Baire product theorem for separately open sets and separate continuity

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Abstract

Our main result is a generalisation of the Baire category theorem: if $X_1, \dots, X_k$ is a finite collection of topological spaces so that X_1 is Baire, and when $k > 1$ each X_i except possibly X_k has a countable pseudo-base and each X_i except possibly X_1 is quasi-regular and strongly countably complete, if $\langle C_n \rangle$ is a sequence of separately semi-closed subsets of the product $\prod_{i=1}^k X_i$ and $O \subset \prod_{i=1}^k X_i$ is a non-empty open set such that $O \subset \bigcup_{n=1}^\infty C_n$, then there is an integer m such that $O \cap \mathring{C}_m \neq \varnothing$.
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Throughout we assume that all topological spaces are non-empty.

1 Separately open sets

Let $X_1, \dots, X_k$ be a finite collection of topological spaces and let $X = \prod_{i=1}^n X_i$. Say that a subset $S \subset X$ is *separately open* (called linearly open in [12]) provided that for each $x = (x_i) \in X$ and each $j = 1, \dots n$ there is a neighbourhood N_j of x_j in X_j such that $\prod_{i=1}^n N_i \subset S$, where $N_i = \{x_i\}$ when $i \neq j$. A *separately closed* set is one whose complement is separately open. The separately open sets form a topology, see [6] and [8], for example; we will call this the *topology of separately open sets*. By the *separate closure* of the set $S \subset X$ we mean the set $S^+ \subset X$ which is the closure of S when we use the topology of separately open sets: note that $S^+ \subset \overline{S}$. Similarly we can define the *separate interior* and note that for any set S the interior of the separate interior of S is the same as the interior of S.

Note that we cannot expect to obtain S^+ by taking the union of the closures of each 'slice' in the factor spaces. For example if we take S to be the open unit square in $\mathbb{R}^2$ then S^+ is the closed unit square but the union of the closures of all of the slices will miss the corners of the square.

In [3] the reviewer states that 'although it seems to be of importance and non-trivial, the author fails to offer an example of a linearly open set which is not open.' Perhaps the simplest example of such a set involves the familiar separately continuous but discontinuous

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function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$: then $f^{-1}(-\varepsilon, \varepsilon)$ is separately open but not open for any ε between 0 and 1. In the following two examples we exhibit separately open sets which are dense and codense hence have empty interior. In the second example the complement is also separately open.

Example 1 *Let X and Y be Baire, T_1, dense-in-themselves and second countable spaces. Then $X \times Y$ contains a separately open, dense and codense set which therefore has empty interior.*

Let $\{B_n \ / \ n = 1, 2, \ldots\}$ be a countable base for $X \times Y$. Then by [7, Proposition 1.32, page 17] each B_n is uncountable. Pick $(x_1, y_1) \in B_1$. Suppose that $(x_i, y_i) \in B_i$ has been chosen for each $i \leq n$ such that $x_n \neq x_i$ and $y_n \neq y_i$ when $i < n$. Choose $(x_{n+1}, y_{n+1}) \in B_{n+1}$ such that $x_{n+1} \neq x_i$ and $y_{n+1} \neq y_i$ when $i < n+1$: such choice is possible as B_{n+1} is uncountable and open. Now let $S = X \times Y - \{(x_i, y_i) \ / \ i = 1, 2, \ldots\}$. Then S satisfies the conditions demanded. ∎

Example 2 *A product space X may possess a pair of disjoint separately open, dense subsets whose union is all of X.*

Let $X = (\mathbb{Q} \cap [0,1])^2$. Note that a subset $S \subset X$ is separately open if for each point $x \in S$ there is a 'cross' centred at x (ie the union of an open horizontal and an open vertical segment each containing x) which lies in S.

Write $X = \{z_n \ / \ n = 0, 1, \ldots\}$ with $z_0 = (0,0)$. Let $\pi_i : X \to \mathbb{R}$ be projection onto the ith coordinate. We first construct two sequences $\langle A_n \rangle$ and $\langle B_n \rangle$ of subsets of X satisfying the following properties:

(i) $A_{n-1} \subset A_n$ and $B_{n-1} \subset B_n$;

(ii) $A_n \cap B_n = \varnothing$;

(iii) $z_n \in A_n \cup B_n$;

(iv) when $n \geq 1$ each small square in X bounded by the lines $x = \frac{i-1}{2^{n-1}}$, $x = \frac{i}{2^{n-1}}$, $y = \frac{j-1}{2^{n-1}}$ and $y = \frac{j}{2^{n-1}}$ meets both A_n and B_n;

