Covering Properties and Metrisation of Manifolds*

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Abstract

In the context of topological manifolds we consider a number of new topological properties involving covers of a space. Several of these conditions are equivalent to metrisability for manifolds. The recently introduced topological notion of property (a) is related to absolute countable compactness and also to normality. We show that not all manifolds possess property (a), that a manifold is metrisable if it satisfies a stronger version of property (a) and that all manifolds possess a weakened form of property (a). We also consider near metaLindelöfness and combine it with linear Lindelöfness to obtain the condition nearly linearly metaLindelöf.

1 Introduction

Throughout this paper, when we use the term manifold we mean a locally euclidean, connected, Hausdorff space.

Matveev, [6], introduced the following definitions (except [strongly] wa-favourable and wpp) and showed how they are related to one another and to paracompactness, see Proposition 2.1. In this section we investigate these relationships in the context of topological manifolds.

Definition 1.1 *A space X* has property (a) *(respectively,* has property (wa), "weak (a)"*) provided that for every open cover $\mathcal{U}$ of X and every dense subset $D \subset X$ there is a subset $F \subset D$ such that F is a closed and discrete (respectively, is a discrete) subspace of X and $st(F,\mathcal{U}) = X$.*

Definition 1.2 *A space X* is a-favourable *(respectively,* is wa-favourable*) provided that for every open cover $\mathcal{U}$ of X there is a winning strategy for the second player in the following topological game: at the αth step the first player chooses a dense subspace $D_\alpha \subset X$ then the second player chooses a point $x_\alpha \in D_\alpha$; the second player wins if for some α the set $F_\alpha = \{x_\beta \ / \ \beta < \alpha\}$ is closed and discrete (respectively, is discrete) in X and $st(F_\alpha,\mathcal{U}) = X$.*

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Definition 1.3 *A space X* is strongly a-favourable *(respectively,* is strongly wa-favourable*) provided that for every open cover* $\mathcal{U}$ *of* X *there is a winning strategy for the first player in the following topological game: at the* α*th step the first player chooses a non-empty open set* $O_\alpha \subset X$ *then the second player chooses a point* $x_\alpha \in O_\alpha$*; the first player wins if for some* α *the set* $F_\alpha = \{x_\beta \ / \ \beta < \alpha\}$ *is closed and discrete (respectively, is discrete) in* X *and* $st(F_\alpha, \mathcal{U}) = X$*.*

Definition 1.4 *A space* X has property pp *(respectively,* has property wpp*) provided that every open cover* $\mathcal{U}$ *of* X *has an open refinement* $\mathcal{V}$ *consisting of non-empty sets such that for every choice function* $f : \mathcal{V} \to X$ *(ie* $f(V) \in V$ *for each* $V \in \mathcal{V}$*) the set* $f(\mathcal{V})$ *is closed and discrete (respectively, is discrete) in* X*.*

The next two definitions are found in [4] and [1] respectively.

Definition 1.5 *A space* X *is* nearly metaLindelöf *if every open cover of* X *has an open refinement which is point-countable at the points of some dense subset.*

Definition 1.6 *A space* X *is* linearly Lindelöf *if every open cover of* X *which is a chain has a countable subcover.*

Combining these two we propose the following:

Definition 1.7 *A space* X *is* nearly linearly metaLindelöf *if every open cover of* X *which is a chain has an open refinement which is point-countable at the points of some dense subset.*

In this paper we prove the following result

Theorem 1.8 (Main Theorem) *Let* M *be a manifold. Then the following are equivalent.*

(i) M *is metrisable;*

(ii) M *has property pp;*

(iii) M *has property wpp;*

(iv) every open cover of M *has an open refinement* $\mathcal{V}$ *consisting of non-empty sets such that for every choice function* $f : \mathcal{V} \to M$ *the set* $f(\mathcal{V})$ *is closed in* M*;*

(v) M *is nearly metaLindelöf;*

(vi) there is a base $\mathcal{B}$ *for the topology on* M *and a dense set* $D \subset M$ *such that for each* $x \in D$ *the family* $\{B \in \mathcal{B} \ / \ x \in B\}$ *is countable.*

While we show that property (a) is strictly weaker than metrisability we exhibit a manifold which does not have property (a). By way of contrast we find that every manifold has property (wa); indeed, every manifold is strongly wa-favourable.

