Covering Properties and Metrisation of Manifolds 2*

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Abstract

There are many conditions equivalent to metrisability for a topological manifold which are not equivalent to metrisability for topological spaces in general. What are the weakest such? We show that a number of weak covering properties which are equivalent to metrisability for a manifold, for example metaLindelöf, may be further weakened by considering only covers of cardinality the first uncountable ordinal. Extensions to higher cardinals are discussed.

1 Introduction and Definitions.

By a topological manifold we mean a connected Hausdorff space each point of which has a neighbourhood homeomorphic to euclidean space. In [4] there is a list of over 50 conditions which are equivalent to metrisability for a manifold but not for a topological space in general. As one might expect, some of these conditions are strictly stronger than metrisability and some are strictly weaker than metrisability in a general space. In this paper we investigate just how weak covering properties can be made while still being equivalent to metrisability for a manifold.

All cardinals are assumed infinite. We denote the cardinality of a set X by $|X|$. If $x \in X$ and $\mathcal{F}$ is a family of subsets of X then $\mathrm{ord}(x, \mathcal{F})$ is the *order* of $\mathcal{F}$ at x, ie $|\{F \in \mathcal{F} \mid x \in F\}|$. When X is a topological space, we denote by $\chi(x, X)$ the *character* of x in X, ie the least infinite cardinality of a local basis at x. A good reference for the set theory used in this paper is [10].

The following properties are studied in [1] where Theorem 4.1 states that every locally metrisable, linearly Lindelöf space is hereditarily Lindelöf. They observe that their proof may be modified to show that every locally metrisable ω_1-Lindelöf space is hereditarily Lindelöf. (As noted in [1] and in Proposition 15 below, every linearly Lindelöf space is ω_1-Lindelöf.) Setting $\kappa = \omega_1$ in Proposition 12 shows that local metrisability can be replaced by local hereditary Lindelöfness.

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Definition 1 *A space X is* linearly Lindelöf *provided that every open cover of X which is a chain has a countable subcover. A family $\mathcal{F}$ of subsets of a set X is a* chain *provided that $\forall F, G \in \mathcal{F}$ either $F \subset G$ or $G \subset F$.*

A space X is ω_1-Lindelöf *provided that every open cover of X of cardinality ω_1 has a countable subcover.*

Recall also the following definition.

Definition 2 *Let κ and λ be two cardinal numbers. A topological space X is* $[\kappa, \lambda]$-compact, [12], *if and only if every open cover of X of cardinality at most λ has a subcover of cardinality less than κ.*

If $\kappa = \omega$ then $[\kappa, \lambda]$-compact is also called initially λ-compact. *If $\lambda \geq |X|$ then $[\kappa, \lambda]$-compact is also called* finally κ-compact.

Motivated by these definitions we formulate the following definitions, where κ and λ are two cardinal numbers:

Definition 3 *A space X is* linearly $[\kappa, \lambda]$-compact *provided that every open cover $\mathcal{U}$ of X which is a chain and satisfies $|\mathcal{U}| \leq \lambda$ has a subcover $\mathcal{V}$ with $|\mathcal{V}| < \kappa$.*

A space X is (linearly) $[\kappa, \lambda]$-metacompact *provided that every open cover $\mathcal{U}$ of X which (is a chain and) satisfies $|\mathcal{U}| \leq \lambda$ has an open refinement $\mathcal{V}$ such that* $\mathrm{ord}(x, \mathcal{V}) < \kappa$ *for each $x \in X$. If $\lambda \geq |X|$ then $[\kappa, \lambda]$-metacompact is also called* finally κ-metacompact.

A space is nearly (linearly) $[\kappa, \lambda]$-metacompact *if we merely demand that* $\mathrm{ord}(x, \mathcal{V}) < \kappa$ *for each point x in some dense subset of X.*

An $[\omega_1, \omega_1]$-metacompact space may also be called an *ω_1-metaLindelöf* space, and is a weak form of metaLindelöfness as it requires point-countability of a refinement only for open covers of cardinality ω_1. Theorem 13 tells us that under appropriate conditions, which all manifolds satisfy, an ω_1-metaLindelöf space is in fact metaLindelöf. *(Nearly) linearly metaLindelöf* and *nearly ω_1-metaLindelöf* are defined analogously. The ultimate must be the following: a space is *(nearly) linearly ω_1-metaLindelöf* provided that for every open cover $\mathcal{U}$ which is a chain and which satisfies $|\mathcal{U}| \leq \omega_1$ there is an open refinement $\mathcal{V}$ which is point-countable (on a dense subset).