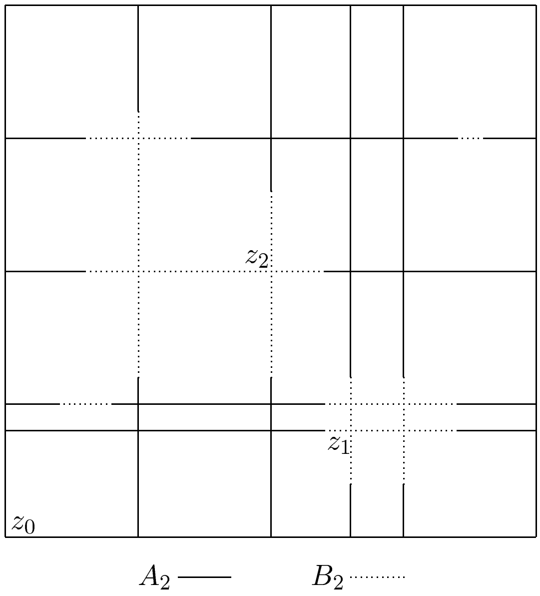
(v) for each m and each $x \in A_m \cup B_m$ there is n such that either A_n or B_n contains a cross centred at x.

Let $A_0 = \{0,1\} \times \mathbb{Q} \cup \mathbb{Q} \times \{0,1\}$ and $B_0 = \varnothing$.

Now suppose that A_{n-1} and B_{n-1} have been constructed to satisfy (i)-(iv). Let $H_0 = \pi_2^{-1}(\pi_2(z_n))$ and $V_0 = \pi_1^{-1}(\pi_1(z_n))$ and let $H_i = \pi_2^{-1}(\frac{2i-1}{2^n})$ and $V_i = \pi_1^{-1}(\frac{2i-1}{2^n})$ when $1 \leq i \leq 2^{n-1}$: then each H_i is a horizontal line and each V_i is a vertical line in X. We set $A_n \cup B_n = A_{n-1} \cup B_{n-1} \cup (\cup_{i \leq 2^{n-1}}(H_i \cup V_i))$ and now indicate which points of $\cup_{i \leq 2^{n-1}}(H_i \cup V_i)$ are in A_n and which are in B_n, (i) dictating the fate of points of $A_{n-1} \cup B_{n-1}$.

If $H_i \not\subset A_{n-1} \cup B_{n-1}$ then $A_{n-1} \cup B_{n-1}$ subdivides H_i into a number of segments: we need to show how each such segment is assigned to A_n and B_n. If one end point of the segment is in A_{n-1} and the other is in B_{n-1} then choose a point p in the segment such that $\pi_1(p)$ is irrational and assign all points of the segment between p and the end point which is in A_{n-1}

to A_n and all other points to B_n. If both end points are in the same one of A_{n-1} and B_{n-1}, say A_{n-1}, then choose two distinct points p, q in the segment with $\pi_1(p)$ and $\pi_1(q)$ both irrational and assign all points between p and q to B_n and all others to A_n, interchanging the roles of A_n and B_n if the end points are in B_{n-1} instead. Similar allocation rules apply to segments of V_j but now all we require is that the allocation of points within the segment should be consistent with the allocation of the end points to A_{n-1} or B_{n-1} and of the point $H_i \cap V_j$ to A_n or B_n.



Conditions (i)-(iv) are easily verified. Given $x \in A_m \cup B_m$, then $x = z_n$ for some n and A_n or B_n contains a cross centred at z_n, so (v) is also satisfied.

Now let $A = \bigcup_{n=0}^{\infty} A_n$ and $B = \bigcup_{n=0}^{\infty} B_n$. Then from (i) and (ii) A and B are disjoint. Their union is all of X from (iii). By (iv) each of A and B is dense in X and from (v) A and B are separately open. ■

For constructions of separately open sets in 'big' products, such as X^k where $k \geq \omega$, see [10], [11] and [13].

When the product space is $\mathbb{R}^n$ for some $n > 1$ we can also consider the following notion: a subset $S \subset \mathbb{R}^n$ is *linearly open* if its intersection with every straight line is relatively open. Contrast this with the definition of 'linearly open' in [12], where the only lines considered are those parallel with the axes. It is not difficult to see that there are linearly open sets in $\mathbb{R}^2$ which are not open: for example take $\mathbb{R}^2 - \{(\frac{1}{n}, \frac{1}{n^2}) \ / \ n = 1, 2, \ldots\}$ or

$$\{(x,y) \in \mathbb{R}^2 \ / \ |y| > x^2\} \cup \{(x,y) \in \mathbb{R}^2 \ / \ |y| < \frac{x^2}{2}\} \cup \{(0,0)\}.$$

Of course we could define notions of openness along curves of higher order, for example say that a set $S \subset \mathbb{R}^n$ is quadratically open provided that the intersection of S with any parabola is relatively open. By taking the curves determining the sets above to be cubic instead of parabolic we get sets which are quadratically open but not open. We could even take sets such as

$$\{(x,y) \in \mathbb{R}^2 \ / \ |y| > e^{-\frac{1}{x^2}}\} \cup \{(x,y) \in \mathbb{R}^2 \ / \ |y| < \frac{e^{-\frac{1}{x^2}}}{2}\} \cup \{(0,y) \ / \ y \in \mathbb{R}\}.$$

2 Other definitions

A space X is *Baire* provided that the intersection of countably many dense open (equivalently, dense G_δ) subsets is dense.

