak{sl}_s$ component.

Other types?

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I'll finish with some (older) results about type E and subregular nilpotent orbit.

More examples: for $\mathfrak{g} = E_8$ we can take $u = 24, 25, 26, 27, 28$ or 29.

$W(E_8(a_1), 31/24) \cong \mathbb{C}\,|0\rangle$

$W(E_8(a_1), 31/25) \cong \mathrm{Vir}_{2,5}$

$W(E_8(a_1), 31/26) \cong \mathrm{Vir}_{2,13}$

$W(E_8(a_1), 31/27) \cong$? But same fusion rules as $L_5(G_2)$

$W(E_8(a_1), 31/28) \cong$? But same fusion rules as $L_5(F_4)$

$W(E_8(a_1), 31/29) \cong$? ???

The coincidences of fusion rules on the previous slide are explained by the following

Theorem (vE, Nakatsuka)

There exist isomorphisms of vertex algebras

$$W(E_8(a_1), 31/27) \cong W(G_2, 9/7),$$
$$W(E_8(a_1), 31/28) \cong W(F_4, 14/13)^*.$$

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The remaining mystery is the case $W(E_8(a_1), 30/29)$. It has 44 irreducible modules and we know the S-matrix.

Proposition (Gannon)

Let (Φ, S, T) be the modular datum associated with a modular tensor category, and suppose the matrices S and T are defined over the cyclotomic number field $\mathbb{Q}(\zeta_N)$. Let $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ defined by $\sigma(\zeta_N) = \zeta_N^\ell$, let ℓ^{-1} denote the multiplicative inverse of ℓ modulo N, and let

$$G = S^3 T^{\ell^{-1}} S T^{\ell} S T^{\ell^{-1}}.$$

Then G is a signed permutation matrix, and thus defines a permutation Σ of Φ by $G_{i,\Sigma(i)} \neq 0$, and a function $\varepsilon : \Phi \to \pm 1$ by $G_{i,\Sigma(i)} = \varepsilon(i)$. Furthermore

$$S_{\Sigma(i),j} = \varepsilon(i)\sigma(S_{i,j}) \quad \text{and} \quad S_{i,\Sigma(j)} = \varepsilon(j)\sigma(S_{i,j}).$$

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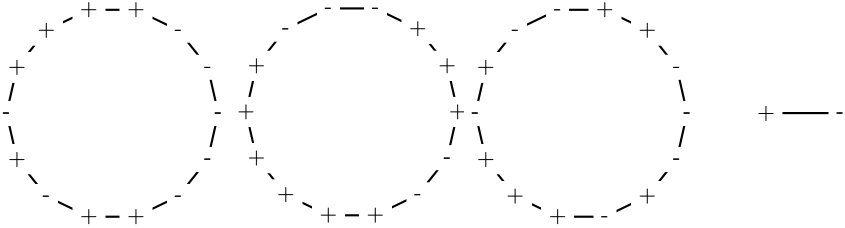
So we just have to specify the signed permutation, and a minimal polynomials of matrix entries between different cycles.



Some of the matrix entries are simple enough:

$$29S_{1,5} = \sqrt{29 + 2\sqrt{29}}.$$

But others are very weird. The minimal polynomial of $(29S_{43,25})^2$ is

$$x^{14} - 319x^{13} + 43094x^{12} - 3257193x^{11} + 153546416x^{10} - 4761488859x^9 + 99637649928x^8$$
$$- 1416589774627x^7 + 13554181725866x^6 - 84852777409590x^5 + 328679573925815x^4$$
$$- 708019388632942x^3 + 682823267578703x^2 - 200943809123541x + 17249876309$$

Here are some of the fusion rules:

24	12	14	7	19	29	22	32	26	37	43	15	16	44	24	27	32	5	7	38	23	27	9	35	47	38	13	22	25	12	13	1	18	13	15	18	3	5	4	2	2	0	0	0
12	6	7	3	9	14	11	15	13	18	21	7	7	22	11	13	15	2	3	19	11	13	4	17	22	19	6	10	12	5	6	0	9	6	8	9	1	2	2	1	1	0	0	0
14	7	8	4	11	16	13	18	14	21	24	8	9	25	14	15	18	3	4	22	13	16	5	19	27	21	7	13	14	7	8	1	10	8	8	10	2	3	2	1	1	0	0	0
7	3	4	1	5	8	6	8	8	10	12	4	4	12	5	8	8	1	1	11	6	7	2	10	12	11	4	5	7	3	3	0	5	3	5	5	1	1	2	1	1	0	0	0
19	9	11	5	14	23	17	24	21	28	34	12	12	34	17	22	24	4	5	30	18	20	7	28	35	30	11	16	20	10	10	1	14	9	13	14	3	4	4	2	2	0	0	0
29	14	16	8	23	32	27	37	30	45	50	16	20	52	29	30	38	5	9	45	25	33	9	39	56	45	14	28	28	12	17	1	22	17	17	23	3	8	5	2	4	0	1	0
22	11	13	6	17	27	20	29	24	34	40	15	15	40	22	26	30	6	7	35	22	24	9	33	43	35	13	21	24	13	13	2	16	12	15	17	4	5	4	3	2	1	0	0
32	15	18	8	24	37	29	40	35	49	57	20	21	57	29	36	41	7	8	50	29	34	11	46	60	51	18	28	33	16	17	2	24	16	22	25	5	7	7	4	4	1	1	0
26	13	14	8	21	30	24	35	28	42	47	16	19	48	28	29	37	5	9	41	24	30	9	37	53	42	14	27	27	12	17	1	20	16	16	22	3	8	5	2	4	0	1	0
37	18	21	10	28	45	34	49	42	58	69	26	26	68	36	45	51	10	11	59	36	40	15	57	73	61	23	35	41	22	22	3	28	20	27	30	7	9	8	5	5	1	1	0
43	21	24	12	34	50	40	57	47	69	78	27	31	79	44	49	60	10	14	68	40	49	15	62	86	70	24	43	46	22	27	3	33	25	29	36	7	12	9	5	6	1	1	0
15	7	8	4	12	16	15	20	16	26	27	8	12	28	17	15	22	2	6	24	13	19	4	20	32	25	7	18	15	5	11	0	12	11	9	14	1	6	3	1	3	0	1	0
16	7	9	4	12	20	15	21	19	26	31	12	12	30	16	21	23	5	5	26	16	18	7	26	33	28	11	16	19	11	10	2	13	9	13	14	4	4	4	3	2	1	0	0
44	22	25	12	34	52	40	57	48	68	79	28	30	80	43	51	60	10	14	69	41	48	16	64	85	71	25	42	47	23	27	3	33	24	30	36	7	12	9	5	6	1	1	0
24	11	14	5	17	29	22	29	28	36	44	17	16	43	20	30	31	7	6	38	22	25	9	37	45	40	16	21	27	15	14	2	18	12	19	20	5	6	6	4	4	1	1	0
27	13	15	8	22	30	26	36	29	45	49	15	21	51	30	30	41	5	11	43	24	33	8	38	57	46	14	31	30	12	20	1	22	18	18	26	3	10	6	2	5	0	1	0
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5	2	3	1	4	5	6	7	5	10	10	2	5	10	7	5	9	0	3	9	5	8	1	7	13	10	2	8	6	1	5	0	5	5	3	6	0	3	1	0	1	0	0	0
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