Given a set X and a collection $\mathcal{S}$ of subsets of X, a *choice function* is a function $f : \mathcal{S} \to X$ such that $f(S) \in S$ for each $S \in \mathcal{S}$.

Definition 4 *A space X has* property (ω_1)pp, *[7], provided that each open cover $\mathcal{U}$ of X (with $|\mathcal{U}| = \omega_1$) has an open refinement $\mathcal{V}$ such that for each choice function $f : \mathcal{V} \to X$ with $f(V) \in V$ for each $V \in \mathcal{V}$ the set $f(\mathcal{V})$ is closed and discrete in X.*

The main result in this paper is the following.

Theorem 5 *Let M be a manifold. Then the following are equivalent:*

(a) M is metrisable;

(b) M is nearly linearly ω_1-metaLindelöf;

(c) for every open cover $\mathcal{U}$ of M with $|\mathcal{U}| = \omega_1$ there is an open refinement $\mathcal{V}$ such that for every choice function $f : \mathcal{V} \to M$ the set $f(\mathcal{V})$ is closed and discrete;

(d) *for every open cover* $\mathcal{U}$ *of* M *with* $|\mathcal{U}| = \omega_1$ *there is an open refinement* $\mathcal{V}$ *such that for every choice function* $f : \mathcal{V} \to M$ *the set* $f(\mathcal{V})$ *is closed;*

(e) *for every open cover* $\mathcal{U}$ *of* M *with* $|\mathcal{U}| = \omega_1$ *there is an open refinement* $\mathcal{V}$ *such that for every choice function* $f : \mathcal{V} \to M$ *the set* $f(\mathcal{V})$ *is discrete.*

Of course with the Continuum Hypothesis this tells us no more than what we already know from [4], ie that every (nearly) meta-Lindelöf manifold (equivalently, manifold with property pp) is metrisable, as every manifold has the cardinality of the continuum, by [9, Theorem 2.9].

2 Finally κ-metacompact Spaces.

Recall that the *character* of a space X is the least cardinal κ for which every point of X has a local base of cardinality at most κ.

We say that a sequence $\langle V_\alpha \rangle$ of subsets of a space is *strongly increasing* provided that $\overline{V_\alpha} \subset V_{\alpha+1}$ for each α.

Lemma 6 *Let* κ *be a regular cardinal. Suppose that* X *is a space such that* $\chi(x, X) < \kappa$ *for each* $x \in X$ *and* $\langle V_\alpha \rangle$ *is a strongly increasing* κ*-sequence of subsets of* X*. Then* $\cup_{\alpha<\kappa} V_\alpha$ *is closed in* X*.*

Proof. Suppose that $x \in \overline{\cup_{\alpha<\kappa} V_\alpha}$. Let $\{U_\beta \mid \beta \leq \theta\}$ be a neighbourhood base at x, where $\theta < \kappa$. For each β we have $U_\beta \cap (\cup_{\alpha<\kappa} V_\alpha) \neq \varnothing$ so $U_\beta \cap V_{\alpha_\beta} \neq \varnothing$ for some $\alpha_\beta < \kappa$. Let $\alpha = \sup\{\alpha_\beta \mid \beta \leq \theta\}$. Then $\alpha < \kappa$ and $U_\beta \cap V_\alpha \neq \varnothing$ for all β, and hence $x \in \bar{V}_\alpha \subset V_{\alpha+1}$. Thus $\overline{\cup_{\alpha<\kappa} V_\alpha} \subset \cup_{\alpha<\kappa} V_\alpha$. ∎

Lemma 7 *Let* κ *be a regular cardinal. Suppose that* X *is a connected space and that* $\mathcal{V}$ *is an open cover of* X *such that* $\mathrm{ord}(x, \mathcal{V}) < \kappa$ *for each* $x \in X$ *and each member of* $\mathcal{V}$ *has density* $< \kappa$*. Then* $|\mathcal{V}| < \kappa$*.*

Proof. We may assume that $\varnothing \notin \mathcal{V}$.

