Simple Character Formulas for Finite W-Superalgebras of Type A

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January 18, 2026

Based on a joint work with Weiqiang Wang

Whittaker Modules

$\mathfrak{g}$ = semisimple Lie algebra over $\mathbb{C}$

Let $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$

$\mathfrak{h}$ Cartan, $\mathfrak{n}^+, \mathfrak{n}^-$ nilradical, opposite nilradical

Π : simple roots in $\mathfrak{n}^+$

$\xi : \mathfrak{n}^+ \longrightarrow \mathbb{C}$ character

determined by $\xi(E_\alpha)$, $\alpha \in \Pi$.

Assume $\xi(E_\alpha) \neq 0$, $\forall \alpha \in \Pi$, i.e., ξ regular

$Z(\mathfrak{g})$ center of $U(\mathfrak{g})$

$\chi_\lambda : Z(\mathfrak{g}) \longrightarrow \mathbb{C}$ central character $(\lambda \in \mathfrak{h}^*)$

$\mathbb{1}_{\chi_\lambda,\xi} = \mathbb{C}\,\mathbb{1}$: $Z(\mathfrak{g})\,\mathcal{U}(\mathfrak{n}^+)$-module with

$$E_\alpha \cdot \mathbb{1} := \xi(E_\alpha)\,\mathbb{1}, \qquad \alpha \in \Pi$$
$$z \cdot \mathbb{1} := \chi_\lambda(z)\,\mathbb{1}, \qquad z \in Z(\mathfrak{g})$$

<u>Theorem.</u> (Kostant, 1977)

- $\mathrm{Ind}_{Z(\mathfrak{g})\mathcal{U}(\mathfrak{n}^+)}^{\mathcal{U}(\mathfrak{g})} \mathbb{1}_{\chi_\lambda,\xi} := L(\lambda,\xi)$ is simple
- Every simple $\mathfrak{g}$-module with locally nilpotent $(x-\xi(x))$-action, $x \in \mathfrak{n}^+$, is of this form.

M : $\mathfrak{g}$-module is a (MMS) Whittaker module :

- M is f.g. over $\mathcal{U}(\mathfrak{g})$
- M is $Z(\mathfrak{g})\,\mathcal{U}(\mathfrak{n}^+)$-locally finite

$\mathcal{N}$: Category of such modules.

$$\mathcal{N} = \bigoplus_{\xi} \mathcal{N}(\xi) \qquad , \; \xi \in \left(\mathfrak{n}^+/[\mathfrak{n}^+,\mathfrak{n}^+]\right)^*$$

subcategory on which $\mathfrak{n}^+$ transforms by ξ.

Note. $L(\lambda,\xi) \in \mathcal{N}(\xi)$, ξ regular (Kostant)

For general ξ :
- $\Pi(\xi) := \{ \alpha \in \Pi \mid \xi(E_\alpha) \neq 0 \}$
- $\mathfrak{l}_\xi$ Levi corresp. to $\Pi(\xi)$

$\mathfrak{p}_\xi$ parabolic of $\mathfrak{g}$ w/ Levi $\mathfrak{l}_\xi$

$M(\lambda,\xi) := \mathrm{Ind}_{\mathfrak{p}_\xi}^{\mathfrak{g}} L_{\mathfrak{l}_\xi}(\lambda,\xi)$ (standard Whittaker module)

<u>Theorem A</u> (McDowell)

- $M(\lambda,\xi) \in \mathcal{N}(\xi)$.
- $M \in \mathcal{N}(\xi)$ has finite length.
- $M(\lambda,\xi)$ has unique cosocle $L(\lambda,\xi)$
- Every simple module in $\mathcal{N}(\xi)$ is of this form.

Note:
- $M(\lambda,\xi) = M(w\cdot\lambda,\xi)$, $w \in W_\xi$ Weyl group of $\mathfrak{l}_\xi$.
- If $\xi = 0$ (i) $M(\lambda,0) = M(\lambda)$ Verma
  (ii) $L(\lambda,0) = L(\lambda)$ simple
  in $\mathcal{O}$ BGG

Theorem B. (Backelin) $\mathfrak{g}$ semisimple Lie algebra

For λ, μ W_ξ-anti-dominant we have

$$[M(\lambda, \xi) : L(\mu, \xi)] = [M(\lambda) : L(\mu)].$$

Thus, the multiplicity is computed by KL polynomial.

Note. Thm B also obtained by Milicic - Soergel when λ, μ integral

Whittaker for Superalgebras

$\mathfrak{g}$: basic Lie superalgebra

$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ & $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$

$\xi : \mathfrak{n}^+ \longrightarrow \mathbb{C}$ character ($\Rightarrow \xi(\mathfrak{n}^+_{\bar{1}}) = 0$!)

