THE MAPPING CLASS GROUP OF POWERS OF THE LONG RAY AND OTHER NON-METRISABLE SPACES

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Abstract. We identify the mapping class group, ie the space of homeomorphisms modulo isotopy, of powers of the long ray and long line as well as generalisations of the long plane obtained by taking copies of the first octant of the long plane and identifying them along their boundaries. We show that, except for the case of five small finite groups, every countable group is the mapping class group of such a space. We also consider homotopy classes of continuous functions between these spaces.

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1. Introduction

As usual we denote by ω_1 the set of countable ordinals with the order topology. Let $\mathbb{L}_+$ denote the *closed long ray*, ie the set $\omega_1 \times [0,1)$ with the lexicographic order topology, and let $\mathbb{L}$ denote the *long line* which is obtained from two copies of the long ray, denoted by $\mathbb{L}_+$ and $\mathbb{L}_-$, with their initial points identified to 0. Order $\mathbb{L}$ by reversing the order in $\mathbb{L}_-$ and declaring $a \leq b$ whenever $a \in \mathbb{L}_-$ and $b \in \mathbb{L}_+$.

In Section 2 we explore the groups of homeomorphisms of $\mathbb{L}_+^n$ and $\mathbb{L}^n$, where n is a positive integer; we denote these groups by $\mathcal{H}(\mathbb{L}_+^n)$ and $\mathcal{H}(\mathbb{L}^n)$ respectively. The main result in this section identifies the structure of the corresponding mapping class groups $\mathcal{H}(\mathbb{L}_+^n)/\cong$ and $\mathcal{H}(\mathbb{L}^n)/\cong$, where we use $\cong$ to denote isotopy. As a result we are able to identify the torsion in the original groups $\mathcal{H}(\mathbb{L}_+^n)$ and $\mathcal{H}(\mathbb{L}^n)$.

Denote by N the set $\{1,\dots,n\}$. For any subset $I \subset N$ let $|I|$ be the cardinality of I.

If $\pi : N \to N$ is a permutation let $\widehat{\pi} : \mathbb{L}_+^n \to \mathbb{L}_+^n$ be the homeomorphism obtained by permuting the coordinates according to π, ie

$$\widehat{\pi}(x_1,\dots,x_n) = (x_{\pi(1)},\dots,x_{\pi(n)}).$$

As might be expected, if $\pi \neq \pi'$ then $\widehat{\pi} \not\cong \widehat{\pi'}$, cf also Corollary 2.3. In $\mathbb{L}^n$ we can also reverse the direction of any one of the coordinates. If $s = (s_1,\dots,s_n)$ is an ordered n-tuple of signs, $+$ or $-$, then by $\widehat{s\pi} : \mathbb{L}^n \to \mathbb{L}^n$ we mean the homeomorphism given by

$$\widehat{s\pi}(x_1,\dots,x_n) = (s_1 x_{\pi(1)},\dots,s_n x_{\pi(n)}).$$

A generic homeomorphism of this form will be represented by $\widehat{\pm^n\pi}$.

Theorem 1.1. *Given any homeomorphism $h : \mathbb{L}_+^n \to \mathbb{L}_+^n$ there is a unique permutation $\pi : N \to N$ such that $h \cong \widehat{\pi}$.*

Corollary 1.2. *An integer q is the order of a homeomorphism $h : \mathbb{L}_+^n \to \mathbb{L}_+^n$ if and only if there is an element of order q in the symmetric group of order n.*

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Proof. Suppose that $h : \mathbb{L}_+^n \to \mathbb{L}_+^n$ is a homeomorphism of order q. By Theorem 1.1 there is a permutation $\pi : N \to N$ such that $h \cong \widehat{\pi}$. Then $\widehat{\pi}^q \cong h^q = 1$ so $\widehat{\pi}^q = 1$ and hence $\pi^q = 1$. □

Theorem 1.3. *Given any homeomorphism $h : \mathbb{L}^n \to \mathbb{L}^n$ there are a unique permutation $\pi : N \to N$ and a unique n-tuple of signs s such that $h \cong \widehat{s\pi}$.*

Corollary 1.4. *An integer q is the order of a homeomorphism $h : \mathbb{L}^n \to \mathbb{L}^n$ if and only if there is an element of order r in the symmetric group of order n, where either $q = r$ or $q = 2r$ and r is odd.*

In Section 3 we extend the results of Section 2 to generalisations of the long plane, 2-dimensional objects obtained by gluing together possibly infinitely many copies of the first octant of the long plane $\mathbb{L}^2$. To a countable directed graph Γ we associate a "long complex" X_Γ and we prove the following result.

