IRREGULARITY

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Abstract. Regular and irregular pretopologies are studied. In particular, for every ordinal there exists a topology such that the series of its partial (pretopological) regularizations has length of that ordinal. Regularity and topologicity of special pretopologies on some trees can be characterized in terms of sets of intervals of natural numbers, which reduces studied problems to combinatorics.

Contents

1. Introduction

By a *convergence* we understand a relation $x \in \lim \mathcal{F}$, between filters $\mathcal{F}$ and points x, such that $\mathcal{F} \subset \mathcal{G}$ implies $\lim \mathcal{F} \subset \lim \mathcal{G}$, and for which the principal ultrafilter of x converges to x for every point x. A convergence ζ is *finer* than a convergence ξ (in symbols, $\zeta \geq \xi$) if $\lim_\zeta \mathcal{F} \subset \lim_\xi \mathcal{F}$ for each filter $\mathcal{F}$. A map f from a convergence space to another is *continuous* provided that $f(\lim \mathcal{F}) \subset \lim f(\mathcal{F})$ for every filter $\mathcal{F}$ ([1]). The class of convergences is a category (with continuous maps as morphisms). A convergence is *Hausdorff* if the limit of every filter is at most a singleton.

The notion of regularity was generalized from topological to convergence spaces in two ways, by Fischer [13] and by Grimeisen [15][16]. A convergence is *regular* (in the sense of Fischer) if the limit of a filter $\mathcal{F}$ is included in the limit of the filter generated by the family of the adherences of the elements of $\mathcal{F}$. The definitions

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[1]We denote by $f(\mathcal{F})$ the filter generated by $\{f(F) : F \in \mathcal{F}\}$.

of Fischer and Grimeisen coincide for pseudotopological spaces, and *a fortiori* for pretopological spaces, which are the framework of this paper ([2]).

Regular convergences form a concretely reflective subcategory of the category of convergences; we denote its reflector by R. In particular, for every convergence ξ there exists a regular convergence $R\xi$, which is the finest among the regular convergences that are coarser than ξ. The convergence $R\xi$ is the *regular reflection* of ξ (the *regularization* of ξ).

To our knowledge, there exists no explicit description of the filters that converge in $R\xi$ in terms of those convergent in ξ. It is possible however to define, explicitly and simply, a *partial regularization* $r\xi$ of ξ so that $\xi \geq r\xi \geq R\xi$ for every convergence ξ, and a convergence τ is regular if and only if $\xi \leq r\xi$ ([3]). Moreover r is a concrete functor, and for each convergence ξ there is a least ordinal α (the *irregularity* of ξ) such that $R\xi$ is equal to the α-th iteration ([4]) of r applied to ξ.

In this paper we show that for each ordinal α there exists a Hausdorff pretopology the irregularity of which is precisely α. Our result is more precise (and our construction is much simpler) than that of Kent and Richardson [18][19] who proved that for every ordinal β there exists a pretopology ξ such that β is the least ordinal for which $(r^{\omega})^{\beta}\xi = R\xi$.

We call an element x *regular* for a convergence ξ if $x \in \lim_{r\xi} \mathcal{F}$ implies $x \in \lim_{\xi} \mathcal{F}$ for every filter $\mathcal{F}$, and *irregular* otherwise. We witness an interesting phenomenon of "propagation of irregularities" concerning the regularity of elements: an element can be regular for a convergence ξ but irregular for its partial regularization $r\xi$, which, by construction, is "more regular" than ξ. This observation leads to a notion of irregularity spectrum.

The *irregularity* of x with respect to ξ is the least ordinal β such that x is regular for $r^{\beta}\xi$. The *irregularity spectrum* of an element x with respect to a convergence ξ is the set of ordinals α for which x is irregular for $r^{\alpha}\xi$. Consequently, an element is irregular if and only if 0 is in its spectrum. It is amazing that for every subset A of an ordinal, one can construct a Hausdorff pretopology such that the irregularity spectrum of an element with respect to this convergence is precisely A.

Study of regularity (and irregularity) of some special pretopologies on sequential trees (*standard pretopologies*) led us to a concept of *states* (sets of intervals of an ordinal). Each standard pretopology is completely determined by its state, and the functors r, R are transferred to the space of states. In this way, each investigation concerning regularity of such a pretopology can be reduced to a combinatorial problem concerning states.

We have observed that an element x of a pretopology of *countable character* ([5]) is irregular (thus of irregularity ≥ 1), then there exists a homeomorphic embedding "at x" of an irregular standard pretopology (on a tree of rank 2). On the other hand, the fact that an element x is of irregularity 2 does not imply the existence of a homeomorphic embedding "at x" of an irregular standard pretopology on a tree of rank 3.

[2]Pseudotopologies and pretopologies are subclasses of convergences; we will define them in Preliminaries.

[3]Kent and Richardson [18][19] introduced another functor of partial regularization, which in our terminology is equal to r^{ω} .

[4]to be defined later.

[5]also called *first-countable*

This discovery led us to a concept of *ramified standard pretopologies* and to our main result that *if x is an element of finite irregularity of a pretopology of countable character, then there is a homeomorphic embedding "at x" of a ramified standard pretopology of the same irregularity.*

2. Preliminaries

Families $\mathcal{F},\mathcal{H}$ (of subsets of a given set) *mesh* (in symbols, $\mathcal{F}\#\mathcal{H}$) if $F\cap H\neq\varnothing$ for every $F\in\mathcal{F}$ and each $H\in\mathcal{H}$. A systematic use of the operation $\#$ in conjunction with other operations, like that of contour, has led to a versatile calculus (see, for example, [8],[9],[3],[11],[4]). The operation $\#$ is related to the notion of *grill* $\mathcal{H}^{\#}$ of a family $\mathcal{H}$, which was defined by Choquet [1] as $\mathcal{H}^{\#}=\bigcap_{H\in\mathcal{H}}\{G : G\cap H\neq\varnothing\}$ (denoted also $\sec(\mathcal{H})$ in [17]); of course,

$$\mathcal{F}\#\mathcal{H}\Leftrightarrow\mathcal{F}\subset\mathcal{H}^{\#}\Leftrightarrow\mathcal{H}\subset\mathcal{F}^{\#}.$$

The *adherence* of a filter $\mathcal{H}$ with respect to a convergence ξ is defined by

$$\operatorname{adh}_\xi\mathcal{H}=\bigcup_{\mathcal{F}\#\mathcal{H}}\lim_\xi\mathcal{F}.$$

In particular, $\operatorname{adh}_\xi H$ denotes the adherence of the principal filter of H. If $\mathcal{F}$ is a filter on the underlying set $|\xi|$ of a convergence ξ, then the symbol $\operatorname{adh}^{\natural}_\xi\mathcal{F}$ denotes the filter generated by $\{\operatorname{adh}_\xi F : F\in\mathcal{F}\}$. The infimum $\mathcal{V}_\xi(x)$ of all filters that converge to x, is called the *vicinity filter* of x with respect to ξ.

A convergence ξ is *regular* (in the sense of Fischer) if

$$\lim_\xi\mathcal{F}\subset\lim_\xi(\operatorname{adh}^{\natural}_\xi\mathcal{F}) \tag{2.1}$$

for every filter $\mathcal{F}$ ([6]). If ξ is a topology, then $x\in\lim_\xi\mathcal{F}$ is equivalent to $\mathcal{N}_\xi(x)\subset\mathcal{F}$ and $\operatorname{adh}_\xi H$ is equal to the closure $\operatorname{cl}_\xi H$ of H, and thus, for topologies ξ, (2.1) is equivalent to

$$\mathcal{N}_\xi(x)\subset\operatorname{cl}^{\natural}_\xi\mathcal{N}_\xi(x),$$

that is, a topology is regular if each neighborhood filter admits a base of closed sets.

More generally, if θ is any convergence on $|\xi|$, then a convergence ξ is said to be *θ-regular* if

$$\lim_\xi\mathcal{F}\subset\lim_\xi(\operatorname{adh}^{\natural}_\theta\mathcal{F}) \tag{2.2}$$

for every filter $\mathcal{F}$. If now J is a concrete functor, then ξ is called *J-regular* if it is $J\xi$-regular.

If $\mathcal{W}(y)$ is a family of subsets of X for every $y\in Y$, and if $\mathcal{A}$ is a family of subsets of Y, then the *contour* of $\mathcal{W}$ along $\mathcal{A}$ is defined by

$$\int_{\mathcal{A}}\mathcal{W}=\mathcal{W}(\mathcal{A})=\bigcup_{A\in\mathcal{A}}\bigcap_{y\in A}\mathcal{W}(y). \tag{2.3}$$

It seems that this notion, introduced by Kowalsky for filters in [20] (the so called *diagonal operation*), and used by many authors under various names (in [2] Cook and Fischer call it the *compression operator*, Frolík uses it for ultrafilters in [14] as the *sum of ultrafilters*), appears in the full generality of (2.3) for the first time in [7].

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[6]Notice that a regular convergence need not be Hausdorff.