$\mathcal{N}$: Category of $\mathfrak{g}$-modules M with

- M is f.g. over $\mathcal{U}(\mathfrak{g})$
- M is $Z(\mathfrak{g}_{\bar{0}})\mathcal{U}(\mathfrak{n}^+)$-locally finite

$$\mathcal{N} = \bigoplus_{\xi} \mathcal{N}(\xi), \qquad \xi \in \left(\mathfrak{n}^+ / [\mathfrak{n}^+, \mathfrak{n}^+]\right)^*$$

<u>Theorem C.</u> $\mathfrak{g}$ basic Lie superalgebra, $\xi: \mathfrak{n}^+ \longrightarrow \mathbb{C}$

Analogues of Theorems A and B hold.

In particular, we have

$$[M(\lambda,\xi) : L(\mu,\xi)] = [M(\lambda) : L(\mu)],$$

for λ, μ W_ξ-anti-dominant.

<u>Note.</u>

- Theorem C due to C.W. Chen for $\mathfrak{g}$ of Type I. For Type II [Chen-C]
- An analogue can be formulated and proved for Type Q [Chen-C, 2026].

Categorification via MMS Whittaker Modules

Focus on $\mathfrak{g} = \mathfrak{gl}(m|n)$ & integral weights

$\mathcal{U} = \mathcal{U}_q(\mathfrak{gl}_\infty)$ (∞ indexed by $\mathbb{Z}$)

- $\mathbb{V}$ natural $\mathcal{U}$-module
- $\mathbb{W}$ restricted dual

Recall: $\xi : \mathfrak{n}^+ \hookrightarrow \mathbb{C} \longmapsto \mathfrak{l}_\xi$ Levi

- $$\mathfrak{l}_\xi = \underbrace{\mathfrak{gl}(\lambda_1) \oplus \cdots \oplus \mathfrak{gl}(\lambda_\ell)}_{\subseteq\, \mathfrak{gl}(m)} \oplus \underbrace{\mathfrak{gl}(\mu_1) \oplus \cdots \oplus \mathfrak{gl}(\mu_t)}_{\subseteq\, \mathfrak{gl}(n)}$$
- $\lambda \vdash m$, $\mu \vdash n$

$$S^{\lambda}(\mathbb{V}) = S^{\lambda_1}(\mathbb{V}) \otimes \dots \otimes S^{\lambda_k}(\mathbb{V})$$

$$S^{\mu}(\mathbb{W}) = S^{\mu_1}(\mathbb{W}) \otimes \dots \otimes S^{\mu_t}(\mathbb{W})$$

$$S^{\lambda|\mu} = S^{\lambda}(\mathbb{V}) \otimes S^{\mu}(\mathbb{W})$$

<u>Theorem D.</u> [Chen - C - Mazorchuk 2023]

$S^{\lambda|\mu}$ is categorified by a properly stratified subcategory $\mathcal{W}(\xi)$ of $\mathcal{N}(\xi)$.

Under this categorification the canonical / dual canonical basis in $S^{\lambda|\mu}$ correspond to tilting and irreducibles

$\mathcal{W}(\xi)$ contains all standard & simple objects of $\mathcal{N}(\xi)$.

<u>Theorem E</u>. (Chen-C 2024) Type BCD analog of the Thm D holds. (need i-quantum groups)

M : tensor product with factors

$$S^k(\mathbb{V}),\ S^l(\mathbb{W}),\ \Lambda^s(\mathbb{V}),\ \Lambda^t(\mathbb{W})$$

<u>Theorem F</u>. (C-Wang 2025)

M is categorified by a (parabolic) properly stratified category $\mathcal{W}(\xi)^{\mathfrak{p}} \subseteq \mathcal{W}(\xi)$ of $\mathfrak{gl}(m|n)$-modules.

Furthermore, the canonical & dual canonical bases correspond to tilting & irreducible modules.

Finite W-superalgebras

$\mathfrak{g} = \mathfrak{gl}(m|n)$

$e \in \mathfrak{g}$ even nilpotent element.

$e = e_1 + e_2$ $[\, e_1 \in \mathfrak{gl}(m), e_2 \in \mathfrak{gl}(n) \,]$

$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(j)$ good grading for e

$(e \in \mathfrak{g}(2))$

Assume grading is even, i.e., $j \in 2\mathbb{Z}$

$\chi : \bigoplus_{j<0} \mathfrak{g}(j) \longrightarrow \mathbb{C}$ character

$\chi(x) := \mathrm{Str}(e|x)$

$\mathfrak{m}_\chi := \{ x - \chi(x) \mid x \in \bigoplus_{j<0} \mathfrak{g}(j) \}$

Finite W-superalgebra of $\mathfrak{g}$ corresp. to e:

$$\mathcal{U}(\mathfrak{g},e) = \operatorname{End}_{\mathcal{U}(\mathfrak{g})}\left(\mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})\mathfrak{m}_\chi\right)^{\mathrm{opp}}$$

Assume e is of standard Levi type

So e is principal in a Levi $\mathfrak{l}$.