2 Property (a) and related properties

The following result, with the exception of (b)⇒(c), is discussed in [6]. We note that every space with property pp is also T_1.

Proposition 2.1 *For any space X each of the following conditions implies the next:*
(a) X is paracompact and T_1;
(b) X has property pp;
(c) X is strongly a-favourable;
(d) X is a-favourable;
(e) X has property (a).

Proof: We prove only (b)⇒(c). Let $\mathcal{U}$ be any open cover of X and suppose that $\mathcal{V}$ is an open refinement as in the definition of property pp. Index $\mathcal{V}$ by ordinals: $\mathcal{V} = \{V_\alpha \ / \ \alpha < \delta\}$ for some ordinal δ, with $V_\alpha \neq V_\beta$ whenever $\alpha \neq \beta$. Then the strategy of the first player should be to choose the set V_α at stage α. The second player will then be forced to construct a partial choice function $f_\alpha : \mathcal{V}_\alpha \to X$, where $\mathcal{V}_\alpha = \{V_\beta \ / \ \beta < \alpha\}$. As every extension of f_α to a full choice function f is such that $f(\mathcal{V})$ is closed and discrete it follows that $f_\alpha(\mathcal{V}_\alpha)$ is also closed and discrete. The game must stop at some stage as $st(f_\delta(\mathcal{V}_\delta), \mathcal{V}) = X$; at this stage the first player has won. ■

Remark. We may insert the symbol w in each of conditions (b) to (e).

When $\kappa = \aleph_0$, the following proposition gives a partial converse to the implication (a)⇒(b) of Proposition 2.1.

Proposition 2.2 *Let κ be any cardinal with $\kappa > 1$. Suppose that X is a space with property pp, with character at most κ and having at most κ many isolated points. Then any open cover of X has an open refinement which is locally $< \kappa$.*

Proof: Let $\mathcal{U}$ be an open cover of X and let $\mathcal{V}$ be an open refinement as in the definition of property pp. We firstly show that $\mathcal{V}$ is locally $< \kappa$ at non-isolated points.

Suppose instead that $x \in X$ is a non-isolated point such that every neighbourhood of x meets at least κ many members of $\mathcal{V}$. Let $\mathcal{N}_x$ be a neighbourhood basis at x of cardinality at most κ. Then there is an injection $g : \mathcal{N}_x \to \mathcal{V}$ such that $g(N) \cap N \neq \varnothing$ for each $N \in \mathcal{N}_x$.

Let $f : \mathcal{V} \to X$ be any choice function such that $f(g(N)) \in g(N) \cap N - \{x\}$ if $N \in \mathcal{N}_x$ and $f(V) \in V - \{x\}$ if $V \notin g(\mathcal{N}_x)$. The sets $g(N) \cap N - \{x\}$ and $V - \{x\}$ are non-empty as x is not an isolated point. Then $x \notin f(\mathcal{V})$ but each neighbourhood of x meets $f(\mathcal{V})$. Thus $\overline{f(\mathcal{V})} - f(\mathcal{V})$ is non-empty so $f(\mathcal{V})$ is not even closed, giving a contradiction.

