Microbundles, Manifolds and Metrisability*

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Abstract

The notion of a microbundle was introduced in the 1960s but the theory came to an abrupt halt when it was shown that for a metrisable manifold microbundles are equivalent to fibre bundles. In this paper we consider microbundles over non-metrisable manifolds. In some cases microbundles are equivalent to fibre bundles but in others they are not. In particular we show that a manifold is metrisable if and only if its tangent microbundle is equivalent to a fibre bundle. We also illustrate that for some non-metrisable manifolds every trivial microbundle contains a trivial fibre bundle whereas other manifolds may support a trivial microbundle not containing a trivial fibre bundle.

1 Definitions and Notation

Throughout this paper by a *manifold* we mean a connected Hausdorff space in which each point has a neighbourhood homeomorphic to euclidean space. It is well-known (cf [20, p 637]) that a manifold is metrisable if and only if it satisfies any one (and hence all) of the following properties: paracompact; σ-compact; second countable; meta-Lindelöf.

In 1964 Milnor [19] introduced the notion of a microbundle as a means of transferring some of the procedures applicable to bundles over smooth manifolds to manifolds which are not necessarily smooth. The development came to an abrupt halt when Kister [16] showed that over metrisable manifolds every microbundle is equivalent to a fibre bundle. At the time the major effort in the study of manifolds was concentrated on compact manifolds and, as far as we know, the study of non-metrisable manifolds did not begin systematically until the late 1970s.

Definition [19]. A *microbundle*, denoted $B \xrightarrow{i} E \xrightarrow{j} B$, consists of topological spaces B and E, called the *base space* and the *total space* respectively, and continuous

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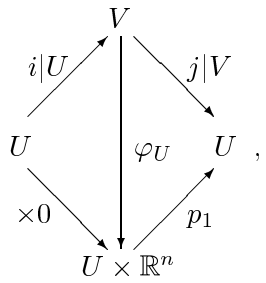
functions i and j, called the *injection* and *projection maps* respectively, such that the following conditions hold:

- $ji = 1_B$, the identity map on B; and
- there is an open cover $\mathcal{U}$ of B so that for each $U \in \mathcal{U}$ there are a set $V \subset j^{-1}(U)$, with $i(U) \subset V$, and a homeomorphism

$$\varphi_U : V \longrightarrow U \times \mathbb{R}^n$$

such that the following diagram commutes:



where $\times 0 : U \longrightarrow U \times \mathbb{R}^n$ is defined by $(\times 0)(u) = (u, 0)$, and p_1 is projection on the first coordinate.

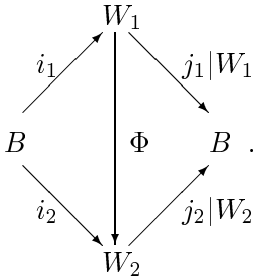
Definition. For any base space B, and any $n \geq 0$, $B \xrightarrow{\times 0} B \times \mathbb{R}^n \xrightarrow{p_1} B$ is called the *standard trivial microbundle*.

Definition. For any manifold M the diagram

$$M \xrightarrow{\Delta} M \times M \xrightarrow{p_1} M,$$

where Δ is the diagonal map, constitutes a microbundle; it is called the *tangent microbundle*.

Definition. Two microbundles $B \xrightarrow{i_1} E_1 \xrightarrow{j_1} B$ and $B \xrightarrow{i_2} E_2 \xrightarrow{j_2} B$ are *equivalent* if there exist neighbourhoods W_1 of $i_1(B)$ in E_1 and W_2 of $i_2(B)$ in E_2, and a homeomorphism $\Phi : W_1 \longrightarrow W_2$ such that the following diagram commutes:



Definition. A microbundle which is equivalent to the standard trivial bundle is called a *trivial microbundle*.

We will denote the fibre bundle with base space B, total space E, 0-section e and bundle projection p by $B \xrightarrow{e} E \xrightarrow{p} B$. In the following, fibre bundles will be assumed to have a 0-section and fibre $\mathbb{R}^n$.