A subset $A \subset X$ of a topological space is *semi-open* provided that $A \subset \overline{\mathring{A}}$ and *semi-closed* provided that its complement is semi-open, ie that $\mathring{\overline{A}} \subset A$. Let $X_1, \ldots, X_k$ be a finite collection of topological spaces and let $X = \prod_{i=1}^n X_i$. Say that a subset $S \subset X$ is *separately semi-closed* provided that S is semi-closed when we use the topology of separately open sets.

Note that we cannot expect a set S to be separately semi-open if and only if each 'slice' is open in the factor spaces. For example take $S = \mathbb{R}^2 - \mathbb{Q} \times \{0\}$, a separately semi-open subset of $\mathbb{R}^2$ but the slice consisting of the part of S on the x-axis has projection onto the first factor just the irrationals and that set has empty interior.

A function $f : X \to Y$ is *quasi-continuous*, [5, definition (32.1)], provided that the inverse image of every open subset of Y is semi-open in X; equivalently that the inverse image of every closed set is semi-closed. In the case where $X = \prod_{i=1}^k X_i$ then f is *separately quasi-continuous* if it is quasi-continuous when the topology of separately open sets is used on X.

A *pseudo-base* for a topological space X is a collection $\mathcal{B}$ of non-empty open sets such that for any non-empty open set $U \subset X$ there is $B \in \mathcal{B}$ such that $B \subset U$. In what follows we will assume that a space has a countable pseudo-base. In the latter case we may assume that the base is indexed as $\langle B_n \rangle$ so that for each non-empty open $U \subset X$ and each positive integer m there is an integer $n > m$ such that $B_n \subset U$; we will say that such an ordered countable pseudo-base is *nicely ordered.* Note that an open subspace of a space with a countable pseudo-base has a countable pseudo-base and a finite product of spaces each with a countable pseudo-base also has a countable pseudo-base.

Following J.C. Oxtoby [9], we say that a topological space is *quasi-regular* if for every non-empty open set U, there is a non-empty open set V such that $\overline{V} \subset U$. Obviously, every regular space is quasi-regular.

Now let $\mathcal{A}$ be an open covering of a space X. Then a subset S of X is said to be $\mathcal{A}$-*small* if S is contained in a member of $\mathcal{A}$. A space X is said to be *strongly countably complete* [4], if there is a sequence $\langle \mathcal{A}_i \ / \ i = 1, 2, \ldots \rangle$ of open coverings of X such that a sequence $\langle F_i \rangle$ of non-empty closed sets of X has a non-empty intersection provided that $F_{i+1} \subset F_i$ for all i and each F_i is $\mathcal{A}_i$-small.

Remark 1. The class of strongly countably complete spaces includes locally compact Hausdorff spaces and complete metric spaces. This follows from the theorem of Arhangel'skii and Frolík which states that a completely regular space is strongly countably complete if and only if it is Čech-complete, see [2, p. 252].

Remark 2. If X is a quasi-regular, strongly countably complete space, then whenever the closed sets F_i mentioned in the definition of strong countable completeness are such that $F_{n+1} \subset \mathring{F}_n$, then $\bigcap_{i=1}^\infty \mathring{F}_i = \bigcap_{i=1}^\infty F_i \neq \varnothing$.

3 Preliminary Results

Lemma 3 *Every quasi-regular, strongly countably complete space is Baire.*

Proof. Suppose that X is a quasi-regular, strongly countably complete space, say $\langle \mathcal{A}_i \rangle$ is a sequence of open covers as in the definition of strongly countably complete. Let $\langle D_i \rangle$ be a sequence of open dense subsets and let $U \subset X$ be any non-empty open set. As X is quasi-regular and strongly countably complete there is a non-empty open $\mathcal{A}_2$-small set U_1 such that $\overline{U_1} \subset U \cap D_1$. Continue inductively to construct a sequence $\langle U_n \rangle$ of non-empty open sets such that $\overline{U_n} \subset U_{n-1} \cap D_{n-1}$ and U_n is $\mathcal{A}_{n+1}$-small.