Now let

$$S = \{x \in X \ / \ \text{each neighbourhood of } x \text{ meets at least } \kappa \text{ many members of } \mathcal{V}\}.$$

By what we have just shown, S consists entirely of isolated points. Thus S has cardinality at most κ so there is a choice function $f : \mathcal{V} \to X$ such that

$S \subset f(\mathcal{V})$. As $f(\mathcal{V})$ is closed and discrete, it follows that S is closed. Let

$$\mathcal{W} = \{V - S \ / \ V \in \mathcal{V}\} \cup \{\{x\} \ / \ x \in S\}.$$

Then $\mathcal{W}$ is an open refinement of $\mathcal{U}$ such that each point of X has a neighbourhood meeting fewer than κ many members of $\mathcal{W}$. ∎

We can now prove the equivalence of conditions (i) and (ii) in Theorem 1.8. By [7, Theorem 2.5] and Proposition 2.1 it suffices to show that property pp implies paracompactness for a manifold. If M has property pp then M has no isolated points and is first countable, and hence by Proposition 2.2 M is paracompact.

Remark. In the part of the proof of Proposition 2.2 dealing with non-isolated points we did not use the full strength of property pp; we need only that the image of the choice function should be closed. In fact by the following lemma we could just as well replace the word "closed" by the word "discrete." This proves the equivalence of conditions (ii), (iii) and (iv) of Theorem 1.8.

Lemma 2.3 *Suppose that M is a manifold. Then the following three conditions are equivalent:*
(a) M has property pp;
(b) every open cover of M has an open refinement $\mathcal{V}$ such that for every choice function $f : \mathcal{V} \to M$ the set $f(\mathcal{V})$ is closed in M;
(c) every open cover of M has an open refinement $\mathcal{V}$ such that for every choice function $f : \mathcal{V} \to M$ the set $f(\mathcal{V})$ is discrete in M.

Proof: It suffices to show that (b) and (c) are equivalent.
(b)⇒(c): Suppose that $\mathcal{V}$ is an open refinement as in (b) and $f : \mathcal{V} \to M$ is a choice function. If $f(\mathcal{V})$ were not discrete then there would be $x \in f(\mathcal{V})$ such that each neighbourhood of x meets $f(\mathcal{V})$ in some point other than x. Define a new choice function $g : \mathcal{V} \to M$ by setting $g(V) = f(V)$ whenever $f(V) \neq x$ and choosing $g(V) \in V - \{x\}$ whenever $f(V) = x$. Then $g(\mathcal{V})$ cannot be closed as $x \in \overline{g(\mathcal{V})} - g(\mathcal{V})$.
(c)⇒(b): Suppose that $\mathcal{V}$ is an open refinement as in (c) and $f : \mathcal{V} \to M$ is a choice function. If $f(\mathcal{V})$ were not closed then there would be $x \in \overline{f(\mathcal{V})} - f(\mathcal{V})$. Choose some $V_x \in \mathcal{V}$ such that $x \in V_x$ and define a new choice function $g : \mathcal{V} \to M$ by setting $g(V) = f(V)$ whenever $V \neq V_x$ and $g(V_x) = x$. Then $x \in g(\mathcal{V})$ but every neighbourhood of x meets $g(\mathcal{V})$ in some point other than x so $g(\mathcal{V})$ is not discrete. ∎

Example 2.4 *The long line, L, is a non-metrisable manifold which is strongly a-favourable.*

Denote by Λ the set of limit ordinals of ω_1. Intervals are intended to be in L rather than just ω_1. Let $\mathcal{U}$ be an open cover of L. Define $f : \Lambda \to \omega_1$ by

$$f(\lambda) = min\{\alpha \in \omega_1 \ / \ \exists U \in \mathcal{U} \text{ such that } (\alpha, \lambda] \subset U\}.$$

As $\mathcal{U}$ is an open cover of L it follows that $f(\lambda) < \lambda$ so by the Pressing Down Lemma there is $\delta \in \omega_1$ such that $f^{-1}(\delta)$ is uncountable. Then $(\delta, \omega_1) \subset st(x, \mathcal{U})$ for any $x > \delta$. Similarly there is $\gamma \in L$ such that $-\gamma \in \omega_1$ and $(-\omega_1, \gamma) \subset st(x, \mathcal{U})$ for any $x < \gamma$. As $[\gamma, \delta]$ is compact there is a finite subfamily $\{U_1, \ldots, U_n\} \subset \mathcal{U}$ which covers $[\gamma, \delta]$.