A fibre bundle satisfies all of the requirements of a microbundle. Kister's result [16] shows that if the base manifold is metrisable then any microbundle admits a fibre bundle which is microbundle equivalent to the original microbundle. We address the question in the case where the base manifold is not metrisable.

Some of the results in this paper have been described in the authors' paper [12].

2 Trivial Microbundles may or may not be Equivalent to Trivial Bundles

Example 1 *For a particular non-metrisable manifold it may be the case that every trivial microbundle over the manifold contains a trivial fibre bundle.*

Milnor [19] showed that if the base space B of a trivial microbundle is paracompact then some open subset of the total space is homeomorphic to all of $B \times \mathbb{R}^n$. In this case there is an equivalent fibre bundle. This is also the case for any trivial microbundle over the long ray $\mathbb{L}^+$ since, by the pressing down lemma, any neighbourhood of $\mathbb{L}^+ \times \{0\}$ in $\mathbb{L}^+ \times \mathbb{R}$ must contain a long tube, i.e. a set $(\alpha, \omega_1) \times (-h, h)$ for some $\alpha \in \omega_1$ and some positive $h \in \mathbb{R}$.

Example 2 *There is a manifold supporting a trivial microbundle which does not contain a trivial fibre bundle with the same injection and projection.*

Consider the manifold M (allied to the Prüfer manifold) found in [20, Example 3.6] except that we interchange the x- and y-axes. Let

$$A = \{(x, y) : x, y \in \mathbb{R} \text{ and } x \neq 0\} \text{ and for each } y \in \mathbb{R}, \text{ let } B_y = \{(0, y)\} \times \mathbb{R}.$$

We will write $\langle y, z\rangle$ for a typical member $((0, y), z)$ of B_y. Set $M = A \cup (\cup_{y\in\mathbb{R}} B_y)$. M is topologised by using the usual topology on A and replacing each point, $(0, y)$, of the missing y-axis by the copy, B_y , of the real line. More precisely, suppose that $\langle y, z\rangle \in B_y$. Declare

$$\{\{\langle y, \zeta\rangle \in B_y : |z - \zeta| < 1/n\} \cup T(y, z, n) : n \in \mathbb{N}\}$$

to be a neighbourhood basis at $\langle y, z\rangle$, where

$$T(y, z, n) = \{(\xi, \eta) \in \mathbb{R}^2 : 0 < |\xi| < \frac{1}{n} \text{ and } z - \frac{1}{n} < \frac{\eta - y}{|\xi|} < z + \frac{1}{n}\}.$$

The triangles $T(y,z,n)$ are bounded by segments of rays emanating from $(0,y)$ with slopes $\pm(z-\frac{1}{n})$ and $\pm(z+\frac{1}{n})$, where the $\pm$ sign is determined by which side of the y-axis we are looking at. Because M contains an uncountable discrete closed subset it cannot be Lindelöf, hence cannot be metrisable.

Define an open subset E of $M \times (-1,1)$ by

$$E = [A \times (-1,1)] \cup \{(\langle y,z\rangle, t) \in B_y \times (-1,1) : y \in \mathbb{R} \text{ and } |t| < \min\{1, \frac{1}{|z|}\}\}.$$

Then $M \xrightarrow{\times 0} E \xrightarrow{p_1} M$ is a trivial microbundle which does not contain a trivial fibre bundle.

Suppose to the contrary that this microbundle did contain a trivial fibre bundle. Then there would be a continuous function $f : M \longrightarrow (0,1)$ so that

$$\forall p \in M \text{ and } \forall t \in (-1,1), |t| < f(p) \Rightarrow (p,t) \in E.$$

Such a function may be obtained from a trivial fibre bundle as follows: if $M \xrightarrow{\times 0} D \xrightarrow{p_1} M$ is a trivial fibre bundle then there is a fibre-preserving homeomorphism $h : M \times \mathbb{R} \longrightarrow D$, so we may define f by letting $f(p)$ be the second coordinate of $h(p,1)$.