The following formula belongs to the calculus of grills and contours, mentioned above: if $\mathcal{A}$ and $\mathcal{B}$ are families of sets, then

$$\mathcal{A}\#\mathcal{V}_\xi(\mathcal{B}) \Longleftrightarrow (\operatorname{adh}_\xi^\natural \mathcal{A})\#\mathcal{B}. \tag{2.4}$$

A convergence is a *pseudotopology* if $\lim \mathcal{F} \supset \bigcap_{\mathcal{H}\#\mathcal{F}} \operatorname{adh}\mathcal{H}$ for every filter $\mathcal{H}$. The class of pseudotopologies is a concretely reflective subcategory of the category of convergences. It is known [18] that a pseudotopology ξ is regular if and only if

$$\operatorname{adh}_\xi \mathcal{V}_\xi(\mathcal{H}) \subset \operatorname{adh}_\xi \mathcal{H} \tag{2.5}$$

A convergence is a *pretopology* if $\lim \mathcal{F} \supset \bigcap_{H\#\mathcal{F}} \operatorname{adh} H$. A convergence ξ is a pretopology if and only if $x \in \lim_\xi \mathcal{V}_\xi(x)$ for every $x \in |\xi|$. A pseudotopology is a pretopology if and only if $\bigcap_{H\#\mathcal{G}} \operatorname{adh} H \subset \operatorname{adh}\mathcal{G}$ for every filter $\mathcal{G}$. A pretopology ξ is a topology if and only if $\mathcal{V}_\xi(\mathcal{V}_\xi(x))$ for every $x \in |\xi|$. The classes of topologies, pretopologies and pseudotopologies are concretely reflective subcategories of the category of convergences. We denote by T, P and S the corresponding reflectors.

3. Partial regularizations

The *partial regularization* r associates with each convergence ξ another convergence $r\xi$ as follows: $x \in \lim_{r\xi} \mathcal{F}$ if there exists a filter $\mathcal{G}$ such that $x \in \lim_\xi \mathcal{G}$ and $\mathcal{F} \geq \operatorname{adh}_\xi^\natural \mathcal{G}$.

It is clear that $\xi \geq r\xi$ and that ξ is regular if and only if $\xi \leq r\xi$. The partial regularization can be iterated: for each ordinal $\beta > 1$, set

$$r^\beta \xi = r(\bigwedge_{\alpha<\beta} r^\alpha \xi),$$

where $\bigwedge$ stands for the infimum in the complete lattice of convergences on a fixed (underlying) set. Sometimes we consider intermediate iterations

$$r^{<\beta}\xi = \bigwedge_{\alpha<\beta} r^\alpha \xi.$$

Of course, $r^\beta\xi = r(r^{<\beta}\xi)$, and $r^{<\beta}\xi = r^{\beta-1}$ in case β is an isolated ordinal. As every set can be well-ordered, for every convergence ξ there is a least ordinal β (called the *irregularity* of ξ) such that $r^{\beta+1}\xi = r^\beta\xi$ and thus $r^\beta\xi = R\xi$.

The *irregularity* $\rho(x,\xi)$ of x with respect to ξ is the least ordinal such that x is regular for $r^\beta\xi$ for every $\beta \geq \rho(x,\xi)$. Let us observe that β is the irregularity of x if and only if for every $\alpha < \beta$ there exists a filter $\mathcal{F}_\alpha$ such that $x \notin \lim_{r^\alpha\xi} \mathcal{F}_\alpha$ and $x \in \lim_{r^\beta\xi} \mathcal{F}_\alpha$. The inversion of quantifiers leads to a slightly stronger property: the irregularity $\rho(x,\xi)$ is *strong* if there is a filter $\mathcal{F}$ such that $x \in \lim_{r^{\rho(x,\xi)}\xi} \mathcal{F}$ and $x \notin \lim_{r^\alpha\xi} \mathcal{F}$ for all $\alpha < \beta$. Notice that if the regularity of an element is an isolated ordinal, then it is strong.

By definition, $A \sqcup B$ is defined and equal to $A \cup B$ whenever $A \cap B = \varnothing$; similarly, $\bigsqcup_{A\in\mathcal{A}} A$ is defined and equal to $\bigcup_{A\in\mathcal{A}} A$ whenever $A_0 \cap A_1 = \varnothing$ for every two distinct elements A_0, A_1 of $\mathcal{A}$.

Example 3.1. *Consider $T = \{o\} \sqcup \{t_n : n < \omega\} \sqcup \{t_{n,k} : n,k < \omega\}$. Let τ be the finest topology on T, for which $o = \lim(t_n)_n$, and $t_n = \lim(t_{n,k})_k$ for each $n < \omega$. Consider now a topology ξ on T such that the neighborhood filter $\mathcal{N}_\xi(t) = \mathcal{N}_\tau(t)$ for each $t \neq o$, and $\mathcal{N}_\xi(o)$ is generated by the restriction of $\mathcal{N}_\tau(o)$ to $T\backslash\{t_n : n < \omega\}$.*

We observe that $\mathrm{adh}^{\natural}_{\xi}\mathcal{N}_{\xi}(o)=\mathcal{N}_{\tau}(o)$ *and thus is not equal* $\mathcal{N}_{\xi}(o)$*, which shows that* o *is irregular for* ξ*. Actually,* $r\xi=\tau$ *and so the irregularity of* o *is* 1.

Example 3.2. *Consider the set* T *of Example 3.1 and the following topology* π *on* T*: the sets* $B_m=\{o\}\cup\{t_{n,k}:m\le n<\omega,k<\omega\}$ *with* $m<\omega$ *form a neighborhood base of* o*, the sets* $\{t_n\}\cup\{t_{n,k}:l\le k<\omega\}$ *with* $l<\omega$ *form a neighborhood base of* t_n *for each* $n<\omega$*, and the elements of the form* $t_{n,k}$ *are isolated. Then* $\mathrm{cl}_{\pi}B_m=B_m\cup\{t_n:m\le n<\omega\}$*, so* $\mathrm{adh}^{\natural}_{\pi}\mathcal{N}_{\pi}(o)$ *is not equal to* $\mathcal{N}_{\pi}(o)$*, and thus* o *is irregular for* π*. As* $r\pi$ *is regular, the irregularity of* o *is* 1.

The two examples above are very similar from the point of view of regularity. In fact, all the elements of T have the same irregularity spectra for ξ and for π. An importance difference between ξ and π is that only the second is of countable character (We recall that for a cardinal number κ, a convergence is of *character* κ if $x\in\lim\mathcal{F}$ implies the existence of a filter $\mathcal{E}$ admitting a base of cardinality not greater than κ and such that $x\in\lim\mathcal{E}$ and $\mathcal{E}\le\mathcal{F}$; if $\kappa=\aleph_0$ then we say that a convergence is of *countable character*).

Proposition 3.3. *For every ordinal* β*, there exists a Hausdorff pretopology of irregularity* β *(of cardinality* $|\beta|\vee\aleph_0$*).*

Proof. Actually we will show that this irregularity is attained at an element for which it is strong. The irregularity of each regular pretopology is 0. Examples 3.1 and 3.2 describe a Hausdorff topology of irregularity 1 and of cardinality $\aleph_0$. Suppose that $\beta>1$ and that for each $\alpha<\beta$, there exists a set X_α (of cardinality $|\alpha|\vee\aleph_0$), a Hausdorff pretopology π_α on X_α, an element x_α of X_α, and a free filter $\mathcal{F}_\alpha$ on X_α such that $x_\alpha\in\lim_{r^{\alpha}\pi_\alpha}\mathcal{F}_\alpha\setminus\lim_{r^{\gamma}\pi_\alpha}\mathcal{F}_\alpha$ for each $\gamma<\alpha$. If β is limit, then consider the simple sum ([7]) $\bigoplus_{\alpha<\beta}\pi_\alpha$ on $\bigsqcup_{\alpha<\beta}X_\alpha$ and let $\mathcal{F}$ be the image on $\{x_\alpha:\alpha<\beta\}$ of the coarsest filter on $\beta=\{\alpha:\alpha<\beta\}$ that converges to β in the natural topology. Define a pretopology π on $\bigsqcup_{\alpha<\beta}X_\alpha\sqcup\{o\}$ (which is of cardinality $|\beta|\vee\aleph_0$) by setting $\{o\}=\lim_{\pi}\int_{\mathcal{F}}(\mathcal{F}_{\ \alpha})_{\alpha<\beta}$.

This is a Hausdorff pretopology of cardinality $|\beta|$, and $o\notin\mathrm{adh}_{r^{\gamma}\pi}\mathcal{F}$ for each $\gamma<\beta$ but $o\in\lim_{r^{\beta}\pi}\mathcal{F}$, because $\mathrm{adh}^{\natural}_{r^{\beta}\pi}\int_{\mathcal{F}}(\mathcal{F}_{\ \alpha})_{\alpha<\beta}\le\mathcal{F}$. If β is isolated, then mimic the construction above, on replacing $\{\pi_\alpha:\alpha<\beta\}$ by countable infinite simple sum of copies of $\beta-1$, and $\mathcal{F}$ by the cofinite filter of a countable infinite set of copies of $x_{\beta-1}$. ∎

The construction in the proof of Proposition 3.3 shows the existence of pretopologies of arbitrary irregularity attained at point of strong irregularity. Example 3.4 illustrates a case of irregularity that is not strong. The construction in the proof of Proposition 3.3 uses elements the irregularity of which is strong. Here is an example of an element whose irregularity is ω_0 and is not strong.

Example 3.4. *Let* π_n *be a Hausdorff pretopology on* X_n *of cardinality* $\aleph_0$ *of irregularity* n *attained at* x_n*. Let* $\mathcal{F}_n$ *be a filter such that* $x_n\in\lim_{r^{n}\pi_n}\mathcal{F}_n\setminus\lim_{r^{n-1}\pi_n}\mathcal{F}_n$*. Take the simple sum* $\bigoplus_{n<\omega}\pi_n$ *on* $\bigsqcup_{n<\omega}X_n$ *and take the pretopological quotient* π *by identifying all* x_n *in* o *. Then* $o\in\lim_{r^{n}\pi}\mathcal{F}_k$ *exactly for* $k\le n$*,* $r^{\omega}\pi=r^{<\omega}\pi$*, and* $o\in\lim_{r^{\omega}\pi}\mathcal{F}_n\setminus\lim_{r^{n-1}\pi_n}\mathcal{F}_n$ *for every* $n<\omega$ *, while there is no filter which converges to* o *in* $r^{\omega}\pi$ *but does not converge in* $r^{n}\pi$ *for every* $n<\omega$*, that is, does*

[7]If ξ_i is a convergence on X_i for each $i\in I$ then $x\in\lim_{\bigoplus_{i\in I}\xi_i}\mathcal{F}$ if there is $i\in I$ such that $x\in X_i\in\mathcal{F}$ and $x\in\lim_{\xi_i}\mathcal{F}$.

not converge in $r^{<\omega}\pi$. If we pretopologize $r^{\omega}\pi$ then we get a regular pretopology, the vicinity filter at o of which is equal to $\bigwedge_{n<\omega}\mathcal{F}_n \wedge \{o\}$, where $\{o\}$ stands for the principal ultrafilter of o.

