Boundedly Metacompact or Finitistic Spaces and the Star Order of Covers

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Abstract

After showing that the topological notion of boundedly metacompact (first named finitistic) is equivalent to metrisability for a topological manifold we then study related notions. In particular we study the star order of covers of a space. This leads us to propose a definition of dimension which we call star covering dimension.

1 Introduction

Motivated by the definition of local covering dimension (see, for example, [8, p.188]), Swan in [9] introduced the property of a topological space being finitistic. This concept was introduced to help extend the classical P A Smith theorems to a more general setting. The importance of this concept is illustrated by its frequent appearance in the cohomological theory of transformation groups, see [1], for example. The concept was taken up by Fletcher, McCoy and Slover, [4], who (apparently unaware of its appearance a decade earlier) renamed it boundedly metacompact. The precise definition is as follows:

Definition 1.1 *A space X is* boundedly metacompact *or* finitistic *provided that for every open cover $\mathcal{U}$ of X there is an open refinement $\mathcal{V}$ and a positive integer m such that for each $x \in X$ the set $\mathcal{V}_x$ has cardinality at most m.*

In this definition and elsewhere in this paper we use the following notation. Let $\mathcal{F}$ be a family of subsets of a set X and let $A \subset X$. Then

$$\mathcal{F}_A = \{F \in \mathcal{F} \ / \ A \cap F \neq \varnothing\}.$$

In the case where $A = \{a\}$ we abbreviate $\mathcal{F}_{\{a\}}$ to $\mathcal{F}_a$.

The following property was also introduced in [4].

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Definition 1.2 *We say that a space X is* boundedly paracompact *provided that for every open cover $\mathcal{U}$ of X there is an open refinement $\mathcal{V}$ and a positive integer m such that for each $x \in X$ there is a neighbourhood N of x such that the set $\mathcal{V}_N$ has cardinality at most m.*

In this paper we introduce a third, even stronger, property and study all three.

Definition 1.3 *We say that a space X is* boundedly strongly paracompact *provided that for every open cover $\mathcal{U}$ of X there is an open refinement $\mathcal{V}$ and a positive integer m such that for each $V \in \mathcal{V}$ the set $\mathcal{V}_V$ has cardinality at most m.*
Given an open cover $\mathcal{V}$ of the space X, we will call the supremum of the cardinalities of the sets $\mathcal{V}_V$ the star order *of $\mathcal{V}$.*

Recall [2] that a space X is *strongly paracompact* or *hypocompact* if every open cover of X has a star finite open refinement. Recall also from [2, Example 6.2(ii), p 390] that there are metrisable spaces which are not strongly paracompact.

We first show that for a Hausdorff space the combination of boundedly metacompact and paracompact implies boundedly paracompact. The main result of the early part of the paper is that for a topological manifold all three conditions, boundedly metacompact, boundedly paracompact and boundedly strongly paracompact, are equivalent to one another and to metrisability. As the property boundedly strongly paracompact appears not to have been studied before we then look at it, particularly at the minimum star order for a refinement of an arbitrary open cover. It may be possible to define a notion of dimension in terms of the star order and we explore this briefly.

2 Preliminary Results

In the definitions we allowed the integer m to depend on the open cover $\mathcal{U}$. One might ask whether the definitions are equivalent to those obtained by demanding that m not depend on $\mathcal{U}$. In general the answer is "no", as shown by the following example. However for a (metrisable!) manifold the integer m can be chosen independently of the open cover $\mathcal{U}$, as seen in the proof of Theorem 3.3.

Example 2.1 *A boundedly metacompact space X in which the bound m in the definition depends on the original open cover $\mathcal{U}$.*

For each positive integer n let X_n be a topological space which is homeomorphic to the unit n-sphere and chosen so that if $m \neq n$ then $X_m \cap X_n = \varnothing$. Let X denote the one-point compactification of the topological sum of the spaces X_n; denote by $\infty \in X$ the point added to form the compactification. Then X is boundedly metacompact (indeed, boundedly paracompact by Theorem 2.2) but it is not possible to choose the integer m of the definition independently of the open cover $\mathcal{U}$. That X is boundedly metacompact may be proved as follows: let $\mathcal{U}$ be any open cover of X and choose some $U \in \mathcal{U}_\infty$ containing the point ∞. Then $X - U_\infty$ is compact so U_∞ contains all but finitely many of the subsets X_n. Set $m = \max\{n \ / \ X_n \not\subset U_\infty\} + 1$. For each

$n < m$ we have $\dim X_n = n$ so there is a refinement $\mathcal{V}_n$ of $\{U \cap X_n \ / \ U \in \mathcal{U}\}$ such that for each $x \in X_n$ the set $\{V \in \mathcal{V}_n \ / \ x \in V\}$ has at most $n+1$ members. Let

$$\mathcal{V} = (\cup_{n<m}\mathcal{V}_n) \cup \{U_\infty - \cup_{n<m}X_n\}.$$

Then $\mathcal{V}$ is an open refinement of $\mathcal{U}$ and for each $x \in X$ the family $\{V \in \mathcal{V} \ / \ x \in V\}$ has at most m members. Note, however, that m must depend on $\mathcal{U}$ as we may arrange for $\mathcal{U}$ to contain exactly one member U_∞ containing ∞ such that $\max\{n \ / \ X_n \not\subset U_\infty\}$ is arbitrarily large. ∎

Theorem 2.2 *A Hausdorff space is boundedly paracompact if and only if it is both boundedly metacompact and paracompact.*

Proof: It is clear that a boundedly paracompact space is both boundedly metacompact and paracompact.

Suppose that X is a boundedly metacompact, paracompact, Hausdorff space and let $\mathcal{U}$ be an open cover of X. Because X is boundedly metacompact there is an open refinement $\mathcal{V}$ of $\mathcal{U}$ and an integer m such that for each $x \in X$ the family $\mathcal{V}_x$ has cardinality at most m. We note in passing that the proof is complete if every point of X lies in exactly m members of $\mathcal{V}$ because then given $x \in X$ there are $m+1$ members $\{V_0, \ldots, V_m\}$ of $\mathcal{V}$ containing x and the open neighbourhood $\cap_{i=0}^m V_i$ of x cannot meet any other members of $\mathcal{V}$.