For each $m,n \in \mathbb{N}$ set $S(m,n) = \{y \in \mathbb{R} : f(T(y,0,n)) \subset (\frac{1}{m},1)\}$. Then $\bigcup_{m,n\in\mathbb{N}} S(m,n) = \mathbb{R}$ because for each $y \in \mathbb{R}$ continuity of f implies that there are integers $m,n \in \mathbb{N}$ such that $f(T(y,0,n)) \subset (\frac{1}{m},1)$. Since the union is countable, $S(m,n)$ is uncountable for some $m,n \in \mathbb{N}$; suppose we have chosen such m,n. As an uncountable subset of $\mathbb{R}$, the set $S(m,n)$ contains a point a which is a limit point of $S(m,n) \cap (a,\infty)$. Thus $f(T(a,0,n)) \subset (\frac{1}{m},1)$. By the definition of E we have that $f(\langle a,m\rangle) < \frac{1}{m}$, so there is $l \in \mathbb{N}$ so that $f(T(a,m,l)) \subset (0,\frac{1}{m})$; we may assume that $l \geq n$. Pick $(x,y) \in T(a,m,l)$. Then $y > a$ and as a is a limit point of $S(m,n) \cap (a,\infty)$, there is $b \in S(m,n)$ with $a < b < y$; thus $f(T(b,0,n)) \subset (\frac{1}{m},1)$.

Because $a < b < y$ and $l \geq n$, it follows that $T(b,0,n) \cap T(a,m,l) \neq \varnothing$, say $(u,v) \in T(b,0,n) \cap T(a,m,l)$. On the one hand $(u,v) \in T(b,0,n)$ so $f(u,v) > \frac{1}{m}$. On the other hand $(u,v) \in T(a,m,l)$ so $f(u,v) < \frac{1}{m}$, a contradiction.

Thus such a continuous function as f cannot exist, so the trivial microbundle $M \longrightarrow E \longrightarrow M$ does not contain a trivial fibre bundle.

Example 3 *Two fibre bundles may be equivalent as microbundles while not being equivalent as fibre bundles. Indeed, the following fibre bundle is microbundle equivalent to the standard trivial microbundle but is not fibre bundle equivalent to the trivial fibre bundle.*

Let $\mathbb{L}^+ \xrightarrow{e} \mathbb{L}^+ \times \mathbb{L} \xrightarrow{p} \mathbb{L}^+$, where $\mathbb{L}^+$ is the long ray and $\mathbb{L}$ is the long line, be the fibre bundle in which p maps (x,y) to x if $x \geq |y|$ and to $|y|$ otherwise. Note that

$\{(0,\alpha) : \alpha \in \omega_1\}$ covers $\mathbb{L}^+$ and that each $p^{-1}(0,\alpha)$ is homeomorphic to $(0,\alpha) \times \mathbb{R}$. This bundle is microbundle equivalent to the standard trivial microbundle, for we may let

$$W_1 = W_2 = \{(x,y) \in \mathbb{L}^+ \times \mathbb{L} : |y| < x \leq 1\} \cup \{(x,y) \in \mathbb{L}^+ \times \mathbb{L} : |y| < 1 \leq x\},$$

and $\Phi : W_1 \longrightarrow W_2$ be the identity. However it is not equivalent to the trivial fibre bundle as $\mathbb{L}^+ \times \mathbb{L}$ is not homeomorphic to $\mathbb{L}^+ \times \mathbb{R}$.

3 Microbundles and Metrisability

Recall that a topological space is defined to be meta-Lindelöf if and only if every open cover has a point-countable open refinement. Also (cf [20, Theorem 2.5]) for a manifold the following conditions are equivalent: metrisable, paracompact, meta-Lindelöf, (hereditarily) Lindelöf, second countable and σ-compact.