4. Regularity in the category of pretopologies

We shall concentrate here on regularity in the case of pretopologies. This level of generality, on one hand, enables one to notice several interesting phenomena (like the propagation of irregularities) that are not visible in the realm of topologies, and on the other to avoid certain complexity, which can be qualified as technical, and which is not essential for the phenomena mentioned above.

Proposition 4.1. *If $\mathcal{H}$ is a filter, then*

$$\operatorname{adh}_{r\xi}\mathcal{H} = \operatorname{adh}_{\xi}\mathcal{V}_{\xi}(\mathcal{H}). \tag{4.1}$$

Proof. By definition, $x \in \operatorname{adh}_{r\xi}\mathcal{H}$ if there exists a filter $\mathcal{F} \geq \mathcal{H}$ such that $x \in \lim_{r\xi}\mathcal{F}$, hence there is a filter $\mathcal{G}$ such that $x \in \lim_{\xi}\mathcal{G}$ and $\operatorname{adh}^{\natural}_{\xi}\mathcal{G} \leq \mathcal{F}$, thus $\operatorname{adh}^{\natural}_{\xi}\mathcal{G}$ meshes with $\mathcal{H}$, equivalently $\mathcal{G}$ meshes with $\mathcal{V}_{\xi}(\mathcal{H})$, which means that $x \in \operatorname{adh}_{\xi}\mathcal{V}_{\xi}(\mathcal{H})$. Conversely, if $x \in \operatorname{adh}_{\xi}(\mathcal{V}_{\xi}(\mathcal{H}))$ then there is a filter $\mathcal{G}$ such that $\mathcal{G}\#\mathcal{V}_{\xi}(\mathcal{H})$ and $x \in \lim_{\xi}\mathcal{G}$, hence $\operatorname{adh}^{\natural}_{\xi}\mathcal{G}$ meshes with $\mathcal{H}$ and $\operatorname{adh}^{\natural}_{\xi}\mathcal{G}$ converges to x in $r\xi$, so that $x \in \operatorname{adh}_{r\xi}\mathcal{H}$. ∎

In particular, for every set H,

$$\operatorname{adh}_{r\xi} H = \operatorname{adh}_{\xi}(\mathcal{V}_{\xi}(H)).$$

Proposition 4.2. *If ξ is a pretopology, then $r\xi$ is a pretopology, and*

$$\mathcal{V}_{r\xi}(x) = \operatorname{adh}^{\natural}_{\xi}\mathcal{V}_{\xi}(x). \tag{4.2}$$

Proof. If ξ is a pretopology, then by (4.1) for every set A,

$$\operatorname{adh}_{r\xi} A = \bigcap\nolimits_{V\in\mathcal{V}_{\xi}(A)} \operatorname{adh}_{\xi} V.$$

By definition, a set A meshes with $\mathcal{V}_{r\xi}(x)$ if and only if $x \in \operatorname{adh}_{r\xi} A$, so when ξ is a pretopology, if and only if $x \in \operatorname{adh}_{\xi} V$ for every $V \in \mathcal{V}_{\xi}(A)$, equivalently if $V \in \mathcal{V}_{\xi}(A)$ then $\mathcal{V}_{\xi}(x)\#V$, that is, $\mathcal{V}_{\xi}(A)\#\mathcal{V}_{\xi}(x)$, which amounts to $A\#\operatorname{adh}^{\natural}_{\xi}\mathcal{V}_{\xi}(x)$. Therefore (4.2) holds. Now if ξ is a pretopology $x \in \lim_{r\xi}\mathcal{F}$ whenever $\mathcal{F} \geq \operatorname{adh}^{\natural}_{\xi}\mathcal{V}_{\xi}(x)$, which proves that $r\xi$ is a pretopology. ∎

Corollary 4.3. *If ξ is a pretopology, then $r^n\xi$ is a pretopology for every n.*

In general, the pretopologicity of ξ does not imply that $r^{<\omega}\xi$ is a pretopology. Therefore, starting from ω_0, we need distinguish between *iterated partial regularizations* and *pretopologically iterated partial regularizations*, which are $r_P\xi = r\xi$ and if $\beta > 1$,

$$r^{\beta}_{P}\xi = rP(\bigwedge_{\alpha<\beta} r^{\alpha}_{P}\xi).$$

If Ξ is a set of pretopologies (on a given set), then the infimum $\bigwedge^{P}\Xi$ of Ξ in the category of pretopologies is equal to $P(\bigwedge\Xi)$ (the pretopological reflection of the infimum $\bigwedge\Xi$ of Ξ in the category of convergences). One can easily compute the corresponding adherence for principal filters (which determines the pretopology), namely [7, (2.17)]

$$\mathrm{adh}_{\bigwedge \Xi} A = \bigcup_{\xi \in \Xi} \mathrm{adh}_\xi A.$$

In agreement with our convention, a convergence ξ is *topologically regular* if ξ is $T\xi$-regular ([8]). We observe here a peculiar, but simple fact concerning regular pretopologies, which seems to have passed unnoticed so far.

Proposition 4.4. *Each regular pretopology is topologically regular.*

Proof. Let ξ be a regular pretopology, that is, $\mathcal{V}_\xi(x) \subset \mathrm{adh}^\natural_\xi \mathcal{V}_\xi(x)$. By applying $\mathrm{adh}^\natural_\xi$ to this inclusion, we get $\mathrm{adh}^\natural_\xi \mathcal{V}_\xi(x) \subset \mathrm{adh}^\natural_{\xi^2} \mathcal{V}_\xi(x)$ hence $\mathcal{V}_\xi(x) \subset \mathrm{adh}^\natural_{\xi^2} \mathcal{V}_\xi(x)$. Therefore $\mathcal{V}_\xi(x) \subset \mathrm{adh}^\natural_{\xi^n} \mathcal{V}_\xi(x)$ for every $n < \omega$ and thus $\mathcal{V}_\xi(x) \subset \mathrm{adh}^\natural_{\xi^\beta} \mathcal{V}_\xi(x)$ for every ordinal β, so that $\mathcal{V}_\xi(x) \subset \mathrm{cl}^\natural_\xi \mathcal{V}_\xi(x)$. ∎

The property above does not hold for general convergences ([9]). For each $n, m < \omega$ let $A_{n,m}$ be a countably infinite set such that $A_{n,m+1}$ is a partition of $A_{n,m}$. Let $A_m = \bigsqcup_{n<\omega} A_{n,m}$ and $A = \bigsqcup_{m<\omega} A_m$ will be called a *sink* (of countable character). The map $\pi^m_{m+1} : A_m \to A_{m+1}$ is the quotient defined by A_{m+1} on A_m. The *natural convergence* of a sink is defined by the fact that for each n, m and $p \in A_{n,m+1}$ the cofinite filter $\mathcal{N}(p)$ of $(\pi^m_{m+1})^-(p)$ converges to p. Of course, the natural convergence of a sink is sequential. Let $\mathcal{F}_m$ be the filter generated by $\{\bigcup_{n\geq l} A_{n,m} : l < \omega\}$.

Example 4.5. *Let $A = \bigsqcup_{n,m<\omega} A_{n,m}$ be a sink endowed with its natural convergence. We extend the convergence of A to $X = \{\infty\} \cup A$ so that $\bigwedge_{m\leq k} \mathcal{F}_m$ converges to ∞ for every $k < \omega$. This is a Hausdorff pseudotopology of countable character. It is regular, because $\mathrm{adh}^\natural \mathcal{F}_m = \mathcal{F}_{m+1} \wedge \mathcal{F}_m$, hence $\mathrm{adh}^\natural \left(\bigwedge_{m\leq k} \mathcal{F}_m\right) \geq \bigwedge_{m\leq k+1} \mathcal{F}_m$, and $\mathrm{adh}^\natural \mathcal{N}(p) = \mathcal{N}(p)$ for each n, m and $p \in A_{n,m+1}$. But it is not topologically regular, because $\mathrm{cl}^\natural \mathcal{F}_0 = \bigwedge_{m<\omega} \mathcal{F}_k$, and the latter filter does not converge to ∞.*

Actually, much more can be said if the underlying set is countable. If a convergence is a pretopology, then it is Hausdorff if $\mathcal{V}(x_0)$ does not mesh with $\mathcal{V}(x_1)$ when $x_0 \neq x_1$. It is straightforward that each point of a Hausdorff pretopology is closed, in other words, the pretopology is T_1.