Now $\langle \overline{U_n} \rangle$ is a decreasing sequence of non-empty closed sets and $\overline{U_n}$ is $\mathcal{A}_n$-small. Thus $\varnothing \neq \cap_{n=1}^{\infty} \overline{U_n} \subset U \cap (\cap_{n=1}^{\infty} D_n)$ so X is Baire. ∎

Lemma 4 *Suppose that X is a Baire space, $\langle A_n \rangle$ is a sequence of closed subsets of X and $U \subset X$ is a non-empty open subset such that $U \subset \bigcup_{n=1}^{\infty} A_n$. Then there is m such that $U \cap \mathring{A}_m \neq \varnothing$.*

Proof. Suppose to the contrary that $U \cap \mathring{A}_m = \varnothing$ for all m. Let $B_n = X - (A_n - \mathring{A}_n)$. Then B_n is open and $U \subset \bigcup_{n=1}^{\infty} (A_n - \mathring{A}_n)$ so that $U \cap \bigcap_{n=1}^{\infty} B_n = \varnothing$ and hence $\bigcap_{n=1}^{\infty} B_n$ is not dense in X. As X is Baire it follows that there is some m for which B_m is not dense. Then there is a nonempty open set $V \subset X$ such that $V \cap B_m = \varnothing$. Then $V \subset A_m - \mathring{A}_m$, which is impossible. ∎

Lemma 5 *Let $X_1, \ldots, X_k$ be a finite collection of topological spaces so that X_1 is Baire, and when $k > 1$ each X_i except possibly X_k has a countable pseudo-base and each X_i except possibly X_1 is quasi-regular and strongly countably complete. Suppose that $\langle A_n \rangle$ is an increasing sequence of separately closed subsets of $X = \prod_{i=1}^{k} X_i$ and $U_i \subset X_i$ are non-empty open sets such that $U = \prod_{i=1}^{k} U_i \subset \bigcup_{n=1}^{\infty} A_n$. Then there is an integer m such that $U \cap \mathring{A}_m \neq \varnothing$.*

Proof. Use induction on k, the result following from Lemma 4 when $k = 1$. Suppose that the claim is true for any product of $k-1$ spaces but there are k spaces for which it is false, and assume that the spaces $X_1, \ldots, X_k$ form a collection of such spaces. Let $\langle B_n \rangle$ be a nicely ordered countable pseudo-base for $\prod_{i=1}^{k-1} U_i$ and $\langle \mathcal{A}_i : i = 1, 2, \ldots \rangle$ a sequence of open coverings of X_k exhibiting strong countable completeness.

We will construct a nested sequence $\langle C_n : n = 0, 1, \ldots \rangle$ of closed subsets of X_k having non-empty interior such that for each n the set C_n is $\mathcal{A}_n$-small, where $\mathcal{A}_0 = \{X_k\}$. Given C_n we will also choose a point $a_{n+1} \in \prod_{i=1}^{k-1} X_i$. Begin the inductive construction by using quasi-regularity of X_k to choose a closed subset C_0 of X_k such that $C_0 \subset U_k$ and $\mathring{C}_0 \neq \varnothing$.

Now suppose that C_i, $i < n$, has been constructed. As we are assuming that $\mathring{A}_n$ does not meet $\prod_{i=1}^{k} U_i$ it follows that $B_n \times \mathring{C}_{n-1} \not\subset A_n$ so there is $a_n \in B_n$ and open $P_n \subset \mathring{C}_{n-1}$ such that $P_n \neq \varnothing$ and $(\{a_n\} \times P_n) \cap A_n = \varnothing$. We may assume that P_n is $\mathcal{A}_n$-small. By quasi-regularity of X_k we may choose a closed set $C_n \subset X_k$ such that $\mathring{C}_n \neq \varnothing$ and $C_n \subset P_n$.

Note that C_n is $\mathcal{A}_n$-small, closed and has non-empty interior, $C_n \subset \mathring{C}_{n-1}$, and $(\{a_n\} \times C_n) \cap A_n = \varnothing$.

Consider the sequence $\langle C_n \rangle$. By strong countable completeness of X_k, we have $\bigcap_{n=1}^{\infty} C_n \neq \varnothing$; let $c \in \bigcap_{n=1}^{\infty} C_n$: then $c \in U_k$. Let $A'_n = \{x \in \prod_{i=1}^{k-1} X_i : (x, c) \in A_n\}$. Then A'_n is separately closed in $\prod_{i=1}^{k-1} X_i$ and $\prod_{i=1}^{k-1} U_i \subset \cup A'_n$. Thus by inductive hypothesis there is an integer m so that $(\prod_{i=1}^{k-1} U_i) \cap \mathring{A}'_m \neq \varnothing$. Because the pseudo-base is nicely ordered, there is an integer $n > m$ so that $B_n \subset (\prod_{i=1}^{k-1} U_i) \cap \mathring{A}'_m$.