Pick any $V_0 \in \mathcal{V}$ and set $\mathcal{V}_0 = \{V_0\}$. Assuming that $\mathcal{V}_i \subset \mathcal{V}$ has been defined, let $V_i = \cup \mathcal{V}_i$ and set $\mathcal{V}_{i+1} = \{V \in \mathcal{V} \mid V \cap V_i \neq \varnothing\}$. It suffices to show that $|\mathcal{V}_i| < \kappa$ and that $\mathcal{V} = \cup_{i=0}^{\infty} \mathcal{V}_i$.

(i) We show that $|\mathcal{V}_i| < \kappa$ by induction on i, the result being trivial when $i = 0$. Suppose that $|\mathcal{V}_i| < \kappa$. Then because κ is regular, V_i has a dense subset, say D_i, with $|D_i| < \kappa$. For each $V \in \mathcal{V}_{i+1}$ we have $V \cap V_i \neq \varnothing$ so $V \cap D_i \neq \varnothing$. Again because κ is regular, $\mathcal{V}_{i+1} = \cup_{d \in D_i} \{V \in \mathcal{V} \mid d \in V\}$ has cardinality less than κ since $\mathrm{ord}(x, \mathcal{V}) < \kappa$ for each $x \in X$.

(ii) $\mathcal{V} = \cup_{i=0}^{\infty} \mathcal{V}_i$ follows from connectedness via the fact that any two points of X are chained to each other by members of $\mathcal{V}$: thus for any $x \in V_0 \in \mathcal{V}$ and any $y \in V \in \mathcal{V}$ there is a finite sequence $\langle W_i \rangle$ of members of $\mathcal{V}$ such that $x \in W_0$, $y \in W_n$ and $W_{i-1} \cap W_i \neq \varnothing$ for each $i = 0, \dots n$. We may assume that $W_0 = V_0$ and $W_n = V$. Then for each i, $W_i \in \mathcal{V}_i$. In particular $V \in \mathcal{V}_n$. ∎

Corollary 8 *Let κ be a regular cardinal. Then any connected and finally κ-metacompact space which is locally of density $< \kappa$ is finally κ-compact.*

In particular every connected, locally separable, metaLindelöf space is Lindelöf. We also obtain:

Corollary 9 *Let κ be a regular cardinal and λ any cardinal. Every connected, $[\kappa, \lambda]$-metacompact space of density $< \kappa$ is $[\kappa, \lambda]$-compact.*

Proof. Suppose that X is a connected, $[\kappa, \lambda]$-metacompact space of density $< \kappa$ and let $\mathcal{U}$ be an open cover of X with $|\mathcal{U}| = \lambda$. Let $\mathcal{V}$ be an open refinement of $\mathcal{U}$ such that $\mathrm{ord}(x, \mathcal{V}) < \kappa$ for each $x \in X$. As an open subset of a space of density $< \kappa$, each member of $\mathcal{V}$ has density $< \kappa$. By Lemma 7, $|\mathcal{V}| < \kappa$ and hence $\mathcal{U}$ has a subcover of cardinality less than κ. ■

Let X be a topological space and A a non-empty subset of X. A point $x \in X$ is a *point of complete accumulation* of A if and only if for every neighbourhood N of x we have $|A \cap N| = |A|$.

Proposition 10 [2, page 17] and [13, Theorem 1] *Let κ be a regular cardinal. A space X is $[\kappa, \kappa]$-compact if and only if every $A \subset X$ such that $|A| = \kappa$ has a point of complete accumulation.*

Proposition 11 *Let κ be a regular cardinal. Let X be a space which is not hereditarily finally κ-compact. Then there is a subspace $Y \subset X$ such that $|Y| = \kappa$ and that no subset $Z \subset Y$ of cardinality κ is finally κ-compact.*

Proof (cf [11, Theorem 3.1]). Because X is not hereditarily finally κ-compact there is a strictly increasing sequence $\langle U_\alpha \rangle_{\alpha<\kappa}$ of open sets. For each $\alpha < \kappa$ choose $y_\alpha \in U_{\alpha+1} - U_\alpha$ and set $Y = \{y_\alpha \mid \alpha < \kappa\}$. ■

The following result generalises [1, theorem 4.1]. The proof may be obtained by appropriate generalisation of the proof of that result using Propositions 10 and 11.