Theorem 4.6. *The topological reflection of a Hausdorff regular pretopology on a countable set is normal, hence regular.*

Theorem 4.6 slightly improves [21, Theorem 2.4] by Nyikos and Vaughan who attribute it to Foged. Actually the authors do not mention pretopologies, but talk about weak bases of a topology. A *weak base* of a topology τ on X is a union of filter bases $\mathcal{B}(x)$ where $x \in X$ such that $x \in B$ if $B \in \mathcal{B}(x)$, and O is open whenever $x \in O$ implies the existence of $B \in \mathcal{B}(x)$ such that $B \subset O$. If we define a pretopology π by declaring $\mathcal{B}(x)$ to be a base of the vicinity filter $\mathcal{V}_\pi(x)$, then it is clear that $\tau = T\pi$. In these terms, τ is *weakly* T_2 means that π is Hausdorff, and τ is *weakly* T_3 means that π is topologically regular. Thus by virtue of Proposition

[8] The notions of pretopological and pseudotopological regularity coincide with that of regularity, because $\mathrm{adh}^\natural_{P\xi} = \mathrm{adh}^\natural_{S\xi} = \mathrm{adh}^\natural_\xi$.

[9] Not even for pseudotopologies.

4.4, we could relax the original assumption of topological regularity of [21, Theorem 2.4].

It follows from [10] that the countability assumption in Theorem 4.6 cannot be removed.

5. Standard pretopologies

We found it useful to study regularity problems for some special pretopologies on certain trees (called cascades), which are well-founded with respect to the inverse order. It turns out that, for such pretopologies, properties related to regularity and topologicity can be reduced to some combinatorial properties of subintervals of ordinal numbers ([10]).

Denote by Σ the *sequential tree*, that is, the set of finite sequences of natural numbers (This notation is better adapted to our considerations than the traditional $\omega^{<\omega}$). The empty sequence (in other words, the sequence of length 0) is denoted by o. If $s = (n_1, \ldots, n_p)$ and $t = (m_1, \ldots, m_q)$ are elements of Σ, then the *concatenation* $(n_1, \ldots, n_p, m_1, \ldots, m_q)$ of s and t is denoted by $s \frown t$. The abbreviation (s, n) for $s \frown (n)$ (where $s \in \Sigma$ and $n < \omega$) is a useful abuse of notation. By definition, $s < t$ if there is a non-empty finite sequence r such that $t = s \frown r$. With this partial order Σ becomes a tree. The *level* $l(t)$ of an element $t \in \Sigma$ is the length of the corresponding finite sequence.

Recall that a subset S of a partially ordered set V is *closed downwards* if $v < s \in S$ implies $v \in S$ for each $v \in V$. A subset T of Σ is called *full* if $T \cap \Sigma^+(s) \neq \varnothing$ implies that $\Sigma^+(s) \subset T$ for every $s \in \Sigma$.

The *standard topology* on Σ is defined with the aid of the following neighborhood bases $\mathcal{B}(t) = \{V_{t,m} : m < \omega\}$ of $t \in T$:

$$V_{t,m} = \{t\} \cup \{(t, n, s) \in T : n \geq m\}.$$

Of course, the standard topology is of countable character ([11]).

We observe that $V_{t,m}$ is open and closed. Indeed if $t \neq r \in V_{t,m}$ then there is a finite (possibly empty) sequence p such that $r = (t, n, p)$ with $n \geq m$, hence $V_{r,0} = T^{\uparrow}(r) \subset V_{t,m}$. On the other hand, if $s \notin V_{t,m}$, then let $r = \min\{t, s\}$. If $r < t$ then there is m such that $(r, m) \leq t$ and then $V_{r,m+1}$ is a neighborhood of s and $t \notin V_{r,m+1}$. If $r = t$ then there exist $n < m$ and a finite (possibly empty) sequence p such that $s = (t, n, p)$ and thus $T^{\uparrow}(s) \cap V_{t,m} = \varnothing$. We infer that the standard topology is Hausdorff zero-dimensional, in particular regular.

Given a natural number p, let T be the subset of the sequential tree Σ consisting of the elements of level less or equal to p, considered with the standard topology induced from Σ. We shall say that T is a (standard) *cascade of rank* $r(T) = p$. A *subcascade* S of a cascade is a subset of T such that $o_T \in S$ and for every $s \in S \setminus \max T$, the set $S^+(s)$ is an infinite subset of $T^+(s)$. A subcascade of cascade T of rank p is also a cascade (that is, can be embedded in Σ) and the standard topology induced from T coincides with that induced from Σ.

[10] Cascades are order-isomorphic to full closed-down subsets of the "naturally ordered" sequential tree (of finite subsequences of natural numbers) [5, Theorem 3.1]. As we need here only rather simple cascades, we will restrict ourselves to the corresponding subsets of the sequential tree.

[11] One can perform an analogous reduction to combinatorics also on starting from the *natural topology* of the sequential tree (one can find the definition in [5] for example).

If $\mathcal{N}(t)$ stands for the neighborhood filter of t for the standard topology of T, then denote by $\mathcal{N}_{(k)}^{(l)}(t)$ the restriction of the neighborhood filter of t, of level k, to the level $T^{(l)}$ of T. Of course, this notation is redundant, but spares the necessity of repeating that $l_T(t) = k$.

The closure (from the level l to the level k) is defined by

$$t \in \mathrm{cl}_{(k)}^{(l)} A \Leftrightarrow A \in \left(\mathcal{N}_{(k)}^{(l)}(t)\right)^{\#}.$$

Hence we can decompose the closure

$$\mathrm{cl}\, A = \bigcup_{k \leq l \leq r(T)} \mathrm{cl}_{(k)}^{(l)} A.$$

It is straightforward that

Lemma 5.1. *For the standard topology of a monotone cascade of finite rank,*

$$\left(\mathrm{cl}_{(l)}^{(m)}\right)^{\natural} \mathcal{N}_{(k)}^{(m)}(t) = \mathcal{N}_{(k)}^{(l)}(t) \tag{5.1}$$

for $k < l < m$.

More generally, if $\mathcal{V}(t)$ stands for the vicinity filter of a pretopology defined on T and t is of level k, then $\mathcal{V}_{(k)}^{(l)}(t)$ stands for the restriction of $\mathcal{V}(t)$ to the level $T^{(l)}$ of T.

A pretopology on a standard cascade T is *standard* if its vicinity filters $\mathcal{V}(t)$ have the following property: for every $0 \leq k < l \leq r(T)$ either $\mathcal{V}_{(k)}^{(l)}(t) = \mathcal{N}_{(k)}^{(l)}(t)$ or $\mathcal{V}_{(k)}^{(l)}(t)$ is degenerate for every t of level k.

Example 5.2. *The topology π of Example 3.2 is a standard pretopology on a cascade T of rank 2, which is not equal to the standard topology of T. We observe that $r\pi$ is the standard topology of T. Notice that $\mathcal{V}_{\pi}{}_{(0)}^{(1)}(o)$ is degenerate, $\mathcal{V}_{\pi}{}_{(0)}^{(2)}(o) = \mathcal{V}_{r\pi}{}_{(0)}^{(2)}(o)$ and $\mathcal{V}_{\pi}{}_{(1)}^{(2)}(t) = \mathcal{V}_{r\pi}{}_{(1)}^{(2)}(t)$ for each t of level 1.*

6. States

We denote the interval $\{k, k+1, \ldots, l\}$ of natural numbers by $[k, l]$. A *state* on $[0, n]$ is a collection of intervals $[k, l]$ of $[0, n]$ such that $k < l$ ([12]). We order states by the inverse inclusion, that is, $\mathcal{S} \geq \mathcal{T}$ whenever $\mathcal{S} \subset \mathcal{T}$.

Two intervals $[i, j]$ and $[k, l]$ are called *consecutive* if $j = k$ and *cofinal* if $j = l$. If $i < k$, we define the *cofinal difference*

$$[i, l] \sim [k, l] = [i, k].$$

A state $\mathcal{S}$ is *regular* if $k < l < m$ and $[k, m], [l, m] \in \mathcal{S}$ implies that $[k, l] \in \mathcal{S}$. For a state $\mathcal{T}$ on $[0, n]$, an element k of $[0, n]$ is *regular* (with respect to $\mathcal{S}$) if $[k, m], [l, m] \in \mathcal{T}$ with $k < l < m$ implies that $[k, l] \in \mathcal{T}$; otherwise, we say that k is *irregular*. Of course, a state is regular if and only if every point is regular with respect to it. The least regular state that includes $\mathcal{S}$ is denoted by $R\mathcal{S}$. The partial regularization $r\mathcal{S}$ of a state $\mathcal{S}$ consists of $\mathcal{S}$ and of all (possible) cofinal differences of the elements of $\mathcal{S}$. The *irregularity* of a state $\mathcal{S}$ is the least number n such that $r^n\mathcal{S}$ is regular.

[12] We notice that there are $2^{\frac{n(n+1)}{2}}$ states on $[0, n]$.

A state is *topological* if $[k,l],[l,m]\in\mathcal{S}$, then $[k,m]\in\mathcal{S}$. For a given state $\mathcal{T}$, a point k is *topological* if $[k,l],[l,m]\in\mathcal{T}$ implies that $[k,m]\in\mathcal{T}$. Sure enough, a state is topological if and only if every point is topological with respect to it. If $\mathcal{S}$ is a state, then $T\mathcal{S}$ denotes the least topological state that includes $\mathcal{S}$. It is straightforward that $T\mathcal{S}$ consists of all the finite unions of consecutive intervals from $\mathcal{S}$.

There is a one-to-one correspondence between standard pretopologies on a cascade of rank n and states on $[0,n]$, namely if $\mathcal{V}$ denotes the vicinity system of such a pretopology, then the corresponding state $\mathcal{S}$ is defined by $[k,l]\in\mathcal{S}$ if and only if $\mathcal{V}_{(k)}^{(l)}(t)$ is non-degenerate (for all t of level k).

Let us notice that

Lemma 6.1. *For a standard pretopology,* $\mathcal{V}_{(l)}^{(m)}(\mathcal{V}_{(k)}^{(l)}(t))$ *is non-degenerate if and only if* $\mathcal{V}_{(k)}^{(m)}(t)$ *is non-degenerate.*

Proposition 6.2. *If* $\mathcal{S}$ *is the state of a standard pretopology* π *then* $r\mathcal{S}$ *is the state of* $r\pi$.