Lemma 1 . *Let M be a manifold and suppose that $M = \cup_{n\in\mathbb{N}} U_n$, where for each $n \in \mathbb{N}$ we have that U_n is open and metrisable. Then M is metrisable.*

Proof. Let $\mathcal{V}$ be an open cover of M. For each $n \in \mathbb{N}$ let $\mathcal{U}_n = \{\ V \cap U_n\ :\ V \in \mathcal{V}\ \}$. Then $\mathcal{U}_n$ is an open cover of U_n. As U_n is metrisable it is also metacompact, hence $\mathcal{U}_n$ has a point finite open refinement, say $\mathcal{W}_n$. Let $\mathcal{W} = \cup_{n\in\mathbb{N}} \mathcal{W}_n$. Then $\mathcal{W}$ is a point-countable open refinement of $\mathcal{V}$. Thus M is meta-Lindelöf and hence metrisable.

Theorem 1 . *Let M^m be a manifold. Then the following conditions are equivalent:*
(i) M is metrisable;
(ii) there is an open cover $\mathcal{U}$ of M such that for each $x \in M$ the set $st(x,\mathcal{U})$ is homeomorphic to an open subset of $\mathbb{R}^m$;
(iii) there is an open cover $\mathcal{U}$ of M such that for each $x \in M$ the set $st(x,\mathcal{U})$ is metrisable;
(iv) there is an open cover $\mathcal{U}$ of M such that for each $x \in M$ the set $st(x,\mathcal{U})$ is Lindelöf;
(v) there is a cover $\mathcal{U}$ of M such that for each $x \in M$ the set $st(x,\mathcal{U})$ is open and Lindelöf;
(vi) there is a cover $\mathcal{U}$ of M such that for each $x \in M$ the set $st(x,\mathcal{U})$ is open and metrisable.

Proof. (i)$\Longrightarrow$(ii): suppose that M has a metric. As a metrisable manifold, M is σ-compact, so may be expressed as a union of countably many compact subsets, say $M = \bigcup_{n\in\mathbb{N}} C_n$. We may assume that $C_n \subset \mathring{C}_{n+1}$ for each $n \in \mathbb{N}$. When $n \leq 0$, by C_n we will mean the empty set.

By compactness we may cover $C_n - \mathring{C}_{n-1}$ by a finite family $\mathcal{V}_n$ of sets each of which is homeomorphic to an open subset of $\mathbb{R}^m$ and each of which is a subset of

$\mathring{C}_{n+1} - C_{n-2}$. Let $\varepsilon_n > 0$ be a Lebesgue number for $\mathcal{V}_n$, ie ε_n is such that every ball of radius ε_n and centre in $C_n - \mathring{C}_{n-1}$ lies in some member of $\mathcal{V}_n$. We let $\varepsilon_0 = 1$.

For each $n \in \mathbb{N}$, let $\mathcal{U}_n$ be a cover of $C_n - \mathring{C}_{n-1}$ by balls whose centres lie in $C_n - \mathring{C}_{n-1}$ and whose diameters are $\min\{\varepsilon_{n-1}, \varepsilon_n, \varepsilon_{n+1}\}$. Let $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \mathcal{U}_n$. If $x \in M$, say $x \in C_n - \mathring{C}_{n-1}$, then $st(x,\mathcal{U})$ is a union of balls from $\mathcal{U}_{n-1} \cup \mathcal{U}_n \cup \mathcal{U}_{n+1}$, hence of diameters at most ε_n. Thus $st(x,\mathcal{U}) \subset B(x;\varepsilon_n)$, so $st(x,\mathcal{U})$ lies in some member of $\mathcal{V}_n$ so is homeomorphic to an open subset of $\mathbb{R}^m$.

(ii)$\Longrightarrow$(iii): obvious.

(iii)$\Longrightarrow$(iv): let $\mathcal{U}'$ be a cover as in (iii) and let

$$\mathcal{U} = \{\ U \subset M\ :\ U \text{ is a component of some member of } \mathcal{U}'\ \}.$$

Then each $U \in \mathcal{U}$ is connected and open; thus for each $x \in M$, $st(x,\mathcal{U})$ is connected and open, and hence a manifold which is a subset of the metrisable manifold, $st(x,\mathcal{U}')$. Thus $st(x,\mathcal{U})$ is Lindelöf.