As X is paracompact, the open cover $\mathcal{V}$ has a locally finite open refinement $\mathcal{W}$ and there is a bijection $\varphi : \mathcal{V} \to \mathcal{W}$ such that for each $V \in \mathcal{V}$ we have $\varphi(V) \subset V$.

Every paracompact Hausdorff space is normal and a space is normal if and only if every point finite open cover has an open shrinkage. Thus there is an open cover $\mathcal{S}$ of X and a bijection $\psi : \mathcal{W} \to \mathcal{S}$ such that for each $W \in \mathcal{W}$ we have $\overline{\psi(W)} \subset W$. We show that for each $x \in X$ there is a neighbourhood N_x of x such that the set $\{S \in \mathcal{S} \ / \ N_x \cap S \neq \varnothing\}$ has cardinality at most m.

Suppose given $x \in X$. Let $\{V_1, \ldots, V_l\}$ denote those members of $\mathcal{V}$ which contain x. Set

$$N_x = (\bigcap_{i=1}^{l} V_i) \cap (X - \cup\{\overline{\psi\varphi(V)} \ / \ V \in \mathcal{V} - \{V_1, \ldots, V_l\}\}).$$

Then

1. $x \in N_x$. Certainly $x \in V_i$ for each i and if $V \in \mathcal{V}$ but $V \neq V_i$ for any i then $x \notin V$ so
$$x \in X - V \subset X - \varphi(V) \subset X - \overline{\psi\varphi(V)}.$$

2. N_x is open. Because $\{\varphi(V) \ / \ V \in \mathcal{V} - \{V_1, \ldots, V_l\}\}$ is locally finite, so is $\{\overline{\psi\varphi(V)} \ / \ V \in \mathcal{V} - \{V_1, \ldots, V_l\}\}$. Thus $\cup\{\overline{\psi\varphi(V)} \ / \ V \in \mathcal{V} - \{V_1, \ldots, V_l\}\}$ is closed and hence its complement is open.

3. The cardinality of $\{S \in \mathcal{S} \ / \ N_x \cap S\}$ is at most m. Indeed the only members of $\mathcal{S}$ meeting N_x are some or all of $\psi\varphi(V_1), \ldots, \psi\varphi(V_l)$.

This completes the proof that X is boundedly paracompact. ∎

Proposition 2.3 *Let X be a Hausdorff boundedly paracompact space. Then X has finite covering dimension if and only if the integer m in the definition of boundedly paracompact can be chosen independently of the cover $\mathcal{U}$.*

Proof: Suppose that X has finite covering dimension, say n. As X is paracompact, the following are equivalent (cf [3]):

- every finite open cover of X has an open refinement of order $\leq k$;
- every open cover of X has an open refinement of order $\leq k$.

Since X has covering dimension n, every finite open cover, and hence by Dowker's result every open cover, has an open refinement of order $\leq n$. Conversely, if X has infinite covering dimension then for each integer n there is an open cover $\mathcal{U}$ of X such that each open refinement $\mathcal{V}$ of $\mathcal{U}$ has order at least n. Thus in this case the number in the definition of boundedly paracompact depends on the open cover $\mathcal{U}$. ∎

Lemma 2.4 *Suppose that X is a σ-compact, locally compact, T_2 space. Then there is a proper continuous function $f : X \to \mathbb{R}$.*

Proof. As X is σ-compact, we may write $X = \cup_{n\in\omega} X_n$, where each X_n is compact. Moreover, because X is locally compact we may assume that $X_n \subset \mathring{X}_{n+1}$ for each n. Let $\mathbb{D}$ denote the non-negative dyadic rational numbers. By induction on the exponent of 2 in the denominator of each member of $\mathbb{D}$ we construct compacta X_p ($p \in \mathbb{D}$) such that $X_p \subset \mathring{X}_q$ whenever $p < q$ as follows. X_p has already been defined when the exponent of 2 is 0, such numbers being the non-negative integers. Suppose that $p \in \mathbb{D}$ has exponent $n > 0$: then there are $q, r \in \mathbb{D}$ such that the exponents of 2 in the denominators of q and r are both less than n and q is the largest such rational less than p while r is the smallest such greater than p. As $X_q \subset \mathring{X}_r$ and X_q is compact while X is locally compact and T_2, we can cover X_q by finitely many open, relatively compact subsets of X whose closures all lie in $\mathring{X}_r$: let X_p be the union of the closures of these sets. Then X_p is compact and $X_q \subset \mathring{X}_p \subset X_p \subset \mathring{X}_r$.

Now define $f : X \to \mathbb{R}$ by $f(x) = \inf\{p \in \mathbb{D} \ / \ x \in X_p\}$. It is standard to verify that f is continuous; see, for example, [6, p. 210]. It is also readily verified that f is proper. ∎

Any open cover of a compact metric space has a corresponding Lebesgue number, but usually for non-compact metric spaces such numbers do not exist. We will find the following lemma a useful compensation for this lack.

Lemma 2.5 *Let $\mathcal{U}$ be an open cover of $\mathbb{R}^n$. Then there is a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that the cover $h(\mathcal{U})$ has Lebesgue number 1.*

Proof. We use the metric on $\mathbb{R}^n$ induced by the norm

$$|(x_1, \ldots, x_n)| = max\{|x_i| \ / \ i = 1, \ldots, n\}.$$

Let $\mathbb{B}^n$ denote the closed unit ball in $\mathbb{R}^n$ and $\mathbb{S}^{n-1}$ its boundary.

For each positive integer k let $\varepsilon_k > 0$ be a Lebesgue number for $\{U \cap k\mathbb{B}^n \;/\; U \in \mathcal{U}\}$; we may assume that the ε_k have been chosen so that $\frac{1}{2} \geq \varepsilon_k \geq \varepsilon_{k+1}$. Define the homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ by letting $h|k\mathbb{S}^{n-1}$ be division by the scalar ε_{k+1} and extending linearly on each line segment which joins a point of $k\mathbb{S}^{n-1}$ to a point of $(k+1)\mathbb{S}^{n-1}$ and which lies on a ray emanating from the origin. Then on the annulus between $(k-1)\mathbb{S}^{n-1}$ and $k\mathbb{S}^{n-1}$ the homeomorphism h magnifies by a factor of at least $\dfrac{1}{\varepsilon_{k+1}}$.