Theorem 1.5. *The mapping class group of X_Γ is isomorphic to* $\mathrm{Aut}(\Gamma)$ *if* Γ *countable.*

Using results from the theory of automorphism groups of directed graphs we are able to deduce that every countable group may be realised as the mapping class of such a long complex.

In Section 4 we present an ω-bounded surface with uncountable mapping class group.

Section 5 addresses the problem of identifying the collection of continuous functions from one long complex to another up to homotopy.

The following result is well-known and is found in many books introducing Set Theory but we will use it many times and so include it for completeness.

Lemma 1.6. *If $C_i \subset \omega_1$ is a closed unbounded subset for each $i < \omega$ then $\cap_{i<\omega} C_i$ is also closed and unbounded.*

2. Powers of the long ray and line

We exploit the observation that if $e : \mathbb{L}_+ \to \mathbb{L}_+^n$ is an embedding then $e(\mathbb{L}_+) \cap \Delta_I(c)$ is closed and unbounded for some I, c, where for $I \subset N$ and $c \in \mathbb{L}_+^{n-|I|}$, the *$I$-diagonal at height c* is given by

$$\Delta_I(c) = \{(x_1, \ldots, x_n) \in \mathbb{L}_+^n \ / \ x_i = x_j \text{ if } i, j \in I \text{ and } x_{N-I} = c\}.$$

As might be expected, the equation $x_{N-I} = c$ means that the k^{th} coordinate of the projection of x onto the coordinates $N - I$ is equal to c_k. When $c = 0$ the name $\Delta_I(c)$ is abbreviated to Δ_I.

A fundamental concept which we use is that of the *direction matrix* defined in [3, page 44] for a continuous function $f : \mathbb{L}_+^n \to \mathbb{L}_+^n$. To fit more closely with the Jacobian matrix of Advanced Calculus we use the transpose of the matrix defined by Baillif. This direction matrix for continuous $f : \mathbb{L}_+^n \to \mathbb{L}_+^m$, denoted $D(f)$, is a $(2^m - 1) \times (2^n - 1)$ matrix with (I, J)-entry $D_{I,J}(f)$ defined for each non-empty $I \subset \{1, \ldots, m\}$ and $J \subset N$ as follows. $D_{I,J}(f) = 1$ if there is some $c \in \mathbb{L}_+^{m-|I|}$ such that $f(\Delta_J) \cap \Delta_I(c)$ is unbounded in $\Delta_I(c)$ and $D_{I,J}(f) = 0$ otherwise.

Lemma 2.1. *Suppose that $h : \mathbb{L}_+^n \to \mathbb{L}_+^n$ is a homeomorphism. If $\varnothing \neq I, J \subset N$ then $D_{I,J}(h) = 1$ if and only if there is a bijection $\sigma : I \to J$ so that $D_{i,\sigma(i)} = 1$ for each $i \in I$.*

Proof. This follows from [5, Corollary 3.10] and its proof. □

According to Lemma 2.1, $D(h)$ is determined by the submatrix consisting of those entries corresponding to singleton subsets of N. Taking the singleton subsets in the natural order and abbreviating $D_{\{i\},\{j\}}(h)$ to $D_{i,j}(h)$ we obtain what we will call the *reduced direction matrix* $\check{D}(h)$.

Proposition 2.2. [3, page 44] *Suppose that $f, g : \mathbb{L}_+^n \to \mathbb{L}_+^n$ are both continuous. Then f is homotopic to g if and only if $D(f) = D(g)$.*

Note that if $\pi, \rho : N \to N$ are two distinct permutations then $D(\widehat{\pi}) \neq D(\widehat{\rho})$, so

Corollary 2.3. *If $\pi, \rho : N \to N$ are two permutations then the following are equivalent:*

(a) $\widehat{\pi} \cong \widehat{\rho}$;
(b) $D(\widehat{\pi}) = D(\widehat{\rho})$;
(c) $\check{D}(\widehat{\pi}) = \check{D}(\widehat{\rho})$;
(d) $\widehat{\pi} = \widehat{\rho}$.