(iv)$\Longrightarrow$(v): obvious.

(v)$\Longrightarrow$(vi): take $\mathcal{U}$ as in (v). Then $st(x,\mathcal{U})$ is T_3 and Lindelöf, hence metrisable.

(vi)$\Longrightarrow$(i): suppose that $\mathcal{U}$ is a cover of M such that $st(x,\mathcal{U})$ is open in M and metrisable for each $x \in M$. Firstly we will show that the manifold contains a sequence $\langle V_\alpha : \alpha \in \omega_1\rangle$ of subsets of M satisfying

- V_α is Lindelöf, connected and open for each α,
- $\overline{V_\beta} \subset V_\alpha$ whenever $\beta < \alpha$, and
- $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ for each limit $\lambda \in \omega_1$

(i.e. M is of Type I) as follows.

1. Let V_0 be any open subset of M homeomorphic to $\mathbb{R}^m$.

2. If V_α is defined as above for any ordinal α , then V_α is separable: let D_α be a countable dense subset. Note that $\overline{V}_\alpha \subset \bigcup_{d\in D_\alpha} st(d,\mathcal{U})$, for given $x \in \overline{V}_\alpha$, choose $d \in D_\alpha \cap st(x,\mathcal{U})$; then $x \in st(d,\mathcal{U})$. Thus because V_α, hence $\overline{V}_\alpha$, is a connected set, it is contained in a single component of $\bigcup_{d\in D_\alpha} st(d,\mathcal{U})$. Let $V_{\alpha+1}$ be the component of $\bigcup_{d\in D_\alpha} st(d,\mathcal{U})$ containing $\overline{V}_\alpha$. By Lemma 1, the set $V_{\alpha+1}$ is Lindelöf. Clearly the other conditions required of $V_{\alpha+1}$ are also satisfied.

3. If λ is any limit ordinal and V_α is defined for each $\alpha < \lambda$ then $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ is Lindelöf, connected and open, being an increasing countable union of such sets, and the other two conditions are clearly satisfied.

Note that $M = \bigcup_{\alpha\in\omega_1} V_\alpha$. Indeed, as a union of open sets, $\bigcup_{\alpha\in\omega_1} V_\alpha$ is open. On the other hand, $\bigcup_{\alpha\in\omega_1} V_\alpha$ is closed, for suppose $x \in \overline{\cup_{\alpha\in\omega_1} V_\alpha}$ and let $\{U_n \ : \ n \in \mathbb{N}\}$ be a countable neighbourhood base at x. For each $n \in \mathbb{N}$ choose $\beta_n \in \omega_1$ such that $U_n \cap V_{\beta_n} \neq \varnothing$. Letting $\beta = sup\{\ \beta_n \ : \ n \in \mathbb{N}\}$, we have $U_n \cap V_\beta \neq \varnothing$ for each $n \in \mathbb{N}$ so that $x \in \overline{V}_\beta$ and hence $x \in V_{\beta+1}$, so $x \in \bigcup_{\alpha\in\omega_1} V_\alpha$. As a non-empty open and closed subset of the connected space M, $\bigcup_{\alpha\in\omega_1} V_\alpha$ is all of M.

Let Λ be the set of limit ordinals of ω_1. If for some $\lambda \in \Lambda$ we have $\overline{V}_\lambda - V_\lambda = \varnothing$ then V_λ is open and closed, so is all of M (by connectedness) and hence M is Lindelöf and hence metrisable.

Suppose instead that for each $\lambda \in \Lambda$ we have $\overline{V}_\lambda - V_\lambda \neq \varnothing$. Then M cannot be metrisable but we will obtain a contradiction.

Choose $x_\lambda \in \overline{V}_\lambda - V_\lambda$. Define $f : \Lambda \longrightarrow \omega_1$ by

$$f(\lambda) = \min\{\alpha \in \omega_1 : st(x_\lambda, \mathcal{U}) \cap V_\alpha \neq \varnothing\}.$$

As $st(x_\lambda, \mathcal{U})$ is a neighbourhood of $x_\lambda \in \overline{V}_\lambda$ it follows that $f(\lambda) < \lambda$. Thus by the pressing down lemma we may choose $\alpha \in \omega_1$ such that $A = f^{-1}(\alpha)$ is stationary.