- $\mathfrak{t} := \mathfrak{h}^e \subseteq \mathcal{U}(\mathfrak{g},e)$

  T adjoint group of $\mathfrak{t}$.
- θ : cocharacter of T gives grading

  $$\mathcal{U}(\mathfrak{g},e) = \bigoplus_{j\in\mathbb{Z}} \mathcal{U}(\mathfrak{g},e)_j$$

- When θ is generic [BGK] defined analogue of BGG category $\mathcal{O}(\theta, e)$ with Verma modules

- $\mathcal{O}(\vartheta, e)$ consists of f.g. $\mathbb{Z}$-graded modules M (with t-eigenspaces) such that $\mathcal{U}(\mathfrak{g}, e)_N \cdot v = 0$, for all $v \in M$ & $N \gg 0$.

- M has a top θ-wt space which is a $\mathcal{U}(\mathfrak{l}, e)$-module.

Example. $\mathfrak{g} = \mathfrak{gl}(m)$

e nilpotent of type $\lambda \vdash m$

- Take $m = 5$ and $\lambda = (3,2)$

Visualize e as left-adjusted pyramid

- $$\begin{matrix} \square & \square & \\ \square & \square & \square \end{matrix} \longmapsto \begin{matrix} \boxed{1} & \boxed{3} & \\ \boxed{2} & \boxed{4} & \boxed{5} \end{matrix} \quad \text{labelled}$$

- $e = e_{13} + (e_{24} + e_{45})$

- $h = 2(e_{11} + e_{22}) + 0 \cdot (e_{33} + e_{44}) - 2e_{55}$

Eigenvalues of $\operatorname{ad} h$ gives even good grading.

- $\mathfrak{t} = \mathfrak{g}^e = \mathbb{C}(e_{11}+e_{33}) + \mathbb{C}(e_{22}+e_{44}+e_{55})$
- $\vartheta = \vartheta_1(e_{11}+e_{33}) + \vartheta_2(e_{22}+e_{44}+e_{55})$

  $\vartheta_1, \vartheta_2 \in \mathbb{Z}$ & $\vartheta_1 \neq \vartheta_2$ generic

- $\mathfrak{l} \cong \mathfrak{gl}(2) \oplus \mathfrak{gl}(3)$ "row" algebra

  $\langle e_{13}, e_{31} \rangle \quad \langle e_{24}, e_{45}, e_{42}, e_{54} \rangle$

  $e = e_{13} + e_{24} + e_{45}$

  is principal nilpotent in $\mathfrak{l}$

- [BGK, 2008] conjectured that $\mathcal{O}(\theta, e)$ is equivalent to MMS category $\mathcal{N}(\xi)$. Here e principal in $\mathfrak{l} \cong \mathfrak{l}_\xi$ for Lie algebras
- Conjecture for Lie algebras proved by [Losev, 2012]

Note. This equivalence is NOT Skryabin!

- This equivalence can be generalized to basic Lie superalgebras using results of [Shu - Xiao, 2020]

- Let $\lambda \vdash k$

- $\mathcal{U}$-module homomorphisms

$$\begin{tikzcd} \Lambda^{\lambda'}(\mathbb{V}) \arrow[r] & P^{\lambda}(\mathbb{V}) \arrow[r] & S^{\lambda}(\mathbb{V}) \\ & \text{Irred. polynomial } \mathcal{U}\text{-module corresp. to } \lambda \arrow[u] & \end{tikzcd}$$

There exists standard basis in $P^{\lambda}(\mathbb{V})$ & dual canonical basis compatible w/ $S^{\lambda}(\mathbb{V})$
[Brundan-Kleshchev 2006]

Suppose $\mathfrak{g} = \mathfrak{gl}(m)$ & e is of type λ

- $P^{\lambda}(\mathbb{V})$ is categorified by the category of finite $U(\mathfrak{g},e)$-module.

dual canonical basis $\longleftrightarrow$ f.d. simple

<u>Note.</u> $P^{(m)}(\mathbb{V}) = S^{m}(\mathbb{V})$ [Kostant]
$P^{(1,\dots,1)}(\mathbb{V}) = \Lambda^{m}(\mathbb{V})$ [f.d. $\mathfrak{g}$-modules]

Theorem G. (C - Wang, 2026)

- $\mathfrak{g}$: $\mathfrak{gl}(m|n)$
- M : tensor product of $\mathcal{U}$-modules of the form : $\bigwedge^{\lambda}(\mathbb{V})$, $\bigwedge^{\mu}(\mathbb{W})$

- M is categorified by a parabolic subcategory of the BGK category $\mathcal{O}(M,e) \subseteq \mathcal{O}(\mathfrak{g},e)$.

