Addendum to “The torsion of the group of homeomorphisms of powers of the long line”

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Abstract

In this addendum we correct an argument concerning the torsion of homeomorphisms of finite powers of the long line. In the original paper Alexander-Spanier cohomology was used but instead Alexander-Spanier cohomology with compact supports is required.
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The proofs of the following two results in [2] rely on unproved statements about P.A.Smith theorems:

THEOREM 3.2. *Let L denote either $\mathbb{L}_+$ or $\mathbb{L}_o$. Then the group $\mathcal{H}(\mathbb{L})$ has no torsion.*

THEOREM 3.3. *The group $\mathcal{H}(\mathbb{L})$ has only 2-torsion, that is, any nontrivial element of finite order must be of order 2.*

The proofs of both theorems as given in [2] use P.A.Smith fixed point theorems for p-power groups. The problem arises because all of the spaces $\mathbb{L}_+$, $\mathbb{L}_o$ and $\mathbb{L}$ are locally compact but not paracompact. It is true that all of these spaces are of finite cohomological dimension and so the Smith theorems apply, but the cohomology used should be Alexander-Spanier cohomology with compact supports [1] and [4, p320], rather than the general cohomology. It is not known whether the Smith theorems are true for the long line using Alexander-Spanier cohomology with general supports. In any case the proofs in [2] are easily modified when we use Alexander-Spanier cohomology $\bar{H}_c$ with compact supports.

Following the arguments used in proving [2, Theorem 2.5], one can easily establish the following results:

THEOREM A. *Let L denote either $\mathbb{L}_+$ or $\mathbb{L}_o$. Then $\bar{H}_c^q(L^n; G) = 0$ for all $q \geq 0$.*
THEOREM B: $\bar{H}_c^q(\mathbb{L}^n; G) = \begin{cases} G & \textit{if } q = n \\ 0 & \textit{if } q \neq n. \end{cases}$

PROOF. (THEOREM 3.2): It suffices to show that $\mathcal{H}(\mathbb{L})$ has no p-torsion for any prime p. Suppose to the contrary that there is a homeomorphism h of order p. Then by [3, Lemma 2] the fixed point set L^h contains an unbounded subset of ω_1. If $\alpha < \beta$ are fixed points then $h[\alpha, \beta]$ must be an interval containing α and β so $[\alpha, \beta]$ must be invariant under h. Applying Smith theory to the (Euclidean) interval $[\alpha, \beta]$ we conclude that $[\alpha, \beta]^h$ must be acyclic with respect to Alexander-Spanier cohomology. Thus h fixes every point of $[\alpha, \beta]$. It follows that h is the identity, a contradiction. $\square$

PROOF. (THEOREM 3.3) Suppose that $h \in \mathcal{H}(L)$ is an element of order p^k where p is prime and $k > 0$. We will show that p=2 and k=1. Since $\mathbb{L}$ is locally compact and is of finite cohomological dimension, we apply the Smith theorem for Alexander-Spanier cohomology with compact supports, $\bar{H}_c$ (mod p). By Theorem B, $\mathbb{L}$ is a cohomology 1-disc(mod p) with respect to $\bar{H}_c$ and hence by the Smith theorem the fixed point set $\mathbb{L}^h$ must be an r-disc (mod p) for $\bar{H}_c$, where $0 \leq r \leq 1$.

Suppose that r=1. Then $\mathbb{L}^h$, being closed, must be either unbounded by [3, Lemma 2] or must be compact. In the case that $\mathbb{L}^h$ is unbounded, the argument of the previous theorem shows that h is the identity, a contradiction. Thus $\mathbb{L}^h$ must be compact. It follows that r=0 and hence by the Smith Parity Theorem p=2. It also follows that $\mathbb{L}^h$ is connected, so that $\mathbb{L}^h = [\alpha, \beta]$ for some $\alpha \leq \beta$.

Because h is a homeomorphism of order 2^k we cannot have $\alpha < \beta$. Then it follows as before that h^2 is the identity. $\square$

References

[1] A. Borel, *Seminar on transformation groups*, Ann. of Math. Stud. 46 (Princeton University Press, Princeton, 1960).

[2] S. Deo and D. Gauld, *The Torsion of the Group of Homeomorphisms of Powers of the Long Line*, J. Austral. Math. Soc. 2001, 311-322.

[3] D. Gauld, 'Homeomorphisms of 1-manifolds',in: *Papers on General Topology and Applications* (Ann. New York Acad. Sci., New York, 1993) pp. 142-149.

[4] E. H. Spanier, *Algebraic topology*, (McGraw Hill, New York, 1966).

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