Now $a_n \in B_n \subset A'_m$ and hence $(a_n, c) \in A_m \subset A_n$ which contradicts $(\{a_n\} \times C_n) \cap A_n = \varnothing$. ∎

Lemma 6 *Let $X_1, \ldots, X_k$ be a finite collection of topological spaces so that X_1 is Baire, and when $k > 1$ each X_i except possibly X_k has a countable pseudo-base and each X_i except possibly X_1 is quasi-regular and strongly countably complete. Suppose that $A, B \subset \prod_{i=1}^{k} X_i$ are separately closed subsets, and $U_i \subset X_i$ are non-empty open sets such that $\prod_{i=1}^{k} U_i \subset A \cup B$. Then either $(\prod_{i=1}^{k} U_i) \cap \mathring{A}$ or $(\prod_{i=1}^{k} U_i) \cap \mathring{B}$ is non-empty.*

Proof. Use induction on k, the result following from Lemma 4 when $k = 1$. Suppose that the claim is true for any product of $k-1$ spaces but there are k spaces for which it is false. Let $\{B_n : n = 1, 2, \ldots\}$ be a countable pseudo-base for $\prod_{i=1}^{k-1} U_i$ and let $\langle \mathcal{A}_i : i = 1, 2, \ldots \rangle$ be a sequence of open coverings of X_k exhibiting its strong countable completeness.

We will construct a nested sequence $\langle C_n : n = 0, 1, \ldots \rangle$ of closed subsets of X_k having non-empty interior such that for each n the set C_n is $\mathcal{A}_n$-small, where $\mathcal{A}_0 = \{X_k\}$. Given C_n we will also choose points $a_{n+1}, b_{n+1} \in \prod_{i=1}^{k-1} X_i$. Begin the inductive construction by using quasi-regularity of X_k to choose a closed subset C_0 of X_k such that $C_0 \subset V$ and $\mathring{C}_0 \neq \varnothing$.

Now suppose that C_i, $i < n$, has been constructed. As we are assuming that $\mathring{A}$ does not meet $\prod_{i=1}^{k} U_i$ it follows that $B_n \times \mathring{C}_{n-1} \not\subset A$ so there is $a_n \in B_n$ and open $P_n \subset \mathring{C}_{n-1}$ such that $P_n \neq \varnothing$ and $(\{a_n\} \times P_n) \cap A = \varnothing$. Similarly there is $b_n \in B_n$ and open $Q_n \subset P_n$ such that $Q_n \neq \varnothing$ and $(\{b_n\} \times Q_n) \cap B = \varnothing$. We may assume that Q_n is $\mathcal{A}_n$-small. By quasi-regularity of X_k we may choose a closed set $C_n \subset X_k$ such that $\mathring{C}_n \neq \varnothing$ and $C_n \subset Q_n$.

Note that C_n is $\mathcal{A}_n$-small, closed and has non-empty interior, $C_n \subset \mathring{C}_{n-1}$, and $(\{a_n\} \times C_n) \cap A = (\{b_n\} \times C_n) \cap B = \varnothing$.

Consider the sequence $\langle C_n \rangle$. By strong countable completeness of X_k, we have $\bigcap_{n=1}^{\infty} C_n \neq \varnothing$; let $c \in \bigcap_{n=1}^{\infty} C_n$: then $c \in U_k$. Let

$$A' = \{x \in \prod_{i=1}^{k-1} X_i : (x, c) \in A\} \quad \text{and} \quad B' = \{x \in \prod_{i=1}^{k-1} X_i : (x, c) \in B\}.$$

Then A' and B' are separately closed in $\prod_{i=1}^{k-1} X_i$ and $\prod_{i=1}^{k-1} U_i \subset A' \cup B'$, so there is a non-empty open set $O \subset \prod_{i=1}^{k-1} U_i$ such that either $O \subset A'$ or $O \subset B'$; say $O \subset A'$. Choose n so that $B_n \subset O$.