Proposition 2.4. *Let $h : \mathbb{L}_+^n \to \mathbb{L}_+^n$ be a homeomorphism. Then there is a unique permutation $\pi : N \to N$ such that $\check{D}(h) = \check{D}(\widehat{\pi})$.*

Proof. This follows from [5, Corollary 3.10]. □

For each $\alpha \in \omega_1$ and each $i \in N$ we set $B_{\alpha,i} = [0, \omega_1)^{i-1} \times [0, \alpha] \times [0, \omega_1)^{n-i}$. We denote the unit square matrix of some unspecified size by $\mathbf{I}$.

Lemma 2.5. *Suppose that $h : \mathbb{L}_+^n \to \mathbb{L}_+^n$ is a homeomorphism with $\check{D}(h) = \mathbf{I}$. Then for each $i \in N$, the set $\{\alpha \in \omega_1 \ / \ h(B_{\alpha,i}) = B_{\alpha,i}\}$ is closed and unbounded in ω_1.*

Proof. Let $S = \{\alpha \in \omega_1 \ / \ h(B_{\alpha,i}) = B_{\alpha,i}\}$. It is clear that S is closed, so we need to show that S is unbounded. Suppose that $\alpha_0 \in \omega_1$. We construct an increasing sequence $\langle \alpha_n \rangle$.

Suppose that α_n is given with n even. Choose $\alpha_{n+1} > \alpha_n$ so that $h(B_{\alpha_n,i}) \subset B_{\alpha_{n+1},i}$.

> This is possible. Indeed, suppose not. Then there is an embedding $e : \omega_1 \to B_{\alpha_n,i}$ such that for each $\alpha > \alpha_n$ we have $h(e(\alpha)) \notin B_{\alpha,i}$. On the one hand the point $r \in R(\mathbb{L}_+^n)$ of the ray space given by [5, Lemma 3.6] must have bounded i^{th} coordinate and hence, by [5, Corollary 3.10], so must the point $\bar{h}(r)$. On the other hand, the i^{th} coordinate of he is unbounded leading to a contradiction.

Suppose given α_n with n odd. Choose $\alpha_{n+1} > \alpha_n$ so that $h^{-1}(B_{\alpha_n,i}) \subset B_{\alpha_{n+1},i}$. This is much the same as the case where n is even.

Now let $\alpha = \lim \alpha_n$. Then $\alpha > \alpha_0$. Moreover for each even $n > 0$ we have $h(B_{\alpha_n,i}) \subset B_{\alpha_{n+1},i} \subset B_{\alpha,i}$ so that $h(B_{\alpha,i}) = \overline{\cup_{n \text{ even}} h(B_{\alpha_n,i})} \subset B_{\alpha,i}$. Similarly by considering n odd we conclude that $h^{-1}(B_{\alpha,i}) \subset B_{\alpha,i}$. Thus $h(B_{\alpha,i}) = B_{\alpha,i}$, so $\alpha \in S$. □

By Lemma 1.6 we obtain the following Corollary, in which, for any $\alpha \in S$, we denote by α^+ the least member of $\{\beta \in S \ / \ \beta > \alpha\}$.

Corollary 2.6. *Suppose that $h : \mathbb{L}_+^n \to \mathbb{L}_+^n$ is a homeomorphism with $\check{D}(h) = \mathbf{I}$. Then there is a closed unbounded subset $S \subset \omega_1$ such that for each $\alpha_1, \dots, \alpha_n \in S$ the hypercubes $\prod_{i=1}^n [\alpha_i, \alpha_i^+]$ are invariant under h.*

We can now give a more direct proof of a strengthened version of Corollary 1.2.

Corollary 2.7. *Let $h : \mathbb{L}_+^n \to \mathbb{L}_+^n$ be a homeomorphism with $\check{D}(h) = \check{D}(\widehat{\pi})$ for some permutation π. If $h^q = \mathbf{1}$ for some $q > 0$ then $h^{q'} = \mathbf{1}$ for the smallest $q' > 0$ such that $\pi^{q'} = \mathbf{1}$.*

