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THE HOFFMAN-SINGLETON GRAPH AND ITS AUTOMORPHISMS

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ABSTRACT. We describe the Hoffman-Singleton graph geometrically, showing that it is closely related to the incidence graph of the affine plane over $\mathbb{Z}_5$. This allows us to construct all automorphisms of the graph.

1. INTRODUCTION

The Hoffman-Singleton graph is the unique Moore graph of order 50, degree 7, diameter 2 and girth 5. A number of different constructions of the graph can be found for example in [1, 2, 3, 4, 5, 8],

McKay, Miller and Širáň [10] showed that the Hoffman-Singleton graph fits into a family of vertex-transitive non-Cayley graphs of order $2q^2$ where $q \equiv 1 \mod 4$ is a prime power. Their construction, though expressed in terms of voltage graphs, is a direct generalisation of Robertson's 'pentagons and pentagrams' construction, replacing $\mathbb{Z}_5$ by a finite field GF(q), $q \equiv 1 \mod 4$. We will show that behind the Robertson construction (and its generalisation [10]) lies the incidence graph of the affine plane over $\mathbb{Z}_5$. Once this connection to geometry is made, it is elementary and easy to work with Robertson's construction which until now seems to have been only a curiosity, described by Benson and Losey [1] in the following words:

"Although this construction is elegant, it is not easy to work with algebraically. For example it is not clear what automorphism groups [the graph] admits."

The Hoffman-Singleton graph has girth 5, whereas the girth of all other members of the family of McKay-Miller-Širáň graphs is 3. As a consequence, the automorphism group of the Hoffman-Singleton graph turns out to be richer than the automorphism groups of the McKay-Miller-Širáň graphs in general [6]. We can recover all automorphisms, using the affine geometry and the uniqueness result for the Hoffman-Singleton graph.

2. A CONSTRUCTION OF THE HOFFMAN-SINGLETON GRAPH

For the sake of convenience we recall Robertson's *pentagons and pentagrams* construction of the Hoffman-Singleton graph (cf. Figure 1): the 50 vertices are grouped into 5 pentagons $P_1, \dots, P_5$ and 5 pentagrams $Q_1, \dots, Q_5$ (labeled so that the pentagrams are the complements of the pentagons); there are no edges between any two distinct pentagons, nor between any two distinct pentagrams.

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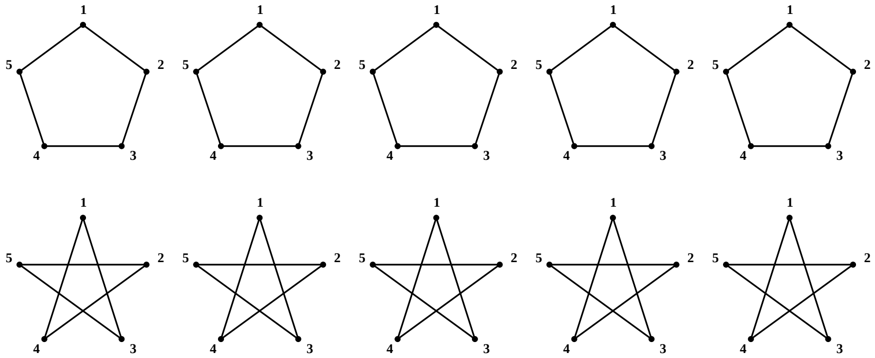

FIGURE 1. Robertson's description of the Hofmann-Singleton graph

Edges between pentagon and pentagram vertices are defined by the rule:

(2.1) vertex i of pentagon P_j is adjacent to vertex $i+jk$ of pentagram Q_k.

Here, $i+jk$ is calculated modulo 5. We will show that the connections between the two halves are given by the edges in the incidence graph of an affine plane over $\mathbb{Z}_5$ after removing all the lines of a distinguished parallel class (but not the points incident with them). We represent the points of the affine plane as triples $(0,x,y) \in \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5$, and the lines $y = mx + c$ as triples $(1,m,c) \in \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5$ (the vertical lines $x = c$ constitute the distinguished parallel class and are omitted). Figure 2 gives a rough indication of the ideas; the hollow dots are a reminder that we have discarded a parallel class of lines—they do not form part of the graph.