Proof. Notice that $\left(\mathrm{adh}_{(l)}^{(m)}\right)^{\natural}\mathcal{V}_{(k)}^{(m)}$ is non-degenerate if and only if $\mathcal{V}_{(k)}^{(m)}$ and $\mathcal{V}_{(l)}^{(m)}$ are non-degenerate, and in this case $\left(\mathrm{adh}_{(l)}^{(m)}\right)^{\natural}\mathcal{V}_{(k)}^{(m)}=\mathcal{N}_{(k)}^{(l)}$ by Lemma 5.1. Hence ${\mathcal{V}_{r\pi}}_{(k)}^{(l)}$ is non-degenerate if and only if ${\mathcal{V}_{\pi}}_{(k)}^{(m)}$ and ${\mathcal{V}_{\pi}}_{(l)}^{(m)}$ are non-degenerate, and thus $[k,l]\in r\mathcal{S}$ whenever there is $m>l>k$ such that $[k,m],[l,m]\in\mathcal{S}$. ∎

Proposition 6.3. *A standard pretopology is topological (resp., regular) if and only if its state is topological (resp., regular).*

Proof. Consider a standard pretopology on T, and let $\mathcal{V}(t)$ be the vicinity filter of t for this pretopology. Let $\mathcal{S}$ be the state corresponding to the pretopology. This pretopology is a topology if and only if $\mathcal{V}(t)\subset\mathcal{V}(\mathcal{V}(t))$ for every t, which holds if and only if $\mathcal{V}_{(l)}^{(m)}(\mathcal{V}_{(k)}^{(l)}(t))$ is non-degenerate, provided that $\mathcal{V}_{(l)}^{(m)}$ and $\mathcal{V}_{(k)}^{(l)}$ are non-degenerate for each $m>l>k$. By Lemma 6.1, this is equivalent to the following condition on $\mathcal{S}$: if $[k,l],[l,m]\in\mathcal{S}$ then $[k,m]\in\mathcal{S}$. The second part of the proposition follows from Proposition 6.2. ∎

Combinatorics related to irregularity is studied in detail in [6]. We mention here only few facts. It is proved that the maximal irregularity of the states on $[0,n]$ is n. A state on $[0,n]$ is *maximally irregular* if its irregularity is n. Maximally irregular states are characterized in [6]. If $\mathcal{S}$ is a maximally irregular state on $[1,n]$, then there are two maximally irregular states on $[0,n]$ whose restriction to $[1,n]$ is equal to $\mathcal{S}$, one is equal to $\mathcal{S}\cup\{[0,2]\}$ and the other is $\mathcal{S}\cup\{[0,k],[1,2]\}$ where $k\leq n$ is a unique natural number such that $[1,k]\in\mathcal{S}$.

There is only one state on $[0,1]$ (it is $\{[0,1]\}$) and it is regular. The states below are on $[0,2],[0,3]$ and $[0,4]$ respectively:

$$[0,2],[1,2]\leftarrow\left\{\begin{array}{l}[0,3],[1,2],[2,3]\leftarrow\left\{\begin{array}{l}[0,4],[1,2],[2,3],[3,4]\\ [0,3],[1,2],[2,4],[3,4]\end{array}\right.\\ \\ [0,2],[1,3],[2,3]\leftarrow\left\{\begin{array}{l}[0,2],[1,4],[2,3],[3,4]\\ [0,2],[1,3],[2,4],[3,4]\end{array}\right.\end{array}\right.$$

The arrows indicate the restriction of $r\mathcal{S}$ to $[0,n]$ of a state $\mathcal{S}$ on $[0,n+1]$. Notice that the only topological state in the table above is that on $[0,2]$. In order to better distinguish graphically the intervals forming a state, we present them as arrows.

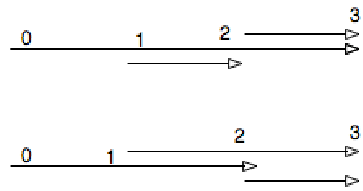

FIGURE 1. Maximally irregular states on $[0,3]$

7. IRREGULARITY SPECTRA

Recall that the *irregularity spectrum* $\sigma(x) = \sigma(x,\xi)$ of an element x with respect to a convergence ξ is the set of ordinals α for which x is irregular for $r^\alpha\xi$. If $\sigma(x) = \varnothing$ then x is called *intrinsically regular*.

Example 7.1. *Let*

$$\mathcal{T} = \{[0,3],[2,4],[3,4],[1,6],[4,5],[5,6]\}.$$

Then

$$\begin{aligned} r\mathcal{T}\backslash\mathcal{T} &= \{[2,3],[1,5]\},\\ r^2\mathcal{T}\backslash r\mathcal{T} &= \{[0,2],[1,4]\},\\ r^3\mathcal{T}\backslash r^2\mathcal{T} &= \{[1,3],[1,2]\},\\ r^4\mathcal{T}\backslash r^3\mathcal{T} &= \{[0,1]\}. \end{aligned}$$

The state $r^4\mathcal{T}$ is already regular. It follows that $\sigma(0) = \{1,3\}$ and the irregularity of 0 is 4. On the other hand, $\sigma(1) = \{0,1,2\}, \sigma(2) = \{0\}$ and $3,4,$ and 5 are intrinsically regular.

Observe a fascinating phenomenon: 0 is regular for $\mathcal{T}$, but irregular for $r\mathcal{T}$, that is, 1 is in its spectrum, but 0 is not. Then again it is regular for $r^2\mathcal{T}$ and irregular for $r^3\mathcal{T}$; in other words, 3 is in its spectrum, but 4 is not. Of course, one could, in a similar way, construct states in which an element has an arbitrarily prescribed spectrum.

Theorem 7.2. *For every finite subset F of natural numbers, there is a state $\mathcal{T}$ such that $\sigma(0,\mathcal{T}) = F$.*

Proof. Use induction on $\max F$. If $\max F = 0$ then the empty state will do. Suppose that $k > 0$, the claim is true for $\max F < k$ and let $\max F = k$. If $\mathcal{T}$ is a state on $\{0,n\}$ such that $\sigma(0,\mathcal{T}) = F\backslash\{k\}$ then $\mathcal{W} = \mathcal{T}\cup\mathcal{U}$ where

$$\mathcal{U} = \{[0,n+1],[n,n+k]\}\cup\{[n+1,n+2],[n+2,n+3],\ldots,[n+k-1,n+k]\}$$

is a required state on $\{0, n+k\}$. Indeed, as all the elements of $\mathcal{T}$ have the ends in $\{0, n\}$, for every $1 \leq p \leq k-1$, we have

$$r^p \mathcal{W} = r^p \mathcal{T} \cup r^p \mathcal{U}$$

and $r^{k-1}\mathcal{U} \backslash r^{k-2}\mathcal{U} = [n, n+1]$ so that $r^k \mathcal{W} = r^k \mathcal{T} \cup \{[0, n]\}$, that is, $[0, n]$ converges to 0 for $r^k \mathcal{U}$ but does not converge for $r^p \mathcal{U}$ as $p < k$. ∎

8. Irregular points for pretopologies of countable character

A convergence ξ is of *countable character (first-countable)* if $x \in \lim_\xi \mathcal{F}$ implies the existence of a countably based filter $\mathcal{G} \subset \mathcal{F}$ such that $x \in \lim_\xi \mathcal{G}$. In particular, a pretopology is of countable character, whenever every vicinity filter is countably based.

Proposition 8.1. *Let π be a pretopology of countable character. An element x is irregular with respect to π if and only if there exists a sequence $(x_n)_n$ such that*

$$x \in \lim_{r\pi} (x_n)_n \backslash \operatorname{adh}_\pi (x_n)_n. \tag{8.1}$$

Proof. An element x is irregular for π if and only if $\operatorname{adh}^\natural_\pi \mathcal{V}_\pi(x)$ does not converge to x, that is, whenever there is $V \in \mathcal{V}_\pi(x)$ and a decreasing filter base (V_n) of $\mathcal{V}_\pi(x)$ such that for every $n < \omega$ there is $x_n \in \operatorname{adh}_\pi V_n \backslash V$. Hence $(x_n)_n$ converges to x in $r\pi$ but $x \notin \operatorname{adh}_\pi(x_n)$, which implies that x is irregular for π. ∎

We observe that no separation axiom has been used in Proposition 8.1. The characterization above cannot be extended to arbitrary convergences (not even pseudotopologies) of countable character ([13]). Proposition 8.1 leads to the following, more explicit, characterization

Proposition 8.2. *Let π be a pretopology of countable character. An element x is irregular with respect to π if and only if there exists a sequence (x_n) and a bisequence $(x_{n,k})$ such that $(x_{n,k})_k$ is free for each $n < \omega$, $x \notin \operatorname{adh}_\pi (x_n)_n$, but $x_n \in \lim_\pi (x_{n,k})_k$ for every $n < \omega$, and*

$$x \in \lim_\pi \int_{(n)} (x_{n,k})_k.$$

Proof. Indeed, by Proposition 8.1 there is a sequence (x_n) such that (8.1) holds. In particular, if (V_m) is a decreasing base of $\mathcal{V}_\pi(x)$ then for every $m < \omega$ there is $n_m > n_{m-1}$ such that $x_n \in \operatorname{adh}_\pi V_m$ for $n \geq n_m$. Consequently, for each such a n there exists a sequence $(x_{n,k})_k$ on V_m for which $x_n \in \lim_\pi (x_{n,k})_k$. Since $\int_{(n)} (x_{n,k})_k$ is finer than $\mathcal{V}_\pi(x)$, it converges to x in π. If $(x_{n,k})_k$ were not free for infinitely many n, then $\int_{(n)} (x_{n,k})_k$ would be coarser than a subsequence of $(x_n)_n$, which must not converge to x in π in view of (8.1). Therefore, $(x_{n,k})_k$ is free for almost all n , hence for all n after having dropped a finite number of them. ∎

Classical simplest examples of non-regular topologies are of countable character.