Set $h(\mathcal{U}) = \{h(U) \;/\; U \in \mathcal{U}\}$. Then $h(\mathcal{U})$ is an open cover of $\mathbb{R}^n$. Moreover $h(\mathcal{U})$ has a Lebesgue number of at least 1. Indeed, suppose that $x \in k\mathbb{B}^n - (k-1)\mathbb{B}^n$. Then either the ε_k-ball centred at x lies in $k\mathbb{B}^n - (k-2)\mathbb{B}^n$ or the ε_{k+1}-ball centred at x lies in $(k+1)\mathbb{B}^n - (k-1)\mathbb{B}^n$ and in either case the ball lies in some member, say U, of $\mathcal{U}$. In the first case h magnifies by a factor of at least $\dfrac{1}{\varepsilon_k}$ and hence the 1-ball centred at $h(x)$ lies inside $h(U)$. In the second case h magnifies by a factor of at least $\dfrac{1}{\varepsilon_{k+1}}$ and hence again the 1-ball centred at $h(x)$ lies inside $h(U)$. Thus $h(\mathcal{U})$ has a Lebesgue number of at least 1. ∎

3 Boundedly Metacompact Manifolds and Metrisability

Where we use the term *manifold* we mean a connected, Hausdorff, locally euclidean, topological space.

An improved version of the following result is proved later as Proposition 4.7.

Lemma 3.1 *Each open cover $\mathcal{U}$ of $\mathbb{R}^n$ has an open refinement $\mathcal{V}$ such that for each $V \in \mathcal{V}$ the family $\mathcal{V}_V$ contains at most 3^n members.*

Proof: Suppose that $\mathcal{U}$ is an open cover of $\mathbb{R}^n$. By Lemma 2.5 there is a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h(\mathcal{U})$ has a Lebesgue number of at least 1.

Let $\mathcal{V}'$ consist of all open 1-balls in $\mathbb{R}^n$ whose centres lie at points having integer coordinates. Then $\mathcal{V}'$ is an open refinement of $h(\mathcal{U})$ and each member of $\mathcal{V}'$ meets m members of $\mathcal{V}'$. Let $\mathcal{V} = \{h^{-1}(V) \;/\; V \in \mathcal{V}\}$. Then $\mathcal{V}$ is an open refinement of $\mathcal{U}$ and each member of $\mathcal{V}$ meets m members of $\mathcal{V}$. ∎

Corollary 3.2 *Suppose that $C \subset \mathbb{R}^n$ is closed. Then every open cover of C has an open refinement whose star order is at most 3^n.*

We now present the main result of this section.

Theorem 3.3 *Let M be a manifold. Then the following conditions are equivalent:*
(1) M is metrisable;
(2) M is boundedly metacompact;
(3) M is boundedly paracompact;
(4) M is boundedly strongly paracompact.

Proof: (1)⇒(4). Suppose that M is metrisable and has dimension n. Then by [5, Theorem 2], M is σ-compact and separable. As a separable metric space of covering dimension n, M embeds in $\mathbb{R}^{2n+1}$ (by [8, Proposition 7.3.9, p. 271]); say $e' : M \to \mathbb{R}^{2n+1}$ is an embedding. Let $f : M \to \mathbb{R}$ be a proper continuous function, as given by Lemma 2.4. Define $e : M \to \mathbb{R}^{2n+2}$ by setting $e(x) = (e'(x), f(x))$. Then e is also an embedding. Moreover, $e(M)$ is closed.

Let $\mathcal{U}$ be any open cover of M. Then $e(\mathcal{U}) = \{e(U) \ / \ U \in \mathcal{U}\}$ is an open cover of $e(M)$. By Corollary 3.2, it follows that there is an open refinement $\mathcal{V}'$ of $e(\mathcal{U})$ such that the star order of each member of $\mathcal{V}'$ is at most 3^{2n+2}. Let $\mathcal{V} = \{e^{-1}(V) \ / \ V \in \mathcal{V}'\}$. Then $\mathcal{V}$ is an open refinement of $\mathcal{U}$ and the star order of each member of $\mathcal{V}$ is also at most 3^{2n+2}.

(4)$\Rightarrow$(3). Trivial.

(3)$\Rightarrow$(2). Trivial.

(2)$\Rightarrow$(1). Every boundedly metacompact space is metacompact and hence metrisable if it is a manifold, by Theorem 2.5 of [7] (see also [5, Theorem 2]). ■

Remark. The implications above do not need the full force of the properties of a manifold. It may be shown that every metaLindelöf space (hence every boundedly metacompact space) which is connected and locally separable is Lindelöf and that every Lindelöf space which is T_3 is metrisable. Every metrisable space is paracompact. Thus the implications (2)$\Rightarrow$(1) and (2)$\Rightarrow$(3) require only that M be connected, locally separable and T_3. The implication (1)$\Rightarrow$(2) requires that M has local covering dimension at most n for some n. The implication (1)$\Rightarrow$(4) requires more as we use the fact that if M is metrisable then it is both separable and σ-compact.

4 The Minimum Star Order of Covers

In this section we assume that all spaces are boundedly strongly paracompact and explore the minimum value of the star order for refinements of arbitrary open covers.

Definition 4.1 *Say that a topological space has* star order at most n *provided that every open cover has an open refinement whose star order is at most n. The smallest integer n for which a space has star order at most n is the* star order of the space. *Denote the star order of a space X by $stor(X)$.*

Proposition 4.2 *Let $C \subset X$ be a closed subset of a topological space. Then $stor(C) \leq stor(X)$.*

Proof: Straightforward. ■

We recall the following definition from [8, p 24].

Definition 4.3 *Let $\{A_\lambda \ / \ \lambda \in \Lambda\}$ and $\{B_\lambda \ / \ \lambda \in \Lambda\}$ be two families of subsets of a topological space X. We say that the two families are* similar *if for each finite subset $F \subset \Lambda$ we have*

$$\cap_{\lambda \in F} A_\lambda \neq \varnothing \iff \cap_{\lambda \in F} B_\lambda \neq \varnothing.$$

Note that for any two similar families $\{A_\lambda \ / \ \lambda \in \Lambda\}$ and $\{B_\lambda \ / \ \lambda \in \Lambda\}$ the order of the star of A_{λ_0} is the same as the order of the star B_{λ_0}. This is because for any $\lambda \neq \lambda_0$ we have $A_{\lambda_0} \cap A_\lambda \neq \varnothing \Leftrightarrow B_{\lambda_0} \cap B_\lambda \neq \varnothing$.