Proof. It suffices to show that if $\check{D}(h) = \mathbf{I}$ then $h = \mathbf{1}$. By Corollary 2.6 there is a closed unbounded subset $S \subset \omega_1$ such that for each $\alpha_1, \dots, \alpha_n \in S$ the hypercubes $\prod_{i=1}^n [\alpha_i, \alpha_i^+]$ are invariant under h. It follows that h leaves each face of any such hypercube invariant. For any such hypercube consider $h | \prod_{i=1}^n [\alpha_i, \alpha_i^+]$. [6, Theorem 2] implies that if the fixed-point set of a periodic homeomorphism of $\mathbb{R}^k$ has non-empty interior then the homeomorphism is the identity, hence any periodic homeomorphism of $[0,1]^k$ which fixes $\partial[0,1]^k$ must be the identity. As h fixes each vertex of the hypercube it follows by induction on the dimension of the faces of the hypercube that h is the identity on $\prod_{i=1}^n [\alpha_i, \alpha_i^+]$ and hence on all of $\mathbb{L}_+^n$. □

For any n let $\mathcal{H}(I^n, \partial I^n) = \{h : I^n \to I^n \ / \ h$ is a homeomorphism and $h|\partial I^n = \mathbf{1}\}$, with the compact-open topology, which is the same as the uniform metric topology in this case. We are denoting the identity function by $\mathbf{1}$.

Proposition 2.8. *There is a continuous function $c : \mathcal{H}(I^n, \partial I^n) \times I \to \mathcal{H}(I^n, \partial I^n)$ such that:*
(i) *$c(h, 0) = h$ for each h;*
(ii) *$c(h, 1) = \mathbf{1}$ for each h;*
(iii) *$c(\mathbf{1}, t) = \mathbf{1}$ for each t.*

Proof. It is easier to work with $(\mathbb{B}^n, \mathbb{S}^{n-1})$, which is homeomorphic to $(I^n, \partial I^n)$. Set

$$c(h,t)(x) = \begin{cases} (1-t)h(\frac{x}{1-t}) & \text{if } t < 1 \text{ and } |x| \le 1-t \\ x & \text{if } t = 1 \text{ or } |x| \ge 1-t. \end{cases}$$

Clearly conditions (i)–(iii) hold. It remains to show that $c(h, t) \in \mathcal{H}(\mathbb{B}^n, \mathbb{S}^{n-1})$ and that c is continuous.

- $c(h, t) \in \mathcal{H}(\mathbb{B}^n, \mathbb{S}^{n-1})$. Because $h|\mathbb{S}^{n-1}$ is the identity we may extend h to a homeomorphism on all of $\mathbb{R}^n$ by setting it to be the identity on $\mathbb{R}^n - \mathbb{B}^n$. For $t < 1$ the function $c(h, t)$ is the composition of three homeomorphisms, division and multiplication by $1-t$ and (extended) h, hence is a homeomorphism. On the other hand $c(h, 1)$ is the identity homeomorphism.
- c is continuous. Continuity of c on $\mathcal{H}(I^n, \partial I^n) \times [0, 1)$ follows from continuity of composition in $\mathcal{H}(I^n, \partial I^n)$. Let $h_0 \in \mathcal{H}(I^n, \partial I^n)$ be given: we show that c is continuous at $(h_0, 1)$. Given $\varepsilon > 0$, assume that $t > 1 - \frac{\varepsilon}{2}$, so that $1 - t < \frac{\varepsilon}{2}$. For all h and all $x \in \mathbb{B}^n$ we have either $c(h, t)(x) = x$ or else $|x| < 1 - t$ and $|h(\frac{x}{1-t})| \le 1$. In the latter case $|c(h, t)(x) - x| \le |c(h, t)(x)| + |x| \le |1 - t| + \frac{\varepsilon}{2} < \varepsilon$. Hence c is continuous at $(h_0, 1)$. □

Proposition 2.9. *There is a continuous function $e : \mathcal{H}(I^{n-1}, \partial I^{n-1})^2 \to \mathcal{H}(I^n)$ such that*
(i) *$e(g, h)(x, 0) = (g(x), 0)$ for each g, h and each $x \in I^{n-1}$;*
(ii) *$e(g, h)(x, 1) = (h(x), 1)$ for each g, h and each $x \in I^{n-1}$;*
(iii) *$e(g, h)(x, s) = (x, s)$ for each g, h, each s and each $x \in \partial I^{n-1}$;*
(iv) *$e(\mathbf{1}, \mathbf{1}) = \mathbf{1}$.*

Proof. Given $g, h \in \mathcal{H}(I^{n-1}, \partial I^{n-1})$ let $e(g, h) : I^n \to I^n$ be defined by

$$e(g, h)(x, s) = (hc(h^{-1}g, s)(x), s),$$

where c is given by Proposition 2.8 but with n replaced by $n - 1$. Note that $(g, h, x, s) \mapsto (hc(h^{-1}g, s)(x), s)$ is continuous because c is continuous, composition is continuous and the action is continuous with respect to the compact-open topology, and hence e is continuous. Further, e satisfies conditions (i)–(iv). □

In $\mathbb{L}^n_+$ we have the following subsets.