Now $a_n \in B_n \subset O \subset A'$ and hence $(a_n, c) \in A$ which contradicts $(\{a_n\} \times C_n) \cap A = \varnothing$. ∎

Corollary 7 *Let $X_1, \ldots, X_k$ be a finite collection of topological spaces so that X_1 is Baire, and when $k > 1$ each X_i except possibly X_k has a countable pseudo-base and each X_i except possibly X_1 is quasi-regular and strongly countably complete. Suppose that $A_1, \ldots, A_n \subset \prod_{i=1}^{k} X_i$ are separately closed subsets, and $U_i \subset X_i$ are non-empty open sets such that $\prod_{i=1}^{k} U_i \subset A_1 \cup \ldots \cup A_n$. Then $(\prod_{i=1}^{k} U_i) \cap \mathring{A}_i$ is non-empty for some i.*

Proof. Use Lemma 6 and induction on n. Note that $A = A_1 \cup \ldots \cup A_{n-1}$ and $B = A_n$ are separately closed so by Lemma 6 the interior of one of them meets $\prod_{i=1}^{k} U_i$. ∎

4 Main Result

The following generalises [12, Theorem 1].

Theorem 8 *Let $X_1, \ldots, X_k$ be a finite collection of topological spaces so that X_1 is Baire, and when $k > 1$ each X_i except possibly X_k has a countable pseudo-base and each X_i except possibly X_1 is quasi-regular and strongly countably complete. Suppose that $\langle C_n \rangle$ is a sequence of separately closed subsets of the product $\prod_{i=1}^k X_i$. Let $O \subset \prod_{i=1}^k X_i$ be a non-empty open set such that $O \subset \bigcup_{n=1}^\infty C_n$. Then there is an integer m such that $O \cap \mathring{C}_m \neq \varnothing$.*

Proof. The proof is by induction on k. When $k = 1$ the result follows from Lemma 4.

Now suppose the result is true for a product of $k-1$ spaces. For each n, let $A_n = \bigcup_{i=1}^n C_i$. Then each A_n is separately closed and $A_n \subset A_{n+1}$. Furthermore, $O \subset \bigcup_{n=1}^\infty A_n$. Moreover $\langle A_n \rangle$ is a nested sequence of separately closed subsets of $\prod_{i=1}^k X_i$ and there are non-empty open subsets $U_i \subset X_i$ such that $\prod_{i=1}^k U_i \subset O$, so that $\prod_{i=1}^k U_i \subset \bigcup_{n=1}^\infty A_n$.

Thus by Lemma 5 there is an integer n such that $(\prod_{i=1}^k U_i) \cap \mathring{A}_m \neq \varnothing$. It now follows from Corollary 7 that $O \cap \mathring{C}_m \neq \varnothing$ for some $m \leq n$. ■

Corollary 9 *Let $X = \prod_{i=1}^k X_i$ be the Cartesian product of Polish spaces. Let $\langle C_n \rangle$ be a sequence of separately closed sets in X, and let $U \subset X$ be a non-empty open subset such that $U \subset \bigcup_{n=1}^\infty C_n$. Then there is an integer m such that $U \cap \mathring{C}_m \neq \varnothing$.*

Corollary 10 *Let $X_1, \ldots, X_k$ be a finite collection of topological spaces so that X_1 is Baire, and when $k > 1$ each X_i except possibly X_k has a countable pseudo-base and each X_i except possibly X_1 is quasi-regular and strongly countably complete. Suppose that $\langle C_n \rangle$ is a sequence of separately semi-closed subsets of the product $\prod_{i=1}^k X_i$. Let $O \subset \prod_{i=1}^k X_i$ be a non-empty open set such that $O \subset \bigcup_{n=1}^\infty C_n$. Then there is an integer m such that $O \cap \mathring{C}_m \neq \varnothing$.*

Proof. Apply Theorem 8 to the sequence $\langle C_n^+ \rangle$. Then $O \subset \bigcup_{n=1}^\infty C_n^+$ as $C_n \subset C_n^+$. Thus by Theorem 8, there is an integer m so that $O \cap intC_m^+ \neq \varnothing$. As C_m is separately semi-closed it follows that $intC_m^+ \subset \mathring{C}_m$ so that $O \cap \mathring{C}_m \neq \varnothing$. ■

5 Applications

Theorem 11 *Let $X_1, \ldots, X_k$ be a finite collection of topological spaces so that X_1 is Baire, and when $k > 1$ each X_i except possibly X_k has a countable pseudo-base and each X_i except possibly X_1 is quasi-regular and strongly countably complete. Let $\mathcal{F}$ be a family of separately quasi-continuous functions from the product space $X = \prod_{i=1}^k X_i$ to a space Y. Let $\{D_n \;/\; n = 1, 2, \ldots\}$ be a closed cover of Y. Suppose that for each $x \in X$ there is n such that $f(x) \in D_n$ for each $f \in \mathcal{F}$. Then for every non-empty open set $O \subset X$ there is a non-empty open set $U \subset O$ and an integer n such that $f(x) \in D_n$ for all $x \in U$ and all $f \in \mathcal{F}$.*