The following result is found in [8, p 24].

Proposition 4.4 *Let* $\{U_\lambda \ / \ \lambda \in \Lambda\}$ *be a locally finite family of open subsets of a normal space* X *and* $\{F_\lambda \ / \ \lambda \in \Lambda\}$ *be a family of closed subsets of* X *such that* $F_\lambda \subset U_\lambda$ *for each* $\lambda \in \Lambda$*. Then there is a family* $\{G_\lambda \ / \ \lambda \in \Lambda\}$ *of open subsets of* X *such that*

$$F_\lambda \subset G_\lambda \subset \bar{G}_\lambda \subset U_\lambda$$

and the families $\{F_\lambda \ / \ \lambda \in \Lambda\}$ *and* $\{\bar{G}_\lambda \ / \ \lambda \in \Lambda\}$ *are similar.*

Proposition 4.5 *Let* X *be a paracompact Hausdorff space. Then for any integer* $s \geq 0$ *the following conditions are equivalent:*

(a) Every open cover of X *has an open refinement of star order* $\leq s$*.*

(b) Every open cover of X *has a closed refinement of star order* $\leq s$*.*

Proof. (a)⇒(b): Let $\mathcal{U}$ be an open cover of X. By (a) there is an open refinement $\mathcal{V} = \{V_\lambda \ / \ \lambda \in \Lambda\}$ of $\mathcal{U}$ having star order at most s. In particular $\mathcal{V}$ is locally finite. Thus, as X is normal, we can shrink $\mathcal{V}$ to an open cover $\mathcal{W} = \{W_\lambda \ / \ \lambda \in \Lambda\}$ such that for each λ we have $W_\lambda \subset \bar{W}_\lambda \subset V_\lambda$. Then $\{\bar{W}_\lambda \ / \ \lambda \in \Lambda\}$ is a closed refinement of $\mathcal{U}$ of star order at most s.

(b)⇒(a): Suppose that $\mathcal{U}$ is an open cover of X. Since X is paracompact, there is a locally finite open refinement $\mathcal{V} = \{V_\lambda \ / \ \lambda \in \Lambda\}$ of $\mathcal{U}$. By (b) there is a closed refinement $\{F_\mu \ / \ \mu \in M\}$ of $\mathcal{V}$ having star order at most s. For each μ we choose V_μ such that $F_\mu \subset V_\mu$. Then $\{V_\mu \ / \ \mu \in M\}$ is again a locally finite open cover of X. Hence by Proposition 4.4 there is an open cover $\{G_\mu \ / \ \mu \in M\}$ of X such that for each μ we have

$$F_\mu \subset G_\mu \subset \bar{G}_\mu \subset V_\mu$$

and the families $\{F_\mu\}$ and $\{\bar{G}_\mu\}$ are similar. Thus the families $\{F_\mu\}$ and $\{G_\mu\}$ are also similar and hence $\{G_\mu\}$ has star order at most s. Of course $\{G_\mu\}$ is an open refinement of $\mathcal{U}$. ■

Lemma 4.6 *Suppose that the space* X *has star order* n*. Then every subset of* X *has star order at most* n *if and only if every open subset of* X *has star order at most* n*.*

Proof. Suppose that every open subset of X has star order at most n and suppose that $S \subset X$ is any subset. Let $\mathcal{U}$ be a cover of S by open subsets of the subspace S. Then there is a collection $\hat{\mathcal{U}}$ of open subsets of X such that $\mathcal{U} = \{U \cap S \ / \ U \in \hat{\mathcal{U}}\}$. Let $\hat{S} = \cup \hat{\mathcal{U}}$. Then $\hat{\mathcal{U}}$ is an open cover of the open subset $\hat{S}$ so there is an open refinement $\hat{\mathcal{V}}$ of star order at most n. The open refinement $\mathcal{V} = \{V \cap S \ / \ V \in \hat{\mathcal{V}}\}$ of $\mathcal{U}$ has star order at most n. ■

The following result improves on Lemma 3.1.

Proposition 4.7 $\mathbb{R}^n$ *has star order is at most* $2^{n+1} - 1$*.*

Proof: As before, we use the metric δ on $\mathbb{R}^n$ defined by

$$\delta((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = max\{|x_i - y_i| \ / \ i = 1,\ldots,n\}.$$

For each n we construct a lattice Λ_n of points in $\mathbb{R}^n$ as follows. Firstly by induction we construct a family of sets $\tilde{\Lambda}_n \subset \mathbb{Z}^n$. Let $\tilde{\Lambda}_1 = \{0\}$. If $\tilde{\Lambda}_n$ has been constructed then let $\tilde{\Lambda}_{n+1}$ be the union of the two sets

$$\{(2x_1,\ldots,2x_n,x_n) \ / \ (x_1,\ldots,x_n) \in \tilde{\Lambda}_n\} \text{ and}$$

$$\{(2x_1,\ldots,2x_{n-1},2x_n+2^{n+1},x_n+2^n) \ / \ (x_1,\ldots,x_n) \in \tilde{\Lambda}_n\}.$$

It is easily shown by induction on n that the ith coordinate of any point of $\tilde{\Lambda}_n$ is at least 0 and less than 2^{n+1} (less than 2^n if $i = n$) and is an integer multiple of 2^{n+1-i}. Now set

$$\Lambda_n = \{(x_1+y_1,\ldots,x_n+y_n) \in \mathbb{R}^n \ / \ (x_i) \in \tilde{\Lambda}_n \text{ and } y_i \in 2^{n+1}\mathbb{Z} \text{ if } i < n \text{ and } y_n \in 2^n\mathbb{Z}\}.$$

We prove the following two facts:

(1) For each $y \in \mathbb{R}^n$ there is $x \in \Lambda_n$ such that $\delta(x,y) \leq 2^{n-1}$;

(2) For each $x \in \Lambda_n$ there are exactly $2^{n+1}-1$ points $y \in \Lambda_n$ such that $\delta(x,y) \leq 2^n$.