Theorem 2.1. *Let G be the graph with vertex set $\mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5$ and adjacencies defined as follows:*

$$(0,x,y) \text{ is adjacent to } (0,x,y') \text{ if and only if } y - y' = \pm 1; \tag{2.2}$$

$$(1,m,c) \text{ is adjacent to } (1,m,c') \text{ if and only if } c - c' = \pm 2; \tag{2.3}$$

$$(0,x,y) \text{ is adjacent to } (1,m,c) \text{ if and only if } y = mx + c. \tag{2.4}$$

Then G is isomorphic to the Hoffman-Singleton graph.

Remark 2.1. Note that the edges between affine points connect vertices which lie on a line of the distinguished (vertical) parallel class; so information about this class is retained in G in a coded form. We will refer to adjacencies of the types (2.2) or (2.3) as *vertical adjacencies.* There are no edges between points lying on distinct vertical lines, nor between lines belonging to distinct parallel classes.

Remark 2.2. Looking at the formulas rather than their geometric interpretation, it is clear that we are dealing with Robertson's construction of the Hoffman-Singleton graph. The form (2.4) of Robertson's rule (2.1) makes explicit that we are dealing with incidence of points and lines. (The reader will

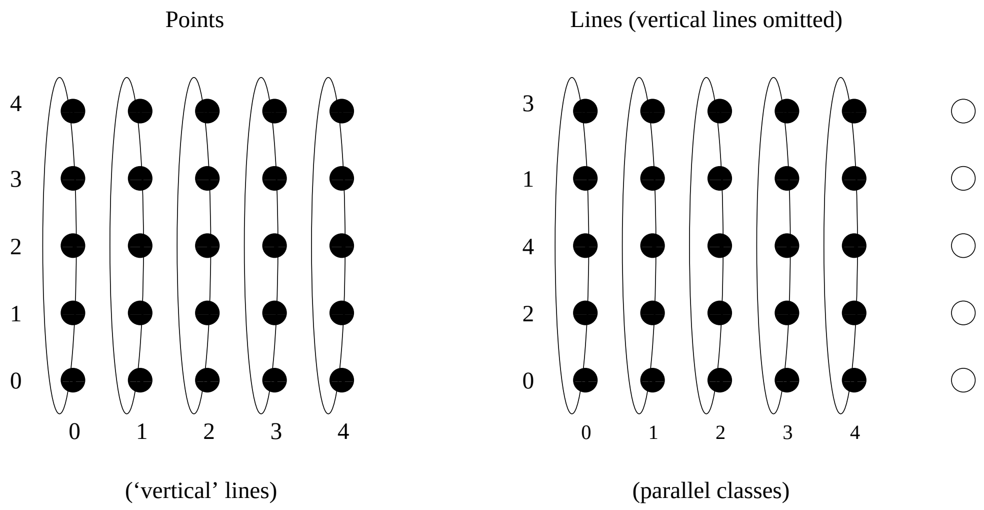

FIGURE 2. Schematic view of the Hoffman-Singleton graph

notice a minor discrepancy between (2.1) and (2.4), which is purely a renumbering of the pentagrams, ensuring that equations of lines have the standard form $y = mx + c$ rather than $c = y + mx$.)

Remark 2.3. With an eye on the more general situation of the McKay-Miller-Širáň graphs, we note that the $\pm 1, \pm 2$ in (2.2), (2.3) should be read as *is a square*, resp. *is a non-square* in $\mathbb{Z}_5$. This identifies the subgraphs in question as a Paley graph, resp. complement of a Paley graph (which are well-known to be isomorphic—and in our case are of course 5-cycles).

Remark 2.4. Let ℓ, ℓ' be two parallel lines with equations $y = mx + c$ and $y' = mx + c'$, respectively. Then $y - y' = c - c'$, and from this it follows that if $p = (0, x, y)$ and $p' = (0, x, y')$ are the points of intersection of ℓ, ℓ' with a distinguished (=vertical) line, then p, p' are adjacent in G if and only if ℓ, ℓ' are *not* adjacent in G (i.e. adjacency of lines is inherited from adjacency of points). It follows that any collineations of the affine plane which respect the vertical adjacencies *of points* automatically also respect the vertical adjacencies of lines.