- For each $k = 0, \dots, n$, the *k-skeleton* is the set

  $S_k = \{(x_i) \in \mathbb{L}^n_+ \ / \ x_i \in \omega_1$ for at least $n - k$ values of $i\}$.
- For each $k = 0, \dots, n$ set $T_k = \cup_{\alpha_{k+1} \in \omega_1} \cdots \cup_{\alpha_n \in \omega_1} \mathbb{L}^k_+ \times \{\alpha_{k+1}\} \times \ldots \times \{\alpha_n\}$.

For each $\alpha \in \omega_1$ there is a natural order-preserving homeomorphism $\overbrace{\ \alpha\ } : [0, 1] \to [\alpha, \alpha + 1]$. When we are given a finite sequence $\alpha_1, \dots, \alpha_k \in \omega_1$ we will denote by $\overbrace{(\alpha_i)_{i=1}^k}$ or just $\overbrace{(\alpha_i)}$ the homeomorphism $[0, 1]^k \to \prod_{i=1}^k [\alpha_i, \alpha_i + 1]$ which is just the product of the homeomorphisms $\overbrace{\alpha_i}$.

Proposition 2.10. *Suppose that $h : \mathbb{L}^n_+ \to \mathbb{L}^n_+$ is a homeomorphism which is the identity on the $(k-1)$-skeleton S_{k-1} for some $k = 1, \dots, n-1$ and leaves each hypercube of the form $\prod_{i=1}^n [\alpha_i, \alpha_i + 1]$, where each $\alpha_i \in \omega_1$, invariant. Then there is an isotopy $h_t : \mathbb{L}^n_+ \to \mathbb{L}^n_+$ such that:*

(i) $h_0 = \mathbf{1}$;
(ii) $h_1 h$ *is the identity on* T_k;
(iii) h_t *is the identity on* $(S_k - T_k) \cup S_{k-1}$.

Proof. We use induction on l to define h_t on T_l, for $l = k, \dots, n$, noting that $T_n = \mathbb{L}^n_+$.

The induction begins at $l = k$ by applying Proposition 2.8 (with n there replaced by k). More precisely when $\alpha_1, \dots, \alpha_n \in \omega_1$ and $x_i \in [\alpha_i, \alpha_i + 1]$ for $i = 1, \dots, k$ set

$$h_t(x_1, \dots, x_k, \alpha_{k+1}, \dots, \alpha_n) = \left(\overbrace{(\alpha_i)_{i=1}^k} c \left(\overbrace{(\alpha_i)_{i=1}^k}^{-1} h^{-1} \overbrace{(\alpha_i)_{i=1}^k}, 1-t\right) \overbrace{(\alpha_i)_{i=1}^k}^{-1} (x_1, \dots, x_k), \alpha_{k+1}, \dots, \alpha_n\right).$$

Now suppose that h_t has been defined on T_{l-1} for some l with $k < l \leq n$. We extend h_t defined so far over T_l using Proposition 2.9 (with n there replaced by l). Again suppose given $\alpha_1, \dots, \alpha_n \in \omega_1$ and $x_i \in [\alpha_i, \alpha_i + 1]$ for $i = 1, \dots, l$. For convenience of notation we set $\beta_i = \alpha_i$ for $i \neq l$ and $\beta_l = \alpha_l + 1$. Set

$$h_t(x_1, \dots, x_l, \alpha_{l+1}, \dots, \alpha_n) = \left(\overbrace{(\alpha_i)_{i=1}^l} e \left(\overbrace{(\alpha_i)_{i=1}^{l-1}}^{-1} h_t \overbrace{(\alpha_i)_{i=1}^{l-1}}, \overbrace{(\beta_i)_{i=1}^{l-1}}^{-1} h_t \overbrace{(\beta_i)_{i=1}^{l-1}}\right) \overbrace{(\alpha_i)_{i=1}^l}^{-1} (x_1, \dots, x_l), \alpha_{l+1}, \dots, \alpha_n\right).$$