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[13] It holds however for paratopologies of countable character. A convergence is a *paratopology* [3] whenever $x \notin \lim \mathcal{F}$ implies the existence of a countably based filter $\mathcal{H}$ that meshes with $\mathcal{F}$ such that $x \notin \operatorname{adh} \mathcal{H}$.

Example 8.3. [12, Example 1.5.6] *Consider the unit interval* $[0,1]$ *in which a basic family of closed sets consists of the closed sets for the natural topology and of* $\{\frac{1}{n} : n < \omega\}$. *In this topology* $x = 0$ *is irregular. Then* $x_n = \frac{1}{n}$ *and* $x_{n,k} = \frac{1}{n} + \frac{1}{k}$ *verify Proposition 8.2.*

Example 8.4. *Consider the unit disc in* $\mathbb{R}^2$, *the interior of which carries the natural topology, while a neighborhood base of an element* x_∞ *of the border is of the form*

$$\{x : \|x\| < 1, \|x - x_\infty\| < \frac{1}{n}\} \cup \{x_\infty\}.$$

To illustrate Proposition 8.2 take any sequence (x_n) *of distinct terms on the border converging to* x_∞ *in the natural topology, and let* $(x_{n,k})_k$ *be a sequence converging to* x_n *from inside. We can also ask that the family* $\{x_{n,k} : k < \omega\}$ *where* $n < \omega$ *be discrete.*

It is well-known that ([14])

Proposition 8.5. *The class of convergences of countable character is a concretely coreflective subcategory of convergences.*

In particular, every infimum of convergences of countable character is of countable character. This however is no longer the case in the category of pretopologies.

Example 8.6. *Consider a countable fan, that is, the disjoint union* $\{\infty\} \cup \{(n,k) : n,k < \omega\}$ *and let* π_m *be a convergence defined by* $\{\infty\} = \lim_{\pi_m} \mathcal{F}$ *for a free filter* $\mathcal{F}$ *whenever* $\mathcal{F}$ *is finer than the cofinite filter of* $\{(n,k) : k < \omega, n \leq m\}$. *The other points are isolated. This defines a descending sequence of Hausdorff pretopologies of countable character (actually sequential), and clearly* $\bigwedge_{m<\omega} \pi_m$ *is a convergence of countable character. But the infimum in the lattice of pretopologies* $\bigwedge^P_{m<\omega} \pi_m = P\left(\bigwedge_{m<\omega} \pi_m\right)$ *is the well-known fan topology, which is Fréchet but not of countable character.*

Proposition 8.7. *Countable character is preserved by the partial regularization.*

Proof. In fact, if ξ is of countable character and if $x \in \lim_{r\xi} \mathcal{F}$, then there is a countably based filter $\mathcal{G}$ such that $x \in \lim_\xi \mathcal{G}$ and $\mathrm{adh}^\natural_\xi \mathcal{G} \leq \mathcal{F}$. Of course, $x \in \lim_{r\xi}(\mathrm{adh}^\natural_\xi \mathcal{G})$ and $\mathrm{adh}^\natural_\xi \mathcal{G}$ is countably based. ∎

It follows from Propositions 8.5 and 8.7 that every iterated partial regularization of a convergence of countable character is of countable character. Hence

Theorem 8.8. [18, Proposition 7.1] *The regularization of a convergence of countable character is of countable character.*

However an infinitely iterated partial pretopological regularization of a pretopology of countable character need not be of countable character, which is due to the fact that the pretopological infimum in general does not preserve the character. Indeed, consider

[14] More generally, the class of convergences of a fixed character is a concretely coreflective subcategory of convergences.

Example 8.9. *Let $A = \bigsqcup_{n,m<\omega} A_{n,m}$ be a sink endowed with its natural convergence. We extend the convergence of A to $X = \{\infty\} \cup A$ so that $\mathcal{F}_0$ converges to ∞. This defines a topology τ of countable character. Notice that* $\mathrm{adh}^{\natural}_{r^k\tau} \mathcal{F}_0 = \bigwedge_{m\leq k} \mathcal{F}_k$*, thus $\bigwedge_{k<\omega} \mathcal{F}_k$ is the coarsest free filter that converges to ∞ in $r_P^{\omega}\tau$, which shows that $r_P^{\omega}\tau$ is not of countable character.*

Proposition 8.10. *If a pretopology π of countable character is Hausdorff, then $r\pi$ is Hausdorff.*

Proof. If x is isolated, then the singleton $\{x\}$ constitutes a base for π at x, hence $\mathrm{adh}_\pi\{x\} = \{x\}$ is a base of x for $r\pi$. If x is not isolated then there is a base $(V_n)_n$ of the vicinity $\mathcal{V}_\pi(x)$ such that $V_n \backslash V_{n+1} \neq \varnothing$ for every $n < \omega$, and $\bigcap_{n<\pi} V_n = \{x\}$. As π is Hausdorff,

$$\{x\} = \lim_\pi \mathcal{V}_\pi(x) = \mathrm{adh}_\pi \mathcal{V}_\pi(x) = \bigcap_{n<\omega} \mathrm{adh}_\pi V_n,$$

hence $r\pi$ is Hausdorff, because the intersection of the base $\{\mathrm{adh}_\pi V_n : n < \omega\}$ of the vicinity filter of x in $r\pi$ is $\{x\}$. ∎

A filter $\mathcal{E}$ on X is sequential if there exists a sequence $(x_n)_n$ of elements of X such that $\{\{x_k : k \geq n\} : n > \omega\}$ is a base of $\mathcal{E}$. Proposition 8.2 will be now extended to

Lemma 8.11. *If π is a pretopology of countable character, and $x_o \in \lim_{r\pi} \mathcal{F}$, then there is $F \in \mathcal{F}$ and for each $x \in F$ there is a sequential filter $\mathcal{E}(x)$ such that $x \in \lim_\pi \mathcal{E}(x)$ and $x_o \in \lim_\pi \mathcal{E}(\mathcal{F})$. If moreover $x_o \notin \mathrm{adh}_\pi \mathcal{F}$, then we can choose $\mathcal{E}(x)$ to be free.*

Proof. If $(V_m)_{m<\omega}$ is a decreasing base of the vicinity filter $\mathcal{V}_\pi(x_o)$, then $x_o \in \lim_{r\pi} \mathcal{F}$ amounts to $\mathrm{adh}^{\natural}_\pi \mathcal{V}_\pi(x_o) \leq \mathcal{F}$, that is, $\mathrm{adh}_\pi V_m \in \mathcal{F}$ for each $m < \omega$. Let $V_\infty = \bigcap_{m<\omega} \mathrm{adh}_\pi V_m$, and decompose $\mathcal{F}$ into $\mathcal{F}_0 = \mathcal{F} \vee V_\infty^c$ and $\mathcal{F}_1 = \mathcal{F} \vee V_\infty$, where either $\mathcal{F}_0$ or $\mathcal{F}_1$ can be degenerate.

If $x \in \mathrm{adh}_\pi V_m \backslash \mathrm{adh}_\pi V_{m+1}$ (we do not exclude the case where the difference is empty), then there is a sequential filter $\mathcal{E}(x)$ such that $V_m \in \mathcal{E}(x)$ and $x \in \lim_\pi \mathcal{E}(x)$. As $\mathrm{adh}_\pi V_m \in \mathcal{F}$ for each $m < \omega$, then $\mathcal{E}(\mathcal{F}_0) \geq \mathcal{V}_\pi(x_o)$ provided that $\mathcal{F}_0$ is non-degenerate.

On the other hand, if $x \in V_\infty = \bigcap_{m<\omega} \mathrm{adh}_\pi V_m = \mathrm{adh}_\pi \mathcal{V}_\pi(x_o)$ (the latter holds because π is a pretopology), then there is a sequential filter $\mathcal{E}(x) \geq \mathcal{V}_\pi(x_o)$ such that $x \in \lim_\pi \mathcal{E}(x)$, hence $\mathcal{E}(V_\infty) \geq \mathcal{V}_\pi(x_o)$. Hence if $\mathcal{F}_1$ is non-degenerate, then $\mathcal{E}(\mathcal{F}_1) \geq \mathcal{E}(V_\infty) \geq \mathcal{V}_\pi(x_o)$. Therefore $\mathcal{E}(\mathcal{F}) = \mathcal{E}(\mathcal{F}_0) \wedge \mathcal{E}(\mathcal{F}_1) \geq \mathcal{V}_\pi(x_o)$.

If $x_o \notin \mathrm{adh}_\pi \mathcal{F}$ and there is $H \in \mathcal{F}^{\#}$ such that $\mathcal{E}(x)$ is not free for every $x \in H$, then the principal filter $\mathcal{N}_\iota(x)$ of x is finer than $\mathcal{E}(x)$, hence $x_o \in \lim_\pi \mathcal{N}_\iota(\mathcal{F} \vee H)$ by the first part of the proof, which yields a contradiction, because $\mathcal{N}_\iota(\mathcal{F} \vee H) = \mathcal{F} \vee H$. ∎

Proposition 8.12. *An element x of a Hausdorff pretopology of countable character is irregular if and only if there is at x a homeomorphically embedded standard irregular pretopology of rank 2.*

Proof. Let π be a Hausdorff pretopology of countable character, and x an irregular point. By Proposition 8.2 there is a sequence (x_n) and a bisequence $(x_{n,k})$ such that $x_n = \lim_\pi (x_{n,k})_k$ for each $n < \omega$, $x = \lim_\pi \int_{(n)} (x_{n,k})_k$ but $x \notin \mathrm{adh}_\pi (x_n)_n$.