Statement (1) is proved by induction on n, it being obvious when $n = 1$. Suppose that statement (1) is true for n and let $y = (y_1,\ldots,y_{n+1}) \in \mathbb{R}^{n+1}$. By inductive hypothesis there is $(x_1/2,\ldots,x_n/2) \in \Lambda_n$ such that

$$\delta((x_1/2,\ldots,x_n/2),(y_1/2,\ldots,y_n/2)) \leq 2^{n-1}, \text{ hence } \delta((x_1,\ldots,x_n),(y_1,\ldots,y_n)) \leq 2^n.$$

Now choose an even integer x_{n+1} within 2^n of y_{n+1} so that $(x_1,\ldots,x_{n+1}) \in \Lambda_{n+1}$; this can be done by the definition of Λ_{n+1}. Then $\delta((x_1,\ldots,x_{n+1}),(y_1,\ldots,y_{n+1})) \leq 2^n$ as required.

Now consider statement (2). Because of the lattice structure, it suffices to show the result when $x = (0,\ldots,0) = 0$, say. The only translates of $(0,\ldots,0)$ within 2^n of 0 are $(0,\ldots,\pm 2^n)$ and 0 itself, a total of 3 points. The last coordinate of each point of $\tilde{\Lambda}_n - \{0\}$ differs from 0 by more than 0 but less than 2^n, so may be decreased by 2^n and still remain within 2^n of 0. Of the other coordinates of a point of $\tilde{\Lambda}_n - \{0\}$, exactly one is 2^n, the others being strictly between -2^n and 2^n or strictly between 2^n and 2^{n+1}; thus there is only one value for all of the other coordinates except that which is 2^n and for the latter there are two possible values: $\pm 2^n$. Thus for each point of $\tilde{\Lambda}_n - \{0\}$ there are two coordinates which may be given two different values while ensuring that the point is within 2^n of 0 while all of the other coordinates have a unique such value, giving a total of 4 different points of Λ_n which are translates of points of $\tilde{\Lambda}_n - \{0\}$ and are at most 2^n from 0. As there are $2^{n-1} - 1$ points in $\tilde{\Lambda}_n - \{0\}$, it follows that the number of points of Λ_n within 2^n of 0 is $3 + 4(2^{n-1} - 1) = 2^{n+1} - 1$.

Consider the collection of open balls centred at points of Λ_n and of radii $2^{n-1}+1$. By statement (1) these balls cover $\mathbb{R}^n$ and by statement (2) each ball meets exactly

$2^{n+1}-1$ members of the collection. By applying the contraction $x \mapsto x/2^n$ we obtain a new cover of $\mathbb{R}^n$ by a regular collection of open balls each of which has radius less than 1 so that it is still the case that the collection covers $\mathbb{R}^n$ and each ball meets exactly $2^{n+1}-1$ members of the collection.

Now suppose given an open cover $\mathcal{U}$ of $\mathbb{R}^n$ and let the homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ be as given by Lemma 2.5. Let $\mathcal{V}'$ consist of all open balls in $\mathbb{R}^n$ of radii less than 1, covering $\mathbb{R}^n$ and with star order $2^{n+1}-1$ as in the first part of the proof. Then $\mathcal{V}'$ is an open refinement of $h(\mathcal{U})$. Let $\mathcal{V} = \{h^{-1}(V) \ / \ V \in \mathcal{V}\}$. Then $\mathcal{V}$ is an open refinement of $\mathcal{U}$ and $\mathcal{V}$ has star order $2^{n+1}-1$. ■

Proposition 4.8 $\mathbb{R}$ *has star order 3.*

Proof: By Lemma 3.1 we know that the star order of $\mathbb{R}$ is at most 3, so it suffices to show that there is an open cover $\mathcal{U}$ of $\mathbb{R}$ all of whose open refinements have star order at least 3.

Consider the open cover $\mathcal{U} = \{(n, n+2) \ / \ n \in \mathbb{Z}\}$ and let $\mathcal{V}$ be any open refinement of $\mathcal{U}$. Clearly $\mathcal{V}$ has infinitely many members.

Suppose there is some non-empty member of $\mathcal{V}$, say V_1, which meets no other member of $\mathcal{V}$. Set $V_2 = \cup\{V \in \mathcal{V} \ / \ V \neq V_1\}$. Then $\{V_1, V_2\}$ is a disconnection of $\mathbb{R}$, giving a contradiction. Thus every member of $\mathcal{V}$ meets at least one other member of $\mathcal{V}$.

Let $V_1, V_2 \in \mathcal{V}$ be such that $V_1 \cap V_2 \neq \varnothing$. Since $\mathcal{V}$ is a refinement of $\mathcal{U}$ it follows that $V_1 \cup V_2 \neq \mathbb{R}$. If no other member of $\mathcal{V}$ meets $V_1 \cup V_2$ then $V_3 = \cup\{V \in \mathcal{V} \ / \ V_1 \neq V \neq V_2\}$ will produce a disconnection $\{V_1 \cup V_2, V_3\}$ of $\mathbb{R}$, giving a contradiction. Thus there is some member $V_3 \in \mathcal{V}$ such that $V_3 \cap (V_1 \cup V_2) \neq \varnothing$. Thus V_3 meets either V_1 or V_2, so that either V_1 or V_2 meets at least 3 members of $\mathcal{V}$. Hence $\mathcal{V}$ has star order at least 3. ■

Corollary 4.9 *The star order of any non-trivial interval in* $\mathbb{R}$ *or in* $\mathbb{S}^1$ *is 3.*

Proposition 4.10 *A subset X of $\mathbb{R}$ or of $\mathbb{S}^1$ has star order 3 if and only if X contains a non-trivial interval. If $X \neq \varnothing$ contains no non-trivial interval then X has star order 1.*

Proof. If X contains a non-trivial interval then by Corollary 4.9 X has star order at least 3 and hence exactly 3 by Proposition 4.8.

Conversely, suppose that X contains no non-trivial interval and let $\mathcal{U}$ be any open cover of X. Then as X has covering dimension 0 it follows that $\mathcal{U}$ has an open refinement $\mathcal{V}$ such that for each $x \in X$ there is a unique member of $\mathcal{V}$ containing x. Thus $\mathcal{V}$ has star order 1. ■

Proposition 4.11 *Let G be a finite graph. Then either G may be embedded in $\mathbb{S}^1$ or G has star order 4.*

Proof: Suppose that G may not be embedded in $\mathbb{S}^1$.

Case (1). G has exactly one vertex of valency more than 2 and that vertex has valency 3. Write $G = A \cup B \cup C$, where $A \cap B \cap C = \{0\}$ is the common vertex and each of A, B and C is connected. As each of A, B and C is homeomorphic to [0,1] it follows that we may transfer the linear order from [0,1] to each of A, B and C in such a way that the common vertex 0 is the least member of each of A, B and C.