Proof. For each n let $C_n = \cap_{f \in \mathcal{F}} f^{-1}(D_n)$. For each $f \in \mathcal{F}$, as f is separately quasi-continuous it follows that $f^{-1}(D_n)$ is separately semi-closed and hence so is C_n. Note that $\cup_{n=1}^\infty C_n = X$. Thus given a non-empty open set $O \subset X$ we have $O \subset \cup_{n=1}^\infty C_n$ so by Corollary 10 there is an integer n such that $O \cap \mathring{C}_n \neq \varnothing$. Set $U = O \cap \mathring{C}_n$. ■

This result generalises [1, Theorem 4]: take $k = 1$, $Y = \mathbb{R}$, $D_n = [-n, n]$ and $O = X$. Another example of the application of this theorem is to the situation where the range is a metric space and $\langle D_n \rangle$ is a sequence of balls of radius n.

Now we attempt generalising [12, Theorem 3]. Our version applies when the domain is a product of two spaces.

Recall that for a function $f : X \to Y$, where X is any topological space and Y is a metric space, the *oscillation* $\omega_f(x)$ of f at x is given by

$$\omega_f(x) = \inf\{\text{diam} f(U) \ / \ U \text{ is a neighbourhood of } x\}.$$

For $A \subset X$ we have $\omega_f(A) = \sup\{\omega_f(x) \ / \ x \in A\}$.

For Theorem 12 we need the following concepts. Call a space X *uniformly first countable* if each point $x \in X$ has a countable neighbourhood base $\{N_i(x) \ / \ i = 1, 2, \ldots\}$ such that $N_{i+1}(x) \subset N_i(x)$ and for each $\xi \in N_i(x)$ there is j such that for each $x' \in N_j(x)$ we have $\xi \in N_i(x')$. Such a collection of neighbourhood bases will be called a *uniform neighbourhood base.* The space X will be called *symmetrically uniformly first countable* if in addition $\xi \in N_i(x)$ if and only if $x \in N_i(\xi)$ for all $x, \xi \in X$ and $i = 1, 2, \ldots$. Such a collection of neighbourhood bases will be called a *symmetrical uniform neighbourhood base.* Note that every metrisable space is symmetrically uniformly first countable and every symmetrically uniformly first countable space is semi-metrisable.

Theorem 12 *Let X and Y be spaces satisfying the following conditions:*

- *X and Y are uniformly first countable with one of them symmetrically so;*
- *X and Y are quasi-regular;*
- *X has a countable pseudo-base;*
- *X is Baire;*
- *Y is strongly countably complete.*

Let (Z, d) be a metric space. Suppose that $\langle f_n : X \times Y \to Z\rangle$ is a sequence of separately continuous functions converging pointwise to a function $f : X \times Y \to Z$. Then the set, $C(f)$, of points of continuity of f is dense in $X \times Y$.

Proof Let $U \subset X \times Y$ be a nonempty open set: we must show that U contains a point at which f is continuous. We first show that for each $\varepsilon > 0$ there is a nonempty open set $V \subset U$ such that $\omega_f(V) \leq \varepsilon$.

For each $p, q = 1, 2, \ldots$ let $A_{p,q} = \{(x, y) \in X \times Y \ / \ d(f_p(x, y), f_q(x, y)) \leq \frac{\varepsilon}{3}\}$ and for $m = 1, 2, \ldots$ let $A_m = \cap_{p,q \geq m} A_{p,q}$. Then

- A_m is separately closed. This follows because the complement of $A_{p,q}$ is separately open as f_p and f_q are separately continuous and the fact that the separately open sets form a topology.
- $\cup_{m=1}^{\infty} A_m = X \times Y$. This is because $f_m \to f$ pointwise.

These two points and Theorem 8 tell us that there is m such that $\mathring{A}_m \neq \varnothing$. As $X \times Y$ is quasiregular there is a closed set $A \subset \mathring{A}_m$ having nonempty interior. We may assume that A is regular closed.

Choose uniform neighbourhood bases for X and Y, with one set of bases being symmetrically so: we will denote the corresponding base at $x \in X \cup Y$ by $\{N_l(x) \ / \ l = 1, 2, \ldots\}$, assuming no confusion between X and Y.

For each $l = 1, 2, \ldots$ let

$$\begin{aligned} B_l \ = \ & \{(x, y) \in A \ / \text{ for each } \xi \in N_l(x) \text{ and } \eta \in N_l(y), \\ & d(f_m(x, y), f_m(\xi, y)) \leq \frac{\varepsilon}{12} \text{ and } d(f_m(x, y), f_m(x, \eta)) \leq \frac{\varepsilon}{12}\}. \end{aligned}$$

Then

- B_l is separately closed. This follows from the fact that the complement is separately open as f_m is separately continuous. *
- $\cup_{l=1}^{\infty} B_l = A$. This also follows from the fact that f_m is separately continuous.