Remark 2.5. While the definition of G is given in algebraic terms, it could equally well have been phrased in more geometric language. It is evident that we are dealing with a modified incidence graph of the affine plane over $\mathbb{Z}_5$. In the following proof, we emphasize this aspect by using geometric language, rather than algebra.

Proof of Theorem 2.1. Clearly, G has order 50 and is regular of valency 7. For fixed a, the vertices $(0, a, b)$ form a 5-cycle, similarly the vertices $(1, m, c)$ for fixed m. This rules out any 3- or 4-cycles involving only points or only lines.

To determine the diameter of G, we need therefore only check the distance between vertices $p = (0, a, b)$ and $\ell = (1, m, c)$. If p is a point on the line ℓ then the distance is 1. If the point of intersection of ℓ with the line $x = a$ is adjacent to p, then we have a path of length 2 from p to ℓ. If the point of intersection of ℓ with the line $x = a$ is not adjacent to p then let ℓ' be the parallel to ℓ through p. Then ℓ and ℓ' are adjacent lines, and we have again a path of length 2 from p to ℓ.

To determine the girth of G we note first of all that there are no triangles in G: a triangle could not consist of 'point' vertices $(0, a, b)$ only, nor of 'line' vertices $(1, m, c)$ only, because any connected set consisting of points only (or consisting of lines only) is part of a 5-cycle without chords. If the points p and p' are adjacent, then they lie on a distinguished line; if both of them are adjacent to a line ℓ then this line has two distinct points of intersection with the distinguished line. Similarly one rules out triangles consisting of two adjacent lines and a point: adjacent lines are parallel and therefore have no point in common.

It remains to rule out 4-cycles. A 4-cycle would have to be of the form $p - \ell - p' - \ell'$ or of the form $p - p' - \ell - \ell'$. In the first case, when it alternates between points and lines, we find that each of ℓ and ℓ' is the line joining the two points p, p', so we don't have a cycle after all (or both p, p' are points of intersection of ℓ, ℓ'). In the second case we have two adjacent lines passing through two adjacent points, contrary to our observation in Remark 2.4.

Now we invoke the uniqueness theorem [8]: any regular graph of valency 7, order 50, diameter 2 and girth 5 is isomorphic to the Hoffman-Singleton graph. □

In section 4 we will use the affine geometry to determine the automorphism group of G.

3. 1260 pentagons, 126 sets of 10 disjoint pentagons

The results in this section are well-known; we do the enumeration as an exercise in the geometric approach, and because we will require the sets of disjoint pentagons later.

To count the pentagons in the graph G we first of all note that there are 10 obvious pentagons, the five pentagons $P_1, \ldots, P_5$ consisting of points, and the 5 pentagons $Q_1, \ldots, Q_5$ consisting of lines. Now we distinguish cases according to how many vertices of a pentagon lie on one of these special pentagons.

It is impossible for a pentagon to have exactly 4 vertices in common with a pentagon P_i, because this implies that 2 vertices in P_i have a line as common neighbour, i.e. the line joining them is not in the distinguished parallel class.

If a pentagon has precisely three vertices in common with P_i then these vertices must form a path of length 2; the endpoints of this path are adjacent to a unique path of length 1 in each of the pentagons Q_j (geometrically: for any non-distinguished direction, there is a pair of parallels through the endpoints of the path of length 2, and since the endpoints are non-adjacent, the lines are adjacent). Since we can reverse the roles of P_i and Q_j, we count $25 \times 5 \times 2 = 250$ possibilities of this kind.

The only other alternative that remains is of the kind $p_1 - p_2 - \ell_1 - p_3 - \ell_2$ (or its mirror-image, lines replacing points and vice versa), which really is characterised by the three points, two of them adjacent on a distinguished line, and the other one on another distinguished line. That's 25 possibilities for the distinguished edge, each combined with 20 possibilities for a third point, and the mirror-image possibilities: $25 \times 20 \times 2 = 1000$.