- Under this correspondence :
simples in $\mathcal{O}(M,e) \longleftrightarrow$ dual canonical basis in M

Examples.

- $\begin{array}{l} \square \\ \square \\ \vdots \\ \square \end{array} \Bigg\} m$ $\quad \begin{array}{ll} \lambda^{(1)} = (1) & P^{(1)}(\mathbb{V}) = \mathbb{V} \\ \lambda^{(2)} = (1) & P^{(1)}(\mathbb{V}) = \mathbb{V} \\ \vdots & \vdots \\ \lambda^{(m)} = (1) & P(\mathbb{V}) = \mathbb{V} \end{array}$

$e = 0 \Rightarrow \mathcal{U}(\mathfrak{g}, e) \cong \mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{gl}(n))$

Standard modules = Verma in BGG $\mathcal{O}$

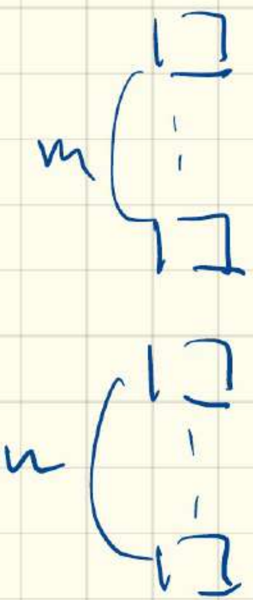
$$P^{(1)}(\mathbb{V}) \otimes \cdots \otimes P^{(1)}(\mathbb{V}) = \mathbb{V}^{\otimes m}$$

Thm G $\Rightarrow$ dual canonical in $\mathbb{V}^{\otimes m}$

$\Updownarrow$

irreducible in BGG $\mathcal{O}$

KL conjectures for type A Lie algebra

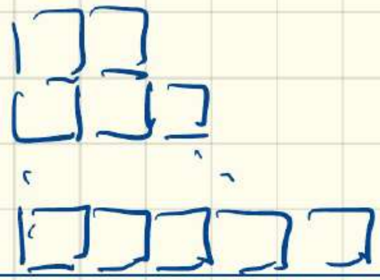
- 

  m : $\lambda^{(1)} = (1)$ $\quad \mathbb{V}$, …, $\lambda^{(m)} = (1)$ $\quad \mathbb{V}$

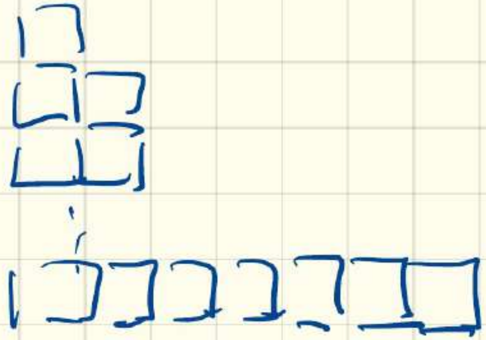
  n : $\mu^{(1)} = (1)$ $\quad \mathbb{W}$, …, $\mu^{(n)} = (1)$ $\quad \mathbb{W}$

$$\mathbb{V}^{\otimes m} \otimes \mathbb{W}^{\otimes n}$$

Thm G $\Longleftrightarrow$ Super KL conj. for type A

- 

  $$\lambda \longmapsto P^{\lambda}(\mathbb{V})$$

  $$\mu \longmapsto P^{\mu}(\mathbb{W})$$

KL polyn. in $P^{\lambda}(\mathbb{V}) \otimes P^{\mu}(\mathbb{W})$ gives irred. character of f.d. $\mathcal{U}(\mathfrak{gl}(m|n),e)$-modules

Comments.

Important ingredients of proof

- Categorification of $S^\lambda(\mathbb{V}) \otimes S^\mu(\mathbb{W})$ via Whittaker modules [CCM, 2023]
- Super version of Losev decomposition [SX, 2020] [CW, 2025]
- GT character formula for Verma [BK, 2008] & [Lu-Peng, 2026]

- Case of one single tensor factor [BK, 2006]
- Comments on Webster's orthodox basis

Thank you for
your attention!