The first player wins the game if the following strategy is followed. Let $O_0 = (-\omega_1, \gamma)$, for $i = 1, \ldots, n$ let $O_i = U_i$, and let $O_{n+1} = (\delta, \omega_1)$. It is claimed that at this stage the first player already wins. Indeed, suppose that $x_i \in O_i$ for each $i = 0, \ldots, n+1$ and let $F_{n+2} = \{x_i \ / \ i < n+2\}$. As a finite subset of L, F_{n+2} is both closed and discrete. Further, $st(F_{n+2}, \mathcal{U}) = L$ because if $x \in L$ then $x \in O_i$ for some i and hence $x \in st(x_i, \mathcal{U})$. ∎

The following result should be compared with [5] where there is constructed a Tychonoff space which does not have property (wa).

Theorem 2.5 *Every manifold is strongly wa-favourable.*

Proof: Let M be an n-manifold and let $\mathcal{U}$ be an open cover of M. We may assume that each $U \in \mathcal{U}$ is homeomorphic to $\mathbb{R}^n$.

Construct a sequence $\langle C_\alpha \rangle_{\alpha<\delta}$ of open n-cells in M as follows. C_0 is any open n-cell in M. Now suppose given an ordinal α such that C_β has been defined for each $\beta < \alpha$ such that each C_β is an open n-cell in M and if $\beta \neq \gamma$ then $C_\beta \cap C_\gamma = \varnothing$. If $\overline{\cup_{\beta<\alpha} C_\beta} = M$ then $\delta = \alpha$ and we end the construction. If $\overline{\cup_{\beta<\alpha} C_\beta} \neq M$ then we may choose an open n-cell $C_\alpha \subset M - \overline{\cup_{\beta<\alpha} C_\beta}$. There is some ordinal δ at which this construction stops.

For each $\alpha < \delta$ pick a homeomorphism $h_\alpha : C_\alpha \to \mathring{B}^n$ (B^n being the ball of radius 1 in euclidean space) and divide C_α into an open cell $A_{\alpha,0} = h_\alpha^{-1}(\frac{1}{2}\mathring{B}^n)$ and concentric open annuli with

$$A_{\alpha,l} = h_\alpha^{-1}(\frac{l+1}{l+2}\mathring{B}^n - \frac{l}{l+1}B^n) \quad (l \in \omega - \{0\}).$$

Of course the open sets $A_{\alpha,l}$ do not cover C_α but the closure of their union contains all of C_α. Choose open cells $B_{\alpha,l,m} \subset A_{\alpha,l}$ ($m \in \omega$) so that

- the sets $B_{\alpha,l,m}$ ($m \in \omega$) are mutually disjoint;
- $\overline{\cup_{m\in\omega} B_{\alpha,l,m}}$ contains $A_{\alpha,l}$;
- $\mathrm{diam} h_\alpha(B_{\alpha,l,m}) \to 0$ as $m \to \infty$;
- $\mathrm{dist}(h_\alpha(B_{\alpha,l,m}), \frac{l+1}{l+2}S^{n-1}) \to 0$ as $m \to \infty$.

For example we may choose a sequence of spheres in the annulus between the spheres of radii $\frac{l}{l+1}$ and $\frac{l+1}{l+2}$, converging outwards to the latter sphere, then subdivide each annulus between consecutive members of the sequence into a finite number of disjoint open cells, where the number of cells in such an annulus increases as we approach the outer sphere.