Therefore, by taking a subsequence if necessary, we can assume that all the terms of $(x_n)_n$ are distinct, because $r\pi$ is Hausdorff and *a fortiori* T_1. As $\{x_n\} \cup \{x_{n,k} : k < \omega\}$ is compact in π for every $n < \omega$, and $x = \lim_\pi \int_{(n)} (x_{n,k})_k$, we can, by taking subsequences of $(x_n)_n$ and of $(x_{n,k})_k$ for $n < \omega$ if necessary, find a neighborhood base $(V_n)_n$ of x such that $x_{n,k} \in V_n \backslash V_{n+1}$ for every $n < \omega$. It is clear that the pretopology induced on $\{x\} \cup \{x : n < \omega\} \cup \{x_{n,k} : n, k < \omega\}$ coincides with the standard irregular topology (of rank 2). ∎

9. Ramified standard cascades

Proposition 8.12 characterizes irregular elements of Hausdorff pretopologies of countable character in terms of a homeomorphically embedded standard irregular pretopology of rank 2. In an attempt at characterizing elements of irregularity $n > 1$ of such spaces, one encounters a new phenomenon already for irregularity 2.

Indeed, let x be an element of irregularity 2 of a Hausdorff pretopology π of countable character on X. This means that x is irregular for $r\pi$, which is of countable character and Hausdorff by Proposition 8.10, and thus by Proposition 8.12, there is a standard irregular pretopological space T of rank 2, and a homeomorphism $f : T \to f(T) \subset X$ such that $f(o) = x$; in particular, $f(n) = \lim_{r\pi} f(n,k)_k$ and $x = \lim_{r\pi} f(\mathcal{V}_{(0)}^{(2)}(o))$, but $x \notin \mathrm{adh}_{r\pi} f(n)_n$.

Case 1. Now, if $x \in \mathrm{adh}_\pi f(\mathcal{V}_{(0)}^{(2)}(o))$, then by taking a subcascade if necessary, we can assume that $x = \lim_\pi f(\mathcal{V}_{(0)}^{(2)}(o))$.

Case 2. Otherwise by Lemma 8.11, T can be extended to a standard cascade S of rank 3, and f to a map $F : S \to X$ so that $F(t,k)_k$ is free and $F(t) = \lim_\pi F(t,k)_k$ for every $t \in \max T$, and $x = \lim_\pi F(\mathcal{V}_{(0)}^{(3)}(o))$.

Consider now another alternative regarding $f : T \to X$.

Case A. If $f(n) = \lim_\pi f(n,k)_k$ for infinitely many n, then by taking a subcascade corresponding to those n, we may suppose that this holds for each $n < \omega$.

Case B. If on the contrary, there is n_0 such that $f(n) \neq \lim_\pi (f(n,k))_k$ for $n \geq n_0$, then by taking a subcascade corresponding to those n, we can assume that the property holds for each $n < \omega$. This means that $f(n)$ is irregular (with respect to π) for each n, and thus by Proposition 8.12, there is an extension V of rank 3 of T, and an extension G of f to V such that $G|_{V\uparrow(n)}$ is a homeomorphically embedded standard irregular bisequence for each $n < \omega$.

If Cases 1. and A. occurred simultaneously, then we would get a characterization of the irregularity 1 of x, that is, $[0,2],[1,2]$. If Cases 1. and B. hold then the cascade G is of the type $[0,2],[1,3],[2,3]$. If Cases 2. and A. hold then the cascade F is of the type $[0,3],[1,2],[2,3]$.

As for the simultaneity of Cases 2. and B., no state corresponds to it. In this case, the map $F \cup G : S \cup V \to X$ presents a new type of embedding, which will be referred to as $[0,3_0],[2,3_0],[1,3_1],[2,3_1]$.

We see that standard pretopologies are not sufficient to reflect possible types of irregularity of points. We need ramified standard pretopologies and their corresponding ramified states.

A *ramified level tree* L is the binary tree of height ω, that is, such that for each $l \in L$, the set $L^+(l)$ of immediate successors of l contains two elements. A ramified

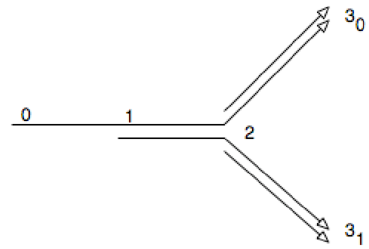

FIGURE 2. The ramified state $[0, 3_0], [2, 3_0], [1, 3_1], [2, 3_1]$

level tree can be represented as the tree of finite sequences, the terms of which are 0 or 1. As every tree, the ramified level tree admits the level (ordinal) function: the root is of level 0, and if the level $h_L(l)$ has been defined till $m < \omega$, then the minimal elements of $\{l \in L : h_L(l) > m\}$ are of level $m + 1$. A *ramified type* is a downwards closed subtree of L with finite branches. Therefore each non-maximal element of a ramified type has either one or two immediate successors.

Let L be a ramified type. A (sequential) *ramified cascade* T of type L is a monotone sequential cascade for which a map $\lambda : T \to L$ is defined so that

$$\lambda(o) = o,$$
$$\lambda(T^+(t)) = L^+(\lambda(t)),$$
$$l \neq o \Rightarrow \operatorname{card} \lambda^{-1}(l) = \infty.$$

If $t \in T$ then $\lambda(t)$ is called the *ramified level* of t.

A *standard pretopology* of a sequential ramified cascade T of type L is defined analogously as for a sequential cascade, that is, for every $r, s \in L$ with $r < s$ either $\mathcal{V}_r^s(t) = \mathcal{N}_r^s(t)$ or $\mathcal{V}_r^s(t)$ is degenerate for every t with $\lambda(t) = r$. Here $\mathcal{V}$ stands for the vicinity system of a pretopology and $\mathcal{N}$ for the neighborhood system of the natural topology of T, while $\mathcal{V}_r^s(t)$ and $\mathcal{N}_r^s(t)$ stand respectively for the restrictions of the vicinity and the neighborhood filters of t with $\lambda(t) = r$ to the elements of ramified level s. Also the definition of *standard* pretopologies obviously extends that for usual sequential cascades.

A map $f : T \to W$ (from one ramified cascade to another) is *level-preserving* if there is a map $\varphi : \lambda_T(T) \to \lambda_W(W)$ (called the *level map* of f) such that

$$\lambda_W \circ f = \varphi \circ \lambda_T.$$

Proposition 9.1. *A level-preserving map $f : T \to W$ is continuous if and only if $f(\mathcal{V}_r^s(t)) \geq \mathcal{V}_{\varphi(r)}^{\varphi(s)}(f(t))$ for every couple $r < s$ of ramified levels of T and for each t of level r, where φ is the level map of f.*

A *state* $\mathcal{S}$ on a ramified level tree L is a finite set of intervals of cardinality at least 2 of L. The (ramified) *type* of $\mathcal{S}$ is the downwards closure of the elements of $\mathcal{S}$. Of course, it is a subtree of L. The *rank* of a state is that of its ramified type.

A state $\mathcal{T}$ associated with a standard pretopology is defined by $[r, s] \in \mathcal{T}$ if and only if $\mathcal{V}_r^s$ is non-degenerate.[15] Regularity, partial regularization and topologicity of

[15]that is, $\mathcal{V}_r^s(t)$ is degenerate for each t of ramified level r.

a state on a ramified type are defined in the same way as for a state on an interval of natural numbers. It is a straightforward generalization of Proposition 6.3 that

Proposition 9.2. *A standard pretopology on a ramified cascade is topological (resp., regular) if and only if its state is topological (resp., regular).*

Let T, W be ramified cascades considered with standard pretopologies, and let $\mathcal{T}, \mathcal{W}$ be the corresponding ramified states.

If a map $f : T \to W$ between sequential cascades is continuous, then

$$f(T^{+}(t)) \backslash W^{+}(f(t) \cup \{f(t)\}$$

is finite for every $t \in T \backslash \max T$. By removing, for every t, the finite number of successors that derogate from that inclusion, we get a restriction of f, which is order-preserving. Consider a level-preserving map $f : T \to W$, and its level map $\varphi : \lambda_T(T) \to \lambda_W(W)$. Then

Proposition 9.3. *If a level-preserving map $f : T \to W$ is continuous then its level map fulfills $\varphi(\mathcal{T}) \subset \mathcal{W}$.*

Proof. Let f be continuous and let $[r, s] \in \mathcal{T}$. This means that $\mathcal{V}_r^s(t)$ is non-degenerate for every $t \in \lambda_T^{-}(r)$. As f is level-preserving, $\lambda_W(f(t)) = \varphi(r)$ and $\lambda_W(f(v)) = \varphi(s)$ for every $v \in \lambda_T^{-}(s)$, and $f(\mathcal{V}_r^s(t)) \geq \mathcal{V}_{\varphi(r)}^{\varphi(s)}(f(t))$, because f is continuous. This implies that $\mathcal{V}_{\varphi(r)}^{\varphi(s)}(w)$ is non-degenerate (for each w of level $\varphi(r)$), hence $[\varphi(r), \varphi(s)] \in \mathcal{W}$. ∎

As we will see, only a special subclass of maximally irregular states is sufficient to characterize finite irregularity of pretopologies of countable character. We define elementary states by induction on the rank. If $\mathcal{T}$ is a state starting at 1 with the property that there is a unique ramified level t such that $[1, t] \in \mathcal{T}$, then

$$\mathcal{T}^{*} = \mathcal{T} \backslash [1, t] \cup [0, t].$$

The *elementary* state of rank 1 is the unique state of rank 1 that is $\{[0, 1]\}$. The *elementary* state of rank 2 is the unique maximally irregular state of rank 2, that is $\{[0, 2], [1, 2]\}$. Suppose that we have defined elementary states $\mathcal{S}$ of rank less than or equal to m with the property that

$$\bigcup \mathcal{S} \cap \{l \in L : h_L(l) \leq 2\} \text{ is a chain,} \tag{9.1}$$

$$\exists!_{s_0, s_1 \in L}\ [0, s_0] \in \mathcal{S}, [1, s_1] \in \mathcal{S}, \tag{9.2}$$

an *elementary* state $\mathcal{S}$ of rank $m + 1$ is of the form

$$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_0^{*}, \tag{9.3}$$

where $\mathcal{S}_0, \mathcal{S}_1$ are elementary states starting from 1, of ranks $1 \leq r(\mathcal{S}_0), r(\mathcal{S}_1) \leq m$ with $m = \max(r(\mathcal{S}_0), r(\mathcal{S}_1))$. It is clear that $\mathcal{S}$ given by (9.3) fulfills (9.1) and (9.2).