By conditions (i) and (ii) of Proposition 2.9 the isotopy h_t just defined extends the inductively assumed isotopy from T_{l-1} to T_l. We now verify conditions (i)–(iii) of the Proposition.
(i) $h_0 = \mathbf{1}$ follows from Proposition 2.9(iv).
(ii) This follows from the inductive assumption as the extended h_t agrees with the original h_t on T_k.
(iii) That h_t is the identity on $(S_k - T_k) \cup S_{k-1}$ follows from Proposition 2.9(iii).

Finally we need to show that the h_t as extended to T_l is an isotopy provided that $h_t|T_{l-1}$ is. The problem points are those with limit ordinals as coordinates. However continuity there follows from continuity of the function $e : \mathcal{H}(I^{n-1}, \partial I^{n-1})^2 \to \mathcal{H}(I^n)$. □

It is evident from the proof of Proposition 2.10 that we may permute the coordinates so that instead of the last $n-k$ coordinates being confined to ω_1 in conditions (ii) and (iii) some other combination of $n-k$ coordinates could do as well. By repeating Proposition 2.10 for all $\binom{n}{k}$ possible combinations and iterating the resulting isotopies we obtain the following Corollary.

Corollary 2.11. *Suppose that $h : \mathbb{L}^n_+ \to \mathbb{L}^n_+$ is a homeomorphism which is the identity on the $(k-1)$-skeleton S_{k-1} for some $k = 0, \dots, n-1$ and leaves each hypercube of the form $\prod_{i=1}^n [\alpha_i, \alpha_i + 1]$, where each $\alpha_i \in \omega_1$, invariant. Then there is an isotopy $h_t : \mathbb{L}^n_+ \to \mathbb{L}^n_+$ such that:*

(i) $h_0 = \mathbf{1}$;
(ii) $h_1 h$ *is the identity on* S_k.

Iterating this isotopy gives the following.

Corollary 2.12. *Suppose that $h : \mathbb{L}^n_+ \to \mathbb{L}^n_+$ which leaves each hypercube of the form $\prod_{i=1}^n [\alpha_i, \alpha_i + 1]$, where each $\alpha_i \in \omega_1$, invariant. Then there is an isotopy $h_t : \mathbb{L}^n_+ \to \mathbb{L}^n_+$ such that:*

(i) $h_0 = h$;
(ii) h_1 *is the identity.*

Proof of Theorem 1.1. By Proposition 2.4 there is a unique permutation π so that $\check{D}(h) = \check{D}(\widehat{\pi})$. It is claimed that $h \cong \widehat{\pi}$. To prove this it suffices to show that $\widehat{\pi}^{-1}h \cong \mathbf{1}$. Since $\check{D}(\widehat{\pi}^{-1}h) = \check{D}(\widehat{\pi}^{-1})\check{D}(h) = \check{D}(\mathbf{1})$ is the identity matrix, we can reduce to the case where $\check{D}(h)$ is the identity matrix and show that $h \cong \mathbf{1}$.

Suppose, then, that h is such that $\check{D}(h)$ is the identity. We may further assume that h fixes the origin; otherwise we may construct an isotopy from h to a homeomorphism which does. By Corollary 2.6 there is a closed unbounded subset $S \subset \omega_1$ such that for each $\alpha_1, \dots, \alpha_n \in S$ the hypercubes $\prod_{i=1}^n [\alpha_i, \alpha_i^+]$ are invariant under h; it may be assumed that $0 \in S$. Note that S is order isomorphic to ω_1 so there is a homeomorphism $\theta : \mathbb{L}_+ \to \mathbb{L}_+$ taking S onto ω_1. Let $\tilde{\theta} : \mathbb{L}_+^n \to \mathbb{L}_+^n$ be the product of θ with itself n times.