For each $\alpha < \delta$ and $l \in \omega$ let $\mathcal{U}_{\alpha,l} = \{U \cap A_{\alpha,l} \ / \ U \in \mathcal{U}\}$. Then $\mathcal{U}_{\alpha,l}$ is an open cover of the paracompact space $A_{\alpha,l}$. Let $\mathcal{V}_{\alpha,l}$ be a locally finite open refinement; we will assume that $\mathcal{V}_{\alpha,l}$ does not contain the empty set. Let $\mathcal{W}_{\alpha,l} = \mathcal{V}_{\alpha,l} \cup \{B_{\alpha,l,m} \ / \ m \in \omega\}$. Then $\mathcal{W}_{\alpha,l}$ is a locally finite collection of open subsets of $A_{\alpha,l}$. Now let $\mathcal{W} = \cup\{\mathcal{W}_{\alpha,l} \ / \ \alpha < \delta \text{ and } l \in \omega\}$.

Suppose that $f : \mathcal{W} \to M$ is a choice function. Because each set $A_{\alpha,l}$ is open and the collection $\mathcal{W}_{\alpha,l}$ is locally finite and all of the sets in it lie in $A_{\alpha,l}$, it follows that $f(\mathcal{W})$ is discrete. Moreover, $st(f(\mathcal{W}),\mathcal{U}) = M$. Indeed, if $x \in M$ then there are three possibilities:

1. $x \in A_{\alpha,l}$ for some $\alpha < \delta$ and $l \in \omega$. In this case,
$$x \in \cup\mathcal{V}_{\alpha,l} = st(f(\mathcal{V}_{\alpha,l}),\mathcal{V}_{\alpha,l}) \subset st(f(\mathcal{W}),\mathcal{U}).$$
2. $x \in \partial A_{\alpha,l}$ for some $\alpha < \delta$ and $l \in \omega$; we avoid possible ambiguity for l by assuming that l is so chosen that $h_\alpha(x)$ is in the sphere of larger radius. Choose $U \in \mathcal{U}$ so that $x \in U$. Then there is $m \in \omega$ so that $B_{\alpha,l,m} \subset U$, so
$$x \in st(f(B_{\alpha,l,m}),\{U\}) \subset st(f(\mathcal{W}),\mathcal{U}).$$
3. $x \in M - \cup_{\alpha<\delta} C_\alpha$. Choose $U \in \mathcal{U}$ so that $x \in U$. Then there is $\alpha < \delta$ so that $U \cap C_\alpha \neq \varnothing$. Because U and C_α are both homeomorphic to $\mathbb{R}^n$ and U contains points in C_α as well as points not in C_α, it follows that there are also $l, m \in \omega$ so that $B_{\alpha,l,m} \subset U$. Again we conclude that $x \in st(f(\mathcal{W}),\mathcal{U})$.

We can now describe a winning strategy for the first player in the game defining strongly wa-favourable. Index the sets in $\mathcal{W}$ by ordinals, say $\mathcal{W} = \{W_\alpha \ / \ \alpha < \gamma\}$ for some γ, so that there is no repetition. When it is the first player's turn that player chooses the non-empty open set W_α. Then the second player, in choosing a point $f(W_\alpha) \in W_\alpha$ is constructing a partial choice function for the collection $\mathcal{W}$. As any complete choice function $f : \mathcal{W} \to M$ is such that $f(\mathcal{W})$ is discrete and satisfies $st(f(\mathcal{W}),\mathcal{U}) = M$, it follows that at some stage the first player wins. ■

Example 2.6 *A manifold which does not have property (a).*

Let $S = \{(x,y) \in \mathbb{R}^2 \ / \ x \neq 0\}$ with the usual topology from $\mathbb{R}^2$. Let $M = S \cup [\{0\} \times \mathbb{R} \times (-1,1)]$. For each $y \in \mathbb{R}$, each $t \in (-1,1)$ and each $r > 0$ let

$$W_{y,t,r} = \{(\xi,\eta) \in S \ / \ t - r < \frac{\eta - y}{\sqrt{\xi^2 + (\eta - y)^2}} < t + r \text{ and } |\xi| < r\}.$$