Proposition 12 *Let κ be a regular cardinal. Every locally hereditarily finally κ-compact, $[\kappa, \kappa]$-compact space is hereditarily finally κ-compact.*

Theorem 13 *Let κ be a regular cardinal. Suppose that X is a space which is of character $< \kappa$, is locally connected, locally hereditarily finally κ-compact and locally hereditarily of density $< \kappa$. If X is $[\kappa, \kappa]$-metacompact then X is the topological direct sum of finally κ-compact spaces.*

Proof. As X is locally connected, every component is open so by looking at each component separately if necessary we may assume that X is connected also. We construct a strongly increasing κ-sequence $\langle V_\alpha \rangle$ of non-empty, connected, open and finally κ-compact subsets of X.

Because X is locally connected and locally hereditarily finally κ-compact we may begin by choosing any non-empty, connected, open, finally κ-compact subset $V_0 \subset X$. For any other limit ordinal α, if V_β has already been constructed for all $\beta < \alpha$, let $V_\alpha = \cup_{\beta<\alpha} V_\beta$.

Suppose that V_α has been constructed. Because V_α is finally κ-compact it also has a dense subset of cardinality $< \kappa$. Thus $\bar{V}_\alpha$ has a dense subset of cardinality $< \kappa$. $\bar{V}_\alpha$ is also connected as V_α is. Furthermore, as a closed subset of a $[\kappa, \kappa]$-metacompact space $\bar{V}_\alpha$ is also $[\kappa, \kappa]$-metacompact. Thus by Corollary 9 $\bar{V}_\alpha$ is $[\kappa, \kappa]$-compact. It now follows from Proposition 12 that $\bar{V}_\alpha$ is finally κ-compact. For each $x \in \bar{V}_\alpha - V_\alpha$ choose $U_x \subset X$ open and finally κ-compact such that $x \in U_x$. Then $\{U_x \mid x \in \bar{V}_\alpha - V_\alpha\}$ is an open cover of the finally κ-compact subset $\bar{V}_\alpha - V_\alpha$ so has a subcover of cardinality $< \kappa$. The collection consisting of this subcover together with V_α is a collection of fewer than κ many open finally κ-compact subsets of X so their union is also open and finally κ-compact and contains $\bar{V}_\alpha$. Let $V_{\alpha+1}$ be the component of this union containing V_α.

Suppose that $\mathcal{U}$ is an open cover of X. Then for each $\alpha < \kappa$, $\mathcal{U}$ is also an open cover of the finally κ-compact set V_α: let $\mathcal{U}_\alpha$ be a subcover of cardinality $< \kappa$. Then $\cup_{\alpha<\kappa}\mathcal{U}_\alpha$ is a subfamily of $\mathcal{U}$ of cardinality at most κ which covers $\cup_{\alpha<\kappa}V_\alpha$, hence the connected space X, by Lemma 6 because this union is non-empty, open and closed. As X is $[\kappa, \kappa]$-metacompact it follows that this subfamily has an open refinement whose order at each point is less than κ and hence so does $\mathcal{U}$. Now it follows from Corollary 8 that X is finally κ-compact. ∎

Remark. The three local properties 'of character $< \kappa$, locally hereditarily finally κ-compact and locally hereditarily of density $< \kappa$' of Theorem 13 are all implied by the single local property: locally of weight $< \kappa$. In the case where $\kappa = \omega_1$ these four properties are, respectively, first countable, locally hereditarily Lindelöf, locally hereditarily separable and locally second countable and in this case, Theorem 13 gives:

Corollary 14 *Every connected, locally connected, locally second countable, ω_1-metaLindelöf space is Lindelöf.*

This corollary has an obvious generalisation to higher regular cardinal κ in place of ω_1.