By Theorem 8 there is l such that $\mathring{B}_l \neq \varnothing$. Pick $(a, b) \in \mathring{B}_l$. Then there is an open set $V = N_k(a) \times N_k(b) \subset \mathring{B}_l$ for some $k \geq l$.

Let $(x, y), (\xi, \eta) \in V$. Then

$$d(f(x, y), f(\xi, \eta)) \leq d(f(x, y), f_m(x, y)) + d(f_m(x, y), f_m(\xi, \eta)) + d(f_m(\xi, \eta), f(\xi, \eta)) \leq \varepsilon.$$

Indeed, $d(f(x, y), f_m(x, y) \leq \varepsilon/3$ and $d(f_m(\xi, \eta), f(\xi, \eta) \leq \varepsilon/3$ follow from the fact that $(x, y), (\xi, \eta) \in A_m$. The inequality $d(f_m(x, y), f_m(\xi, \eta)) \leq \varepsilon/3$ may be deduced as follows, where we have assumed that X has a symmetric neighbourhood base (if instead it is Y then replace (a, y) and (a, η) by (x, b) and (ξ, b) respectively):

$$\begin{aligned} & d(f_m(x, y), f_m(\xi, \eta)) \\ & \leq d(f_m(x, y), f_m(a, y)) + d(f_m(a, y), f_m(a, b)) + d(f_m(a, b), f_m(a, \eta)) + d(f_m(a, \eta), f_m(\xi, \eta)) \end{aligned}$$

Each of these terms is at most $\frac{\varepsilon}{12}$ because of the definition of B_l and the location of the points (a, b), (x, y) and (ξ, η). It follows that $\omega_f(V) \leq \varepsilon$.

We now define a sequence $\langle C_m \rangle$ of nonempty, regular closed subsets of $X \times Y$ such that $C_{m+1} \subset \mathring{C}_m$, $C_1 \subset U$, $\omega_f(C_m) \leq \frac{1}{m}$ and each C_m is $\mathcal{A}_m$-small, where $\langle \mathcal{A}_n \rangle$ is a sequence of open covers exhibiting strong countable completeness of $X \times Y$. Given C_m (or just U to begin the induction) apply what we have proved to the nonempty open set $\mathring{C}_m$ (or U) to get a nonempty open set $U_{m+1} \subset \mathring{C}_m$ (or U) with $\omega_f(U_{m+1}) \leq \frac{1}{m+1}$. We may assume that U_{m+1} is $\mathcal{A}_{m+1}$-small. Let C_{m+1} be a nonempty, regular closed subset of U_{m+1}. Note that C_m is $\mathcal{A}_m$-small for each m, so by choice of $\mathcal{A}_m$ the nested sequence $\langle C_m \rangle$ has nonempty intersection, say $c \in \cap_{m=1}^{\infty} C_m$. Then $c \in C(f) \cap U$. ■

An assumption close to Baireness is needed in Theorem 12. Indeed, if we let $A, B \subset (\mathbb{Q} \cap [0, 1])^2$ be the sets described in Example 2 then the function $f : (\mathbb{Q} \cap [0, 1])^2 \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \in B \end{cases}$ is separately continuous but not continuous at any point.

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* We need to show that $X \times Y - B_l$ is separately open. Suppose that $(x, y) \in X \times Y - B_l$. If $(x, y) \notin A$ then $X \times Y - A$ is an open, so separately open, set containing (x, y) and missing B_l. If instead $(x, y) \in A$ then either there is $\xi \in N_l(x)$ such that $d(f_m(x, y), f_m(\xi, y)) > \frac{\varepsilon}{12}$ or there is $\eta \in N_l(y)$ such that $d(f_m(x, y), f_m(x, \eta)) > \frac{\varepsilon}{12}$: suppose the former. Find n large enough that $\xi \in N_l(x')$ for each $x' \in N_n(x)$. As f_m is separately continuous at (x, y) then (again assuming n large enough) for each $x' \in N_n(x)$ we have

$$d(f_m(x, y), f_m(x', y)) < d(f_m(x, y), f_m(\xi, y)) - \frac{\varepsilon}{12} (> 0)$$

$$\text{so} \qquad d(f_m(x', y), f_m(\xi, y)) \geq d(f_m(x, y), f_m(\xi, y)) - d(f_m(x, y), f_m(x', y)) > \frac{\varepsilon}{12}.$$

Thus $N_n(x) \cap B_l \neq \varnothing$.

Question 1 *Do the hypotheses in Theorem 12 actually imply that the spaces X and Y are metrisable?*

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