If in addition $-1 \leq t - r < t + r \leq 1$ then set

$$\tilde{W}_{y,t,r} = [\{0\} \times \{y\} \times (t - r, t + r)] \cup W_{y,t,r}.$$

Let $D = \{(x,y) \in S \ / \ x, y \in \mathbb{Q}\}$. Let $\{E_y \ / \ y \in \mathbb{R} - \mathbb{Q}\}$ list all subsets of D which are closed and discrete in S.

For $y \in \mathbb{R}-\mathbb{Q}$ let $L_y = \{(x, y) \,/\, x \neq 0\}$; then $E_y \cap L_y = \varnothing$. As E_y and L_y are both closed subsets of S there is a neighbourhood of L_y which is disjoint from E_y. Using this neighbourhood we may construct a homeomorphism $\theta_y : S \to S$ and a neighbourhood V_y of L_y so that $\theta_y(E_y) \cap V_y = \varnothing$ and V_y is bounded by straight lines passing through $(0, y)$: for example using [3, Lemma 2] we may choose two continuous functions $\mathbb{R} \to \mathbb{R}$ one of whose graphs lies above L_y and the other below and such that all points between the graphs lie in the neighbourhood of L_y and then use these functions to construct θ_y to be a function which moves points parallel with the y-axis in such a way that points of the graphs move to points on the lines bounding V_y. Extend θ_y to a function $\hat{\theta}_y : M \to M$ by letting $\hat{\theta}_y(0, \eta, t) = (0, \eta, t)$.

For $y \in \mathbb{Q}$ define $\hat{\theta}_y : M \to M$ to be the identity function.

Topologise M by declaring $U \subset M$ to be open if and only if $U \cap S$ is open in S and for each $(0, y, t) \in U \cap (M - S)$ there is $r > 0$ so that $\hat{\theta}_y^{-1}(\tilde{W}_{y,t,r}) \subset U$. If each of the functions $\hat{\theta}_y$ were the identity then we would obtain the double of the manifold with boundary constructed as [7, Example 3.6]. Thus M is a 2-manifold and we now show that it does not have property (a).

Clearly D is dense in M. For each $y \in \mathbb{R} - \mathbb{Q}$ choose $r_y \in (0, 1)$ such that $W_{y,0,r_y} \subset V_y$. Then $\hat{\theta}_y^{-1}(\tilde{W}_{y,0,r_y})$ is an open subset of M containing $(0, y, 0)$. Let

$$\mathcal{U} = \{\hat{\theta}_y^{-1}(\tilde{W}_{y,0,r_y}) \,/\, y \in \mathbb{R} - \mathbb{Q}\} \cup \{M - \{(0, y, 0) \,/\, y \in \mathbb{R} - \mathbb{Q}\}\}.$$

Then $\mathcal{U}$ is an open cover of M. We show that whatever subset $E \subset D$ we choose, if E is closed and discrete in M then we do not have $st(E, \mathcal{U}) = M$. Indeed, given any such E, there is $y \in \mathbb{R} - \mathbb{Q}$ such that $E = E_y$. The only member of $\mathcal{U}$ containing $(0, y, 0)$ is $\hat{\theta}_y^{-1}(\tilde{W}_{y,0,r_y})$ but as $\hat{\theta}_y(E) \cap \tilde{W}_{y,0,r_y} \subset \theta_y(E_y) \cap V_y = \varnothing$ it follows that $(0, y, 0) \notin st(E, \mathcal{U})$, hence $st(E, \mathcal{U}) \neq M$. Thus M does not have property (a). ∎

3 Near metaLindelöfness and related properties

In this section we adopt the following notation: suppose that $\mathcal{S}$ is a family of subsets of a set X and $A \subset X$. Denote the subfamily $\{S \in \mathcal{S} \,/\, A \cap S \neq \varnothing\}$ by $\mathcal{S}_A$. If $A = \{a\}$, a singleton, then we will abbreviate $\mathcal{S}_A$ to $\mathcal{S}_a$.