Proposition 15 (cf [1]) *Every linearly ω_1-(meta)Lindelöf space is ω_1-(meta)Lindelöf.*

Proof. We will just consider the metaLindelöf case. Let $\mathcal{U}$ be an open cover of the linearly ω_1-metaLindelöf space X such that $|\mathcal{U}| = \omega_1$. Then we can write $\mathcal{U} = \{U_\alpha \mid \alpha < \omega_1\}$. For each $\alpha < \omega_1$ let $V_\alpha = \cup\{U_\beta \mid \beta < \alpha\}$. Then $\mathcal{V} = \{V_\alpha \mid \alpha < \omega_1\}$ is an open cover of X which is a chain. Thus as X is linearly ω_1-metaLindelöf it follows that there is a point-countable open refinement, say $\mathcal{W}$.

For each $W \in \mathcal{W}$ there is $\alpha(W) < \omega_1$ such that $W \subset V_{\alpha(W)}$. Let $\mathcal{S} = \{W \cap U_\beta \mid W \in \mathcal{W}$ and $\beta \leq \alpha(W)\}$. Then $\mathcal{S}$ is a point-countable open refinement of $\mathcal{U}$. ∎

Proof of the equivalence of (a) and (b) of Theorem 5

As every metrisable space is paracompact, it is also nearly linearly ω_1-metaLindelöf so (a)⇒(b) in Theorem 5. For the converse, suppose that M is a nearly linearly ω_1-metaLindelöf manifold. Clearly one can modify the proof of [5, Lemma 3.2] to conclude that M is linearly ω_1-metaLindelöf. As every manifold is T_3, connected, locally connected and locally second countable, it follows from Corollary 14 and Proposition 15 that M is Lindelöf, hence second countable and therefore metrisable by Urysohn's Metrisation Theorem. ∎

3 Spaces with Property pp.

Lemma 16 *A point $x \in X$ is a limit point of X if and only if for each collection $\mathcal{V}$ of open sets containing x, with $|\mathcal{V}| \geq \chi(x,X)$, there exists a choice function $f : \mathcal{V} \to X$, such that $x \in \overline{f(\mathcal{V})} - f(\mathcal{V})$.*

Proof. $\Rightarrow$: Suppose that $\mathcal{V}$ is a collection of open sets containing x with $|\mathcal{V}| \geq \chi(x,X)$, say $\{V_\alpha \mid \alpha < \chi(x,X)\} \subset \mathcal{V}$ satisfies $V_\alpha \neq V_\beta$ whenever $\alpha \neq \beta$. Let $\{W_\alpha \mid \alpha < \chi(x,X)\}$ be a neighbourhood basis at x. Then we may define $f : \mathcal{V} \to X$ so that $f(V) \in V - \{x\}$ if $V \neq V_\alpha$ for any $\alpha < \chi(x,X)$ and $f(V_\alpha) \in V_\alpha \cap W_\alpha - \{x\}$. Then $x \in \overline{f(\mathcal{V})} - f(\mathcal{V})$.

$\Leftarrow$: Let U be any neighbourhood of x and take $\mathcal{V}$ to be a collection of open neighbourhhods of x forming a neighbourhood basis at x. Then $|\mathcal{V}| \geq \chi(x,X)$. Let $f : \mathcal{V} \to X$ be a choice function such that $x \in \overline{f(\mathcal{V})} - f(\mathcal{V})$. Then $f(U) \in U - \{x\}$, so x is a limit point of X. ■

Lemma 17 *Let $\mathcal{V}$ be an open cover of a T_1 space X. Then the following are equivalent:*

(a) For every choice function $f : \mathcal{V} \to X$, the set $f(\mathcal{V})$ is closed and discrete;

(b) For every choice function $f : \mathcal{V} \to X$, the set $f(\mathcal{V})$ is closed;

(c) For every choice function $f : \mathcal{V} \to X$, the set $f(\mathcal{V})$ is discrete.