Consider the homeomorphism $\tilde{\theta}h\tilde{\theta}^{-1} : \mathbb{L}_+^n \to \mathbb{L}_+^n$. By Corollary 2.12 there is an isotopy $\varphi_t : \mathbb{L}_+^n \to \mathbb{L}_+^n$ such that $\varphi_0 = \tilde{\theta}h\tilde{\theta}^{-1}$ and $\varphi_1 = \mathbf{1}$. Set $h_t = \tilde{\theta}^{-1}\varphi_t\tilde{\theta}$. □

Proof of Theorem 1.3. This is similar to the proof of Theorem 1.1 but we require a new version of Proposition 2.4, viz for a homeomorphism $h : \mathbb{L}^n \to \mathbb{L}^n$ there is a permutation π so that $\check{D}(h) = \check{D}(\widehat{s\pi})$, where $\check{D}(h)$ now applies to homeomorphisms of $\mathbb{L}^n$ and individual entries of this matrix may include -1 as well as 0 and 1. We may then rephrase Lemma 2.5 and Corollary 2.6 to fit this context. For example the analogue of Corollary 2.6 will read

> Suppose that $h : \mathbb{L}^n \to \mathbb{L}^n$ is a homeomorphism with $\check{D}(h) = \mathbf{I}$. Then there is a closed unbounded subset $S \subset \omega_1$ such that the hypercubes $\prod_{i=1}^n I_i$, where each I_i is of one of the forms $[\alpha, \alpha^+]$, $[\alpha^+, \alpha]$ and $[-\alpha_0, \alpha_0]$ with $\alpha \in S$ and $\alpha_0 = \min S$ (assumed > 0), are invariant under h.

The proof now proceeds as in the proof of Theorem 1.1. □

3. Generalisations of the long plane

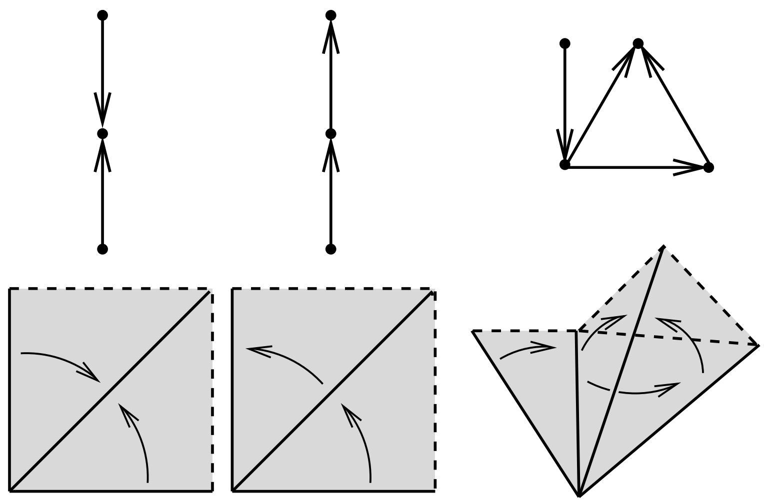
The basic theme of this section is inspired by the observation that $\mathbb{L}_+^2$ (respectively $\mathbb{L}^2$) may be obtained by taking two (respectively eight) copies of the octant $\mathbb{O} = \{(x, y) \in \mathbb{L}_+^2 \ / \ x \geq y\}$ and joining them along their sides. As noted in [7, page 646], though the two sides, i.e. the x-axis $X = \mathbb{L}_+ \times \{0\}$ and the diagonal $\Delta = \{(x, x) \ / \ x \in \mathbb{L}_+\}$, are each homeomorphic to $\mathbb{L}_+$, their embeddings in $\mathbb{O}$ are quite different; indeed, there is no homeomorphism of the octant $\mathbb{O}$ which interchanges X and Δ. We shall denote the side X by $\mathbb{O}(0)$ and the side Δ by $\mathbb{O}(1)$ and we shall always assume the identification of $\mathbb{O}(0)$ and $\mathbb{O}(1)$ with $\mathbb{L}_+$ is by the natural homeomorphism which ignores the second coordinate. Then $\mathbb{L}_+^2$ is obtained from two copies of $\mathbb{O}$, say $\mathbb{O}_1$ and $\mathbb{O}_2$, by identifying the sides $\mathbb{O}_1(1)$ and $\mathbb{O}_2(1)$ by the natural homeomorphism (though the choice of homeomorphism is irrelevant in this case). The long plane $\mathbb{L}^2$ is obtained from eight copies, say $\mathbb{O}_1, \dots, \mathbb{O}_8$, by identifying $\mathbb{O}_{2k-1}(1)$ with $\mathbb{O}_{2k}(1)$ and $\mathbb{O}_{2k}(0)$ with $\mathbb{O}_{2k+1}(0)$, for $k = 1, 2, 3, 4$, with the subscripts counted mod 8. We can build up more surfaces by taking a sequence $\langle \mathbb{O}_i \rangle$ of octants and joining $\mathbb{O}_i$ to $\mathbb{O}_{i+1}$ by identifying either $\mathbb{O}_i(0)$ or $\mathbb{O}_i(1)$ with either $\mathbb{O}_{i+1}(0)$ or $\mathbb{O}_{i+1}(1)$.