Lemma 3.1 *Suppose that the topological space X has a base $\mathcal{B}$ and a dense set $D \subset X$ such that for each $x \in D$ the family $\mathcal{B}_x$ is countable. Then X is nearly metaLindelöf.*

Proof. Let $\mathcal{U}$ be an open cover of X. For each $x \in X$ choose $U_x \in \mathcal{U}$ and $B_x \in \mathcal{B}$ such that $x \in B_x \subset U_x$. Let $\mathcal{V} = \{B_x \,/\, x \in X\}$. Then $\mathcal{V}$ is an open refinement of $\mathcal{U}$ which is point-countable at the points of the dense subset D. ∎

Lemma 3.2 *Suppose that the topological space X is locally hereditarily separable and nearly (linearly) metaLindelöf. Then X is (linearly) metaLindelöf.*

Proof. Suppose that $\mathcal{U}$ is an open cover of X (which is a chain). Let $\mathcal{V}$ be an open refinement of $\mathcal{U}$ and $D \subset X$ a dense subset such that $\mathcal{V}$ is point countable at each point of D. We show that $\mathcal{V}$ is point countable.

Let $x \in X$ be arbitrary. Choose a hereditarily separable open neighbourhood $O \subset X$ of x. Let E be a countable subset of $D \cap O$ which is still dense in O. Choose a function $\varphi : \mathcal{V}_x \to E$ as follows: for each $V \in \mathcal{V}_x$ we have $E \cap V \cap O \neq \varnothing$; let $\varphi(V) = e$ for some $e \in E \cap V \cap O$. Because $\mathcal{V}_e$ is countable it follows that $\varphi^{-1}(e)$ is countable for each $e \in E$. Thus $\mathcal{V}_x = \varphi^{-1}(E)$ is also countable, i.e. $\mathcal{V}$ is point countable. ∎

The following theorem provides a stronger conclusion than that of [2, Theorem 3.1] but requires stronger hypotheses. It also shows the equivalence of conditions (i) and (vi) of Theorem 1.8.

Theorem 3.3 *Let X be a connected, locally second countable, T_3 space. Then X is metrisable if and only if there is a base $\mathcal{B}$ for the topology on X and a dense set $D \subset X$ such that for each $x \in D$ the family $\mathcal{B}_x$ is countable.*

Proof. If X is metrisable then the condition regarding the basis follows from the stronger condition of [2, theorem 3.1].

Conversely suppose that the basis condition holds. By Lemma 3.1, X is nearly metaLindelöf. Then by Lemma 3.2, X is metaLindelöf. Now every connected, locally second countable, metaLindelöf space is second countable and (by Urysohn's metrisation theorem) every T_3 second countable space is metrisable. Thus X is metrisable. ∎

We can now prove the equivalence of (i) and (v) of Theorem 1.8. The implication (i)⇒(v) follows from the fact that every metrisable space is paracompact, hence (nearly) metaLindelöf. The implication (v)⇒(i) uses Lemma 3.2 to deduce that M is metaLindelöf and hence, as in the proof of Theorem 3.3, M is metrisable.

From [1, Theorem 4.1] we conclude that a manifold is metrisable if and only if it is linearly Lindelöf. This leads us to ask whether a manifold is metrisable if and only if it is linearly metaLindelöf or even nearly linearly metaLindelöf. In fact Lemma 3.2 tells us that for a manifold the conditions linearly metaLindelöf and nearly linearly metaLindelöf are equivalent so we are really only asking one question. It would appear that there are two ways one may try to attack this question:

1. In proving that a manifold is metrisable if it is metaLindelöf the simplest way seems to be to note firstly that every metaLindelöf, locally separable, connected space is Lindelöf (see Lemma 3.4 below) and then use Urysohn's Metrisation Theorem. It may be possible to construct a parallel proof using some of the properties of a manifold to show that every linearly metaLindelöf manifold is linearly Lindelöf and then appeal to [1, Theorem 4.1] to conclude that the manifold is then metrisable.