Altogether we have found $10+250+1000 = 1260$ pentagons in the Hoffman-Singleton graph.

If P is any pentagon in G then there are 25 distinct vertices not in P adjacent to some vertex of P, call this set V_1. The complement of V_1 is again a set of 25 vertices, V_0 say (this includes the pentagon P). With little effort one can establish that V_0 and V_1 each consist of five 5-cycles without edges between them. It follows that each pentagon in G uniquely determines a set of 10 disjoint pentagons. Since there are 1260 pentagons altogether, we have 126 sets of 10 disjoint pentagons in G.

4. 252,000 AUTOMORPHISMS

In this section we apply the geometric description of the Hoffman-Singleton graph to determine its automorphisms. It is clear that all affine collineations which fix the distinguished direction and preserve the vertical adjacencies induce an automorphism of our graph G. The number of such collineations can be counted as follows.

To preserve the 'vertical' direction, the second standard basis vector must be an eigenvector; and for the adjacencies on that vertical line to be preserved, the corresponding eigenvalue must be a square. This gives 2 possibilities: ± 1.

The first standard basis vector can be mapped onto any vector which is linearly independent of the second one; that gives $25 - 5 = 20$ possibilities.

Then there are $5^2 = 25$ possible translations. Altogether we have a group H of $2 \times 20 \times 25 = 1000$ affine collineations which fix the distinguished direction and respect vertical adjacencies. H has 2 orbits: the set of 25 points and the set of 25 lines. It is straightforward to write the elements of H down explicitly: if the point (x, y) undergoes a transformation

$$(x, y) \mapsto (x, y) \begin{pmatrix} a & b \\ 0 & d^2 \end{pmatrix} + (e, f)$$

one can calculate the transformed slopes and y-intercepts of the (non-vertical) lines. The result is:

$$\begin{aligned} (x, y) &\mapsto \left(ax + e,\ bx + d^2 y + f\right), \\ (m, c) &\mapsto \left(\frac{b + d^2 m}{a},\ d^2 c + f - \frac{b + d^2 m}{a} e\right). \end{aligned}$$

The following mapping is a correlation (i.e. an incidence preserving mapping which maps points onto lines and vice versa) which also preserves vertical adjacencies:

$$(0, x, y) \mapsto (1, 3x, 2y), \qquad (1, m, c) \mapsto (0, m, 2c). \tag{4.1}$$

Together with the previous automorphisms this generates a group $\hat{H}$ of 2000 automorphisms of the graph G. $\hat{H}$ acts transitively on G.

In order to find *all* automorphisms of G, we revisit the uniqueness theorem for the Hoffman-Singleton graph. Inspection of proofs [4, 9] of this result reveals that a stronger conclusion is actually reached than what might briefly be stated as: there is (up to isomorphism) only one Moore graph of order 50. Just as at the end of section 3 one might start with a 5-cycle, consider a neighbourhood of this cycle (which turns out to consist of 25 vertices, grouped into disjoint 5-cycles), then consider the complement of this neighbourhood (another five disjoint 5-cycles). Then it turns out that there is a unique way to join up the vertices of the 5-cycles. This means that the automorphism group of the Hoffman-Singleton graph acts transitively on the set of all sets of 10 disjoint 5-cycles. As there are 126 such sets, and the subgroup $\hat{H}$ is the stabiliser of a set of 10 disjoint 5-cycles (namely $P_1, \ldots, P_5, Q_1, \ldots, Q_5$), we have established the following theorem.

Theorem 4.1. *The Hoffman-Singleton graph G is vertex-transitive; its automorphism group has order 252,000 and contains a subgroup H of order 1000 of the affine group $AGL(2, 5)$. This subgroup H has two orbits and has an extension $\hat{H}$ of order 2000 which acts transitively on G.*

We note that the isomorphism type of the automorphism group of the Hoffman-Singleton graph is obtained in [7].

5. Conclusion

We have shown how to derive the Hoffman-Singleton graph and its automorphisms from the affine plane over $\mathbb{Z}_5$ via the incidence graph of the plane (after removing a parallel class of lines). A similar discussion is possible for the McKay-Miller-Širáň graphs in general [6].

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