Recall the dense subset D_α of V_α. For each $\lambda \in A$ choose $d_\lambda \in st(x_\lambda, \mathcal{U}) \cap D_\alpha$ and let $cst_\lambda(d_\lambda, \mathcal{U})$ denote the component of $st(d_\lambda, \mathcal{U})$ containing x_λ: note that $cst_\lambda(d_\lambda, \mathcal{U})$ is open and connected. Define $g : A \longrightarrow \omega_1$ by

$$g(\lambda) = \min\{\beta \in \omega_1 : cst_\lambda(d_\lambda, \mathcal{U}) \cap V_\beta \neq \varnothing\}.$$

Again we may apply the pressing down lemma to find $\beta \in \omega_1$ so that $B = g^{-1}(\beta)$ is stationary.

Because $cst_\lambda(d_\lambda, \mathcal{U}) \cap D_\beta \neq \varnothing$ for each $\lambda \in B$, and D_α and D_β are countable whereas B is uncountable, we may choose $d \in D_\alpha$ and $d' \in D_\beta$ such that

$$C = \{\lambda \in B \ : \ d_\lambda = d \text{ and } d' \in cst_\lambda(d_\lambda, \mathcal{U})\}$$

is uncountable. Thus for each $\lambda \in C$, we have that $cst_\lambda(d, \mathcal{U})$ is a component of $st(d, \mathcal{U})$. Since also $d' \in cst_\lambda(d, \mathcal{U})$, it follows that $N = \bigcup_{\lambda\in C} cst_\lambda(d, \mathcal{U})$ is connected (in fact $N = cst_\lambda(d, \mathcal{U})$ for each $\lambda \in C$). Then N is a metrisable manifold, hence Lindelöf.

Let $X = \{x_\lambda \in M \ : \ \lambda \in C\}$. On the one hand, because $X \subset N$ and N is hereditarily Lindelöf, X is Lindelöf. On the other hand, $\{V_\lambda : \lambda \in \Lambda\}$ forms an open cover of X with no countable subcover, and hence X cannot be Lindelöf, a contradiction.

Theorem 2 *The tangent microbundle over a manifold M is microbundle equivalent to a fibre bundle if and only if M is metrisable.*

Proof. $\Longrightarrow$: Suppose that the tangent microbundle, $M \xrightarrow{\Delta} M \times M \xrightarrow{p_1} M$, is equivalent to a fibre bundle $M \xrightarrow{e} E \xrightarrow{p} M$. Then there is an open neighbourhood W of $\Delta(M)$ in $M \times M$, which is fibre-preserving homeomorphic to an open neighbourhood W' of $e(M)$ in E.

For each $x \in M$ let $F_x = \{y \in M : (x, y) \in W\}$, an open subset of M. As part of the fibre $p_1^{-1}(x)$ the set F_x is homeomorphic to an open subset of $\mathbb{R}^m$. Also for each $x \in M$ choose an open subset U_x of M such that $U_x \times U_x \subset W$. Set $\mathcal{U} = \{U_x : x \in M\}$. Then $\mathcal{U}$ satisfies condition (ii) of Theorem 1 as $\forall x \in M$, $st(x, \mathcal{U}) \subset F_x$.
$\Longleftarrow$: If M is metrisable then every microbundle over M admits a fibre bundle by [16, corollary 1].

Remark. We may use the ideas of the proof of Theorem 2 to obtain an alternative proof of the (i)$\Longrightarrow$(ii) part of the proof of Theorem 1. If M is metrisable then by [16, corollary 1] there is a fibre bundle which may be embedded fibrewise as a neighbourhood of the diagonal in $M \times M$. Then the open cover $\mathcal{U}$ is obtained from the fibres of this fibre bundle.

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