Proof. It suffices to show that (b) and (c) are equivalent.
(b)$\Rightarrow$(c). Suppose that $f : \mathcal{V} \to X$ is a choice function but $f(\mathcal{V})$ is not discrete. Then there is $x \in f(\mathcal{V})$ every neighbourhood of which meets $f(\mathcal{V})$ in some point other than x. Define $g : \mathcal{V} \to X$ by $g(V) = f(V)$ if $f(V) \neq x$ and $g(V) \in V - \{x\}$ if $f(V) = x$. Then $x \in \overline{g(\mathcal{V})} - g(\mathcal{V})$ so $g(\mathcal{V})$ is not closed.
(c)$\Rightarrow$(b). Suppose that $f : \mathcal{V} \to X$ is a choice function but $f(\mathcal{V})$ is not closed, say $x \in \overline{f(\mathcal{V})} - f(\mathcal{V})$. Pick $V_x \in \mathcal{V}$ such that $x \in V_x$. Define $g : \mathcal{V} \to X$ by $g(V) = f(V)$ unless $V = V_x$ and let $g(V_x) = x$. Because X is T_1 it follows that every neighbourhood of x meets $g(\mathcal{V})$ in some point other than x so $g(\mathcal{V})$ is not discrete. ■

Proposition 18 *Let κ be a cardinal. Suppose that X has character at most κ and has no isolated points, and that every open cover $\mathcal{U}$ of X with $|\mathcal{U}| = \kappa^+$ has an open refinement $\mathcal{V}$ such that for every choice function $f : \mathcal{V} \to X$ the set $f(\mathcal{V})$ is closed. Then X is $[\kappa^+, \kappa^+]$-metacompact.*

Proof. Let $\mathcal{U}$ be an open cover of X with $|\mathcal{U}| = \kappa^+$. Apply Lemma 16 to the open refinement $\mathcal{V}$ given by hypothesis: then $\mathrm{ord}(x,\mathcal{V}) < \kappa < \kappa^+$ for each $x \in X$. ■

We can now complete the proof of Theorem 5.

By Lemma 17 (c), (d) and (e) are equivalent. By Proposition 18 with $\kappa = \omega$, (d) implies (b). Finally every metrisable manifold is pp and hence satisfies (c).

4 Some Questions.

Are there even weaker covering conditions which are equivalent to metrisability for a manifold?

Using [6, Theorems 1 and 2] (or see [3, Theorem 8.11]) and [9, Theorem 2.5] we find that the following conditions are each equivalent to metrisability for a manifold:

- M is normal and θ-refinable;
- M is normal and subparacompact.

Let X be a space.

X is *θ-refinable* ([14]) (also called *submetacompact*) if every open cover can be refined to an open θ-cover, i.e. a cover $\mathcal{U}$ which can be expressed as $\cup_{n\in\omega}\mathcal{U}_n$ where each $\mathcal{U}_n$ covers X and for each $x \in X$ there is n such that $ord(x, \mathcal{U}_n) < \omega$.

X is *subparacompact*, [8] (where it is called *F_σ-screenable*), if every open cover has a σ-discrete closed refinement.

Our theme suggests the following definition.

Definition 19 *Say that X is ω_1-θ-*refinable *if every open cover $\mathcal{U}$ of X with $|\mathcal{U}| = \omega_1$ has a θ-refinement.*

Question 20 *Is every ω_1-θ-refinable manifold θ-refinable?*

Question 21 *Must a manifold be metrisable if it is normal and every open cover of cardinality at most ω_1 has an open θ-refinement?*

Question 22 *Must a manifold be metrisable if it is normal and every open cover of cardinality at most ω_1 has a σ-discrete closed refinement?*

Comparing Corollary 8 with Corollary 9 leads to the following question.

Question 23 *Let κ be a regular cardinal. Must every connected and $[\kappa, \kappa]$-metacompact space which is locally of density $< \kappa$ be $[\kappa, \kappa]$-compact?*

Note that in Proposition 18 we have only concluded that X is $[\kappa^+, \kappa^+]$-metacompact rather than $[\kappa, \kappa^+]$-metacompact even though the open cover of size κ^+ has been refined to an open cover of order less than κ: we did not carry out a similar reduction of an open cover of cardinality κ because we did not need to. This raises the following question.

Question 24 *Is there a space X with character at most κ and having no isolated points such that every open cover of size κ^+ has an open refinement $\mathcal{V}$ whose order at each point is less than κ but X is not $[\kappa, \kappa^+]$-metacompact?*

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