2. One may try to adapt the proof of [1, Theorem 4.1] from the linearly Lindelöf context to the linearly metaLindelöf context.

Added later: In the paper "Covering Properties and Metrisability of Manifolds 2" the author and M K Vamanamurthy have combined both of these approaches to show that every linearly metaLindelöf manifold is metrisable.

Lemma 3.4 *Every locally separable, connected, metaLindelöf space is Lindelöf.*

Proof. Suppose that X is a locally separable, connected, metaLindelöf space and let $\mathcal{U}$ be an open cover of X. By local separability we may find a collection $\mathcal{V}$ of subsets of X such that each member of $\mathcal{V}$ is separable, the interiors of the members of $\mathcal{V}$ cover X and each member of $\mathcal{V}$ lies in some member of $\mathcal{U}$. It suffices to find a countable subcollection of $\mathcal{V}$ which still covers X.

Let $\mathcal{W}$ be a point countable open cover of X which refines the cover by the interiors of the members of $\mathcal{V}$. Then there is an extending function $E : \mathcal{W} \to \mathcal{V}$ such that $W \subset E(W)$ for each $W \in \mathcal{W}$. There is also a function $D : \mathcal{V} \to \mathcal{P}(X)$ so that $D(V)$ is a countable dense subset of V for each $V \in \mathcal{V}$.

Fix $a \in X$ and set $\mathcal{W}_0 = \mathcal{W}_a$. Note that $\mathcal{W}_0$ is countable. With the countable family $\mathcal{W}_i$ having been defined, set $V_i = \cup\{E(W) \ / \ W \in \mathcal{W}_i\}$ and $\mathcal{W}_{i+1} = \cup\{\mathcal{W}_{DE(W')} \ / \ W' \in \mathcal{W}_i\}$. Note that $\mathcal{W}_{i+1}$ is also countable because each $\mathcal{W}_{DE(W')}$ is countable.

Set $V_\omega = \cup_{i\in\omega} V_i$. We show that $V_\omega = X$. Firstly V_ω is open, for suppose that $x \in V_\omega$, say $x \in V_i$. Choose some $W \in \mathcal{W}$ with $x \in W$. As $x \in V_i$ there is $W' \in \mathcal{W}_i$ with $x \in E(W')$. Then $W \cap DE(W') \neq \varnothing$ so $W \in \mathcal{W}_{i+1}$ and hence $W \subset V_{i+1}$. Secondly V_ω is closed for suppose that $x \in \overline{V_\omega}$. Choose $W \in \mathcal{W}$ with $x \in W$. Again we can show that there is some $i \in \omega$ and $W' \in \mathcal{W}_i$ such that $W \cap DE(W') \neq \varnothing$ and hence $x \in V_{i+1} \subset V_\omega$.

As a non-empty open and closed set, V_ω must be the whole of X. Thus X is covered by the countable subfamily $\cup_{i\in\omega}\mathcal{W}_i$. ■

4 Conclusion

A number of questions arise from the results and conditions considered in this paper.

Question 4.1 *Is the hypothesis that X has at most κ many isolated points needed in Proposition 2.2?*

Question 4.2 *Is there an example of a manifold which is a-favourable but not strongly a-favourable?*

Question 4.3 *Is there an example of a manifold which has property (a) but is not a-favourable?*

Question 4.4 *Must every (nearly) linearly metaLindelöf manifold be metrisable?*

Added later: As noted above, just before Lemma 3.4, the answer to this question is "yes."

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