It is clear that the star order of G is at most 4, because we can cover 0 by a small open set and then the remainders of each of A, B and C may be covered by three series of open sets, each series lying in just one of A, B and C with the order of the star of each member of each series at most 3: the star order of this open cover will be 4 as the order of the star of the set containing 0 is 4.

Now let U_0 be an open subset of G containing 0 and three short open intervals projecting from 0, one into each of A, B and C. Then $\mathcal{U} = \{U_0, A-\{0\}, B-\{0\}, C-\{0\}\}$ is an open cover of G. Let $\mathcal{V}$ be an open refinement of $\mathcal{U}$: we will show that $\mathcal{V}$ has star order at least 4. Pick any $V_0 \in \mathcal{V}$ containing 0 and let $a_0 \in A, b_0 \in B$ and $c_0 \in C$ maximal with respect to the conditions $a_0, b_0, c_0 \in \bar{V}_0$. If a_0, b_0 and c_0 lie in different members of $\mathcal{V}$ then V_0 has star order at least 4 and we are finished. On the other hand if at least two of a_0, b_0 and c_0 lie in the same member of $\mathcal{V}$ then we may assume without loss of generality that there is $V_1 \in \mathcal{V}$ with $a_0, b_0 \in V_1$. Let $a_1 \in A$ and $b_1 \in B$ be maximal with respect to the conditions $a_1, b_1 \in \bar{V}_1$. Again either a_1 and b_1 are in different members of $\mathcal{V}$, in which case the order of the star of V_1 is at least 4, or they lie in the same member, $V_2 \in \mathcal{V}$. Continuing in this way, either we reach a stage where a set $V_n \in \mathcal{V}$ meets V_{n-1}, itself and two other members of $\mathcal{V}$, so the order of its star is at least 4, or else we obtain such sets V_n and points $a_n, b_n \in \bar{V}_n$ for every positive integer n. In the latter case the sequence $\langle a_n \rangle$ must converge, say to $\tilde{a} \in A$, and any member of $\mathcal{V}$ containing $\tilde{a}$ must meet all but finitely many of the sets V_n, so again the star order of $\mathcal{V}$ is at least 4.

Case (2). G has exactly one vertex of valency more than 2. Then we may write $G = \cup_{i=1}^{n} A_i$, where each A_i is an arm of G and the vertex, 0, of higher valency n is the only point common to any two, hence all, of the sets A_i.

As $\cup_{i=1}^{3} A_i \subset G$ is of the form of the graph G in Case (1), it follows from Case (1) and Proposition 4.2 that G has star order at least 4. We prove the reverse inequality by induction on n, Case (1) having started the induction at $n = 3$.

Let $\mathcal{U}$ be any open cover of G: we must show that there is an open refinement of $\mathcal{U}$ whose star order is at most 4. Choose any $U_0 \in \mathcal{U}$ containing 0. As in case (1) we linearly order each A_i so that 0 is the least member of each A_i. For each $i = 1, \ldots, n$, use the order on A_i to choose a sequence of open intervals $\langle I_{i,j} \rangle_{j=0}^{n_i}$ in A_i so that:

(1) $\{I_{i,1}, \ldots I_{i,n_i}\}$ covers A_i;

(2) each of the intervals is a subset of some member of $\mathcal{U}$;

(3) if $j \leq i$ then $I_{i,j} \subset U_0$;

(4) $I_{i,j} \cap I_{i,k} \neq \varnothing$ if and only if $|j-k| \leq 1$.

Now set $\mathcal{V} = \{\cup_{i=j}^{n} I_{i,j} \ / \ j = 1, \ldots, n\} \cup \{I_{i,j} \ / \ j > i\}$. By (1), (2) and (3), $\mathcal{V}$ is an open refinement of $\mathcal{U}$. By (4), the star order of $\mathcal{V}$ is at most 4, sets of the form

$\cup_{i=j}^{n} I_{i,j}$ for $1 < j < n$ being those which meet the largest number of members of $\mathcal{V}$, and the only members of $\mathcal{V}$ that such sets meet are the four sets $\cup_{i=k}^{n} I_{i,k}$ (for $k = j-1, j, j+1$) and $I_{j+1,j}$.

Case (3) General case. Again we need only show that the star order is at most 4. Let $\mathcal{U}$ be an open cover of G. Firstly take an open refinement of $\mathcal{U}$ so that each multi-valent vertex of G is covered by a single member of the refinement and no two of these intersect. Then apply Case (2) to each vertex of G to get a refinement of $\mathcal{U}$ restricted to a neighbourhood of each vertex so that the star order is at most 4. Now fill in between each vertex by a refinement of order at most 3 to get the required open refinement $\mathcal{V}$ of star order at most 4. ∎

Corollary 4.12 $\mathbb{R}^2$ *has star order at least 4.*

Example 4.13 *There is a compact, connected and locally path connected space X having star order $2^{n+1}-1$ and an open cover $\mathcal{U}$ such that every open refinement $\mathcal{V}$ of $\mathcal{U}$ having star order $2^{n+1}-1$ contains disconnected members.*

Indeed, let X be the graph consisting of the 6 vertices 0,1, ...,5 and the 5 edges joining 0 to each of 1,...,5, and let $\mathcal{U}$ consist of the following 6 open subsets of X: each of the 5 edges with 0 deleted and $X - \{1, \ldots, 5\}$. Then from Proposition 4.11, X has star order 4, so $\mathcal{U}$ has open refinements of star order at most 4: let $\mathcal{V}$ be such a refinement and suppose that each member of $\mathcal{V}$ is connected. For each $m = 1, \ldots, 5$ let $\bar{m}$ denote the first point on the edge from 0 to m for which there is no member of $\mathcal{V}$ containing both 0 and m. Then there are distinct members $V_m \in \mathcal{V}$ so that $\bar{m} \in V_m$. Choose $V \in \mathcal{V}$ containing $[0, \bar{1})$. We consider several cases.

1. V contains at least four of the intervals of the form $[0, \bar{m})$. This is impossible as then V meets at least four other members of $\mathcal{V}$, viz at least four of the sets V_m.

2. V contains three of the intervals of the form $[0, \bar{m})$. This is impossible as then there must be at least one other member, say V', of $\mathcal{V}$ containing 0 and then V meets at least four other members of $\mathcal{V}$, viz V' and three of the sets V_m.