We call a standard pretopology on a sequential cascade *elementary* if the corresponding state is elementary.

Theorem 9.4. *If π is a Hausdorff pretopology of countable character, $m \geq 1$ and*

$$x \in \lim\nolimits_{r^m \pi}(x_n)_n \backslash \operatorname{adh}_{r^{m-1}\pi}(x_n)_n, \tag{9.4}$$

then there exists a homeomorphic embedding f of an elementary cascade of rank $m + 1$ and of irregularity m such that $f(o) = x$ and $f(n) = x_n$ for each $n < \omega$.

Proof. This is true for $m=1$ because of Propositions 8.1 and 8.12. So suppose that the claim holds for $m \geq 1$, and let π be a Hausdorff pretopology of countable character on X such that

$$x \in \lim_{r^{m+1}\pi}(x_n)_n \setminus \operatorname{adh}_{r^m\pi}(x_n)_n.$$

Because $r^m\pi$ is of countable character and Hausdorff (by Proposition 8.10), we can apply Proposition 8.12 to $r^m\pi$ to infer the existence of an elementary cascade f and a homeomorphic embedding into X endowed with $r^m\pi$ so that $f(o)=x$ and $f(n)=x_n$. Moreover, by Hausdorffness, we can require that there is a collection $(W_n)_n$ of disjoint subsets of X such that $W_n \in \mathcal{V}_{r^m\pi}(x_n)$ for every $n<\omega$.

Let p be the least natural number such that $x_n \in \lim_{r^p\pi}(x_{n,k})_k$ for almost all $n<\omega$, where $x_{n,k}=f(n,k)$. Because $p \leq m$, by inductive assumption, for every such n there is a homeomorphic embedding f_n of an elementary ramified cascade T_n of rank $p+1$ to W_n, such that $f_n(o)=x_n$ and $f_n(k)=x_{n,k}$ for each $n,k<\omega$. Of course, in case $p=0$, the cascades T_n are of rank 1 hence endowed with a regular topology. Because there are finitely many types of elementary ramified cascades of finite rank, by taking a subsequence of $(n)_n$ if necessary, we can assume that T_n are all of the same type.

Let q be the least natural number such that the filter $\mathcal{F} \approx \{\{f(n,k) : k<\omega\} : n<\omega\}$ converges to x in $r^q\pi$. Then $m=\max(p,q)$ for otherwise $x \in \operatorname{adh}_{r^m\pi}(x_n)_n$, contrary to the assumption.

By Hausdorffness, we can assume that there is a collection $\{W_{n,k} : n,k<\omega\}$ of disjoint sets such that $W_{n,k} \in \mathcal{V}_{r^q\pi}(x_{n,k})$.

If $q=0$ then $x \in \lim_\pi \mathcal{F}$. Otherwise, by Lemma 8.11, for every (n,k) there exists a free sequential filter $\mathcal{E}(n,k) \approx (x_{n,k,l})_l$, which converges to $x_{n,k}$ in $r^{q-1}\pi$, and such that the filter $\mathcal{G} \approx \{\{x_{n,k,l} : k,l<\omega\} : n<\omega\}$ converges to x in $r^{q-1}\pi$.

Let v be the least natural number such that $\mathcal{E}(n,k)$ converges to $x_{n,k}$ in $r^v\pi$ for almost n,k. Of course, $v \leq q-1$. Hence, by inductive assumption, there is a homeomorphic embedding $f_{n,k}$ of an elementary cascade $S_{n,k}$ to $W_{n,k}$ of rank $v+1$ such that $f_{n,k}(o)=x_{n,k}$ and $f_{n,k}(l)=x_{n,k,l}$ for each $n,k,l<\omega$. Of course, in case $v=0$, each $S_{n,k}$ is of rank 1, hence endowed with a regular topology. Because there are finitely many types of elementary cascades of finite rank, by taking a subcascade R of $\{o\} \cup \{(n) : n<\omega\} \cup \{(n,k) : n,k<\omega\}$ if necessary, we can assume that $S_{n,k}$ are all of the same type.

Let w be the least natural number such that $x \in \lim_{r^w\pi}\mathcal{G}$. Of course, $q-1 = \max(v,w)$ for otherwise $x \in \lim_{r^q\pi}\mathcal{F}$. If $w=0$ then we stop the construction. Otherwise we continue on applying Lemma 8.11 to $\mathcal{G}$, and so on.

We construct now a ramified cascade by taking the disjoint union of $R, T_n, S_{n,k}$ and possibly of other cascades resulting from the construction described. Then we quotient so that o_{T_n} coincides with $n \in R$, $T_n^+(o_{T_n})$ coincides with $R^+(n)$, $o_{S_{n,k}}$ coincides $(n,k) \in R$, and so on. The constructed component embeddings coincide at the points that we have identified. Moreover we took care that the individual component cascades have ranges in disjoint vicinities of distinct points. Therefore the constructed mapping is an injection. The pretopology of the constructed cascade is induced with the component cascades with the exception of the vicinity of the least element o. There is only one non-zero ramified level s for which $\mathcal{V}_0^{(s)}(o)$ is non-degenerate. If in our construction $q=0$, then it will correspond to the filter $\mathcal{F}$, if $w=0$ then it will correspond to the filter $\mathcal{G}$, and so on. The constructed cascade is elementary of rank $m+1$.

Because a map between pretopologies of countable character, is continuous if and only if it is sequentially continuous, the constructed injective map is a homeomorphic embedding. ∎

Corollary 9.5. *If m is the irregularity of an element x of a Hausdorff pretopology of countable character, then there exists a homeomorphic embedding f from an elementary cascade of rank $m+1$ and irregularity m such that $f(o)=x$.*

The converse is true only for irregularity 1. More generally, it holds only if the image of a considered homeomorphic embedding is open.

References

[1] G. Choquet. Sur les notions de filtre et de grille. *C. R. Acad. Sci. Paris*, **224**:171–173, 1947.
[2] C. Cook and H. Fischer. Uniform convergence structures. *Math. Ann.*, **173**:290–306, 1967.
[3] S. Dolecki. Convergence-theoretic methods in quotient quest. *Topology Appl.*, **73**:1–21, 1996.
[4] S. Dolecki. Convergence-theoretic characterizations of compactness. *Topology Appl.*, **125**:393–417, 2002.
[5] S. Dolecki. Multisequences. *Quaestiones Mathematicae*, **29**:239–277, 2006.
[6] S. Dolecki and D. Gauld. Combinatorics of irregularity. to appear.
[7] S. Dolecki and G. H. Greco. Topologically maximal pretopologies. *Studia Math.*, **77**:265–281, 1984.
[8] S. Dolecki and G. H. Greco. Cyrtologies of convergences, I. *Math. Nachr.*, **126**:327–348, 1986.
[9] S. Dolecki and G. H. Greco. Cyrtologies of convergences, II: Sequential convergences. *Math. Nachr.*, **127**:317–334, 1986.
[10] S. Dolecki and H.-P. Künzi. Consecutive topologizations and regularizations of pretopologies. to appear.
[11] S. Dolecki and F. Mynard. Hyperconvergences. *Appl. General Topology*, **4**:391–419, 2003.
[12] R. Engelking. *Topology*. Heldermann Verlag, 1989.
[13] H. Fischer. Limesraüme. *Math. Ann.*, **137**:269–303, 1959.
[14] Z. Frolík. Sums of ultrafilters. *Bull. Amer. Math. Soc.*, **73**:87–91, 1967.
[15] G. Grimeisen. Gefilterte Summation von Filtern und iterierte Grenzeprozesse, I. *Math. Annalen*, **141**:318–342, 1960.
[16] G. Grimeisen. Gefilterte Summation von Filtern und iterierte Grenzeprozesse, II. *Math. Annalen*, **144**:386–417, 1961.
[17] H. Herrlich. Topological structures. In P. C. Baayen, editor, *Topological structures*. 1974.
[18] D. C. Kent and G. Richardson. The regularity series of a convergence space. *Bull. Austral. Math. Soc.*, **13**:21–44, 1975.
[19] D. C. Kent and G. Richardson. The regularity series of a convergence space II. *Bull. Austral. Math. Soc.*, **15**:223–243, 1976.
[20] H. J. Kowalsky. Limesräume und Kompletierung. *Math. Nachr.*, **12**:302–340, 1954.
[21] P. Nyikos and J. E. Vaughan. The Scarborough-Stone problem for Hausdorff spaces. *Topology Appl.*, **44**:309–316, 1992.

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