3. V contains two of the intervals of the form $[0, \bar{m})$. This is impossible as then by case 2 there must be at least two other members, say V' and V'', of $\mathcal{V}$ containing 0 and again V meets at least four other members of $\mathcal{V}$, viz V', V'' and two of the sets V_m.

4. V contains no other interval of the form $[0, \bar{m})$. This is impossible as then by case 3 there must be five distinct members of $\mathcal{V}$ containing 0.

As each of the exhaustive cases is impossible, we conclude that not all members of $\mathcal{V}$ can be connected. ∎

We observe that if each vertex of the graph G has valency at most 4 then it is possible to find arbitrarily fine open covers of G whose star order is at most four and whose members are all connected.

Proposition 4.14 *Let $Y \subset \mathbb{I} = [0,1]$ be a 0-dimensional subset with $Y \not\subset \{0,1\}$ and set*

$$X = \{(0,y) \in \mathbb{R}^2 \ / \ y \in \mathbb{I}\} \cup \{(x,y) \in \mathbb{R}^2 \ / \ x \in \mathbb{I} \text{ and } y \in Y\}$$

with the subspace topology. Then X has star order 4.

Proof: Choose any $\bar{y} \in Y - \{0,1\}$. Then by Proposition 4.11 the closed subset $\{0\} \times \mathbb{I} \cup \mathbb{I} \times \{\bar{y}\}$ of X has star order 4, so by Proposition 4.2 the star order of X is at least 4.

For the converse, suppose that $\mathcal{U}$ is an open cover of X. Then there is a partition $\langle 0 = t_0 < \ldots < t_k = 1 \rangle$ of $\mathbb{I}$ and $\varepsilon > 0$ such that for each i there is $U \in \mathcal{U}$ such that $([0,\varepsilon] \times [t_{i-1}, t_i]) \cap X \subset U$. Because Y is 0-dimensional we may assume that each $t_i \in \mathbb{I} - Y$ except possibly when $i = 0$ or n.

For each $y \in Y$ the set $\mathbb{I} \times \{y\}$ is compact so there is an open interval $J_y \subset \mathbb{I}$ and a finite partition of $[\varepsilon, 1]$ such that $y \in J_y$ and for each elementary subinterval I determined by the partition there is $U \in \mathcal{U}$ such that $(I \times J_y) \cap X \subset U$. We may also assume that $J_y \subset [t_{i-1}, t_i]$ for some i. Because Y is 0-dimensional, it follows that the open cover $\{J_y \ / \ y \in Y\}$ of Y has an open refinement, $\mathcal{J}$, of order 0. Each member of $\mathcal{J}$ is the intersection of an open subset of $\mathbb{R}$ with Y, hence a disjoint union of sets of the form $J \cap Y$, where J is an interval: thus we may assume that each member of $\mathcal{J}$ is of this form. Because Y is Lindelöf it follows that $\mathcal{J}$ is countable, say $\mathcal{J} = \{J_n \ / \ n \in \omega\}$. Each J_n extends to an open interval $\hat{J}_n \subset \mathbb{I}$ so that $\hat{J}_n \cap Y = J_n$. We may assume that $\hat{J}_m \cap \hat{J}_n \neq \varnothing$ for $m \neq n$, for if this is not the case then because we cannot have $\hat{J}_m \cap \hat{J}_n \cap Y \neq \varnothing$, we may inductively cut down the ends of $\hat{J}_n$ (and J_n) so as to avoid $\hat{J}_m$ for $m < n$. Note that the end points of each J_n cannot be in $Y - \{0,1\}$, and we may assume that i is an end point of J_i for $i \in \{0,1\} \cap Y$. Thus we have a countable open cover, $\mathcal{J} = \{J_n \ / \ n \in \omega\}$, of Y satisfying the following:

1. each set J_n is of the form $\hat{J}_n \cap Y$, where $\hat{J}_n$ is an open interval in $\mathbb{I}$;
2. for $i = 0,1$ we have that i is an end point of J_i if $i \in Y$ and no other point of Y is an end point of a set J_n;
3. if $m \neq n$ then $\hat{J}_m \cap \hat{J}_n = \varnothing$;
4. for each $n \in \omega$ there is $i = 1, \ldots, k$ such that $J_n \subset [t_{i-1}, t_i]$;
5. for each $n \in \omega$ there is a partition of $[\varepsilon, 1]$ such that for each elementary subinterval I determined by the partition there is $U \in \mathcal{U}$ such that $(I \times J_n) \cap X \subset U$.

Let $\langle a_j \rangle$ enumerate the end points of the intervals $\{\hat{J}_n \ / \ n \in \omega\}$. In what follows, whenever we have an interval of the form (a,b) it is, of course, an open interval in $\mathbb{I}$, but if either $a = 0$ or $b = 1$ then we will assume that the interval contains that point. Similarly the interior of any interval will be taken as the interior in $\mathbb{I}$, so if the interval contains 0 or 1 then so does its interior.

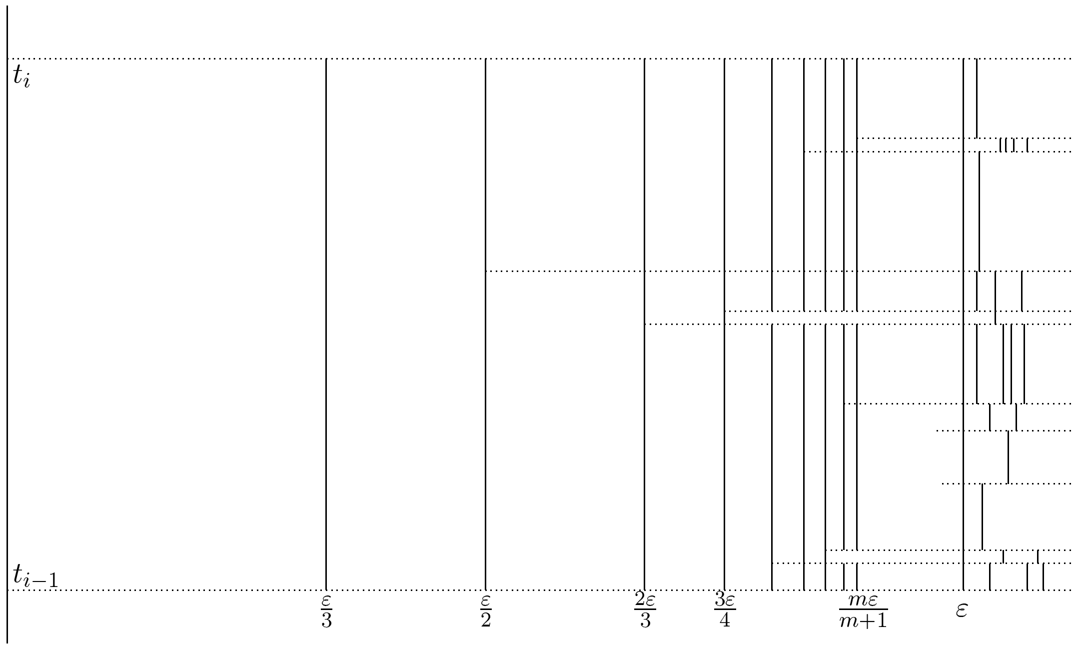
Construct a countable collection, $\mathcal{T}$, of tiles covering X as follows. $\mathcal{T}$ contains all sets of either of the following forms:

$$[0, \frac{\varepsilon}{3}] \times [t_{i-1}, t_i] - [0, \frac{\varepsilon}{3}] \times \{t_{i-1}, t_i\} \quad \text{and} \quad [\frac{\varepsilon}{3}, \frac{\varepsilon}{2}] \times (t_{i-1}, t_i).$$

We add further tiles inductively on $n \in \omega$. For each i reorder

$$(\{a_j \ / \ j \leq n\} \cup \{t_{i-1}, t_i\}) \cap [t_{i-1}, t_i]$$

so as to form a partition of $[t_{i-1}, t_i]$: suppose that J is an elementary subinterval of this partition. If $\mathring{J} = J_m$ for some m and J is also an elementary subinterval from a partition for a smaller n then ignore J; otherwise if $\mathring{J} = J_m$ for some m add to $\mathcal{T}$ a tile of the form $[\frac{(n+1)\varepsilon}{n+2}, \varepsilon] \times \mathring{J}$; if $\mathring{J}$ is not of the form J_m for any m add a tile of the form $[\frac{(n+1)\varepsilon}{n+2}, \frac{(n+2)\varepsilon}{n+3}] \times \mathring{J}$. Finally, for any $n \in \omega$ let I be an elementary subinterval determined by the partition of $[\varepsilon, 1]$ referred to in condition 5 above of the cover $\mathcal{J}$: add a tile of the form $I \times \hat{J}_n$.



Note that any tile from $\mathcal{T}$ meets at most 3 other tiles from $\mathcal{T}$ so the star order of $\mathcal{T}$ is at most 4. Thus by Proposition 4.5, X has star order at most 4. ∎

Remark. In the quest for a subset of the plane of star order greater than 4 we may ask whether adding to X the reflection of the space X of Proposition 4.14 in the y-axis will increase its star order as it seems superficially that each tile (except those at the ends) containing part of $\{0\} \times \mathbb{I}$ will meet 4 other tiles, viz one above, one below and one to either side. However the two tiles to either side may each be split in half vertically and then the two pieces nearest the y-axis combined to give a disconnected member of the open cover $\hat{\mathcal{T}}$.

Another superficially apparent way of obtaining a subset of the plane of star order greater than 4, but which also fails, is as follows. Take two complementary subsets of $\mathbb{R}$ each of dimension 0, for example the rationals $\mathbb{Q}$ and irrationals $\mathbb{P}$, and let

$$X = (\{0\} \times \mathbb{I}) \cup (\mathbb{I} \times (\mathbb{I} \cap \mathbb{Q})) \cup (\mathbb{I} \times ([-1, 0] \cap \mathbb{P})).$$

Using the same procedure as in Proposition 4.14 and the previous paragraph, we choose tiles covering $\{0\} \times \mathbb{I}$ as before with $t_i \in \mathbb{Q}$ but replace their former common boundary $[-\frac{\varepsilon}{4}, 0] \times \{t_i\}$ by a straight line segment which runs from $(0, t_i)$ to a point $(-\frac{\varepsilon}{4}, t_i')$, where $t_i' \in \mathbb{P}$ is near t_i (except that $t_i' = t_i$ when $t_i = 0$ or 1). Here "near" is intended to mean that the resulting tiles still lie in some member of $\mathcal{U}$. Then the construction of tiles to the left of the y-axis uses the partition determined by the numbers t_i' while that to the right uses the partition determined by the numbers t_i.

5 Conclusion

By analogy with covering dimension we may try to define a notion of dimension based on star order. More precisely for a boundedly strongly paracompact space X define the star covering dimension of X to be some appropriately chosen monotonically increasing function of the star order of X. Preferably the function would be chosen so that it gives $\mathbb{R}^n$ a star covering dimension of n. If the answer to Question 2 below is 'yes' we could declare that if the star order of X is p then the star covering dimension of X is $\log_2(p + 1) - 1$. Whatever the case, from Proposition 4.11, star order/dimension does give a way of discriminating between certain classes of graphs.

Question 1 *Does the star analogue of Theorem 2.2 hold, ie is it true that a space (maybe Hausdorff) is boundedly strongly paracompact if and only if it is both boundedly metacompact and strongly paracompact?*

Question 2 *Is the star order of* $\mathbb{R}^n$ *exactly* $2^{n+1} - 1$*?*

Question 3 *Is the star order of every subspace of* $\mathbb{R}^n$ *at most* $2^{n+1} - 1$*?*

Comment
From Propositions 4.2 and 4.7 we note that if $C \subset \mathbb{R}^n$ is closed then the star order of C is at most $2^{n+1} - 1$. It may be possible to answer this question in general by use of Lemma 4.6 to reduce to the open case and then imitate the proof of Proposition 4.7.

Question 4 *Do metrisable n-manifolds have star order* $2^{n+1} - 1$*?*

Comment
Of course the answer is 'no' for non-metrisable manifolds, the long line being an example of a 1-manifold with infinite star order.

Question 5 *In our tentative definition of star covering dimension we began with an arbitrary open cover of X. Is it true, as in the case of covering dimension, that it suffices to refine only finite open covers?*

Question 6 *How does star covering dimension of a boundedly strongly paracompact space compare with other dimensions, especially covering dimension?*

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