A *directed graph* or *digraph* $\Gamma = (V, K, E)$ is given by a a set of vertices V and a pair $E = (E^1, E^2)$ of maps $E^1, E^2 : K \to V$, where K is an index set. $E(k) = (v, w)$ is a *directed edge* or *arrow* between v and w. Γ is finite, countable, etc if both V and K are alike. A *morphism* of digraphs $f : \Gamma = (V, K, E) \to \Gamma' = (V', K', E')$ is given by maps $f_V : V \to V'$ and $f_E : K \to K$ such that for all $k \in K$, $E'^i(f_E(k)) = f_V(E^i(k))$ $i = 1, 2$. We denote by $\mathrm{Mor}(\Gamma, \Gamma')$ the set of morphisms $\Gamma \to \Gamma'$. An *automorphism* of Γ is a morphism such that both f_V and f_E are bijections, we denote by $\mathrm{Aut}(\Gamma)$ the group of automorphisms of Γ. In this way we obtain the category DG of *directed graphs*.

To simplify the definitions, we will consider in this section only graphs $\Gamma = (V, E)$ for which for each $v, w \in V$, there is *at most* one $k \in K$ with $E(k) = (v, w)$. We thus see E *as a subset* *of* $V \times V$, and a morphism $f : \Gamma \to \Gamma$ is entirely specified once f_V is known. We therefore often omit the subscripts E, V on f and suppress the reference to K. We make this restriction only for the sake of clarity; the discussion could also proceed with graphs having more than one directed edge between two vertices.

Given a digraph $\Gamma = (V, E)$, we define the *long complex* X_Γ *associated with* Γ as follows. First, take disjoint copies $\mathbb{L}_v$ of $\mathbb{L}_+$ for each $v \in V$ and disjoint copies $\mathbb{O}_e$ of $\mathbb{O}$ for each $e \in E$. Topologise $(\cup_{v\in V}\mathbb{L}_v) \cup (\cup_{e\in E}\mathbb{O}_e)$ by declaring each of the sets $\mathbb{L}_v$ and $\mathbb{O}_e$ open and giving each of them the usual topology inherited from $\mathbb{L}_+$ and $\mathbb{O}$ respectively. Identify $\mathbb{L}_v$ and $\mathbb{L}_{v'}$ respectively to $\mathbb{O}_e(0)$ and $\mathbb{O}_e(1)$ whenever $e = (v, v') \in E$ and declare X_Γ to be the resulting quotient space. We identify each $\mathbb{O}_e$ with its image in the quotient space. X_Γ is the union $\cup_{\alpha\in\omega_1} U^\Gamma_\alpha$, where $U^\Gamma_\alpha = \cup_{e\in E} U^e_\alpha$ and $U^e_\alpha = [0, \alpha)^2 \cap \mathbb{O}_e$. Note that each of the sets U^Γ_α is open.

The following examples show that going from Γ to X_Γ and vice-versa is easy. Here and elsewhere a curved arrow in an octant runs from the edge $\mathbb{O}(0)$ to the edge $\mathbb{O}(1)$.



Lemma 3.1. *If* Γ *is at most countable,* X_Γ *is type I and locally isomorphic to the geometric realisation of a* 2*-dimensional simplicial complex. If* Γ *is finite,* X_Γ *is* ω*-bounded.*

Proof. Firstly note that each set U^e_α has compact closure $[0, \alpha]^2 \cap \mathbb{O}_e$ and hence the closure of the open set U^Γ_α, which is the set $\cup_{e\in E}[0, \alpha]^2 \cap \mathbb{O}_e$, is Lindelöf. Thus X is of Type I because $X_\Gamma = \cup_{\alpha\in\omega_1} U^\Gamma_\alpha$.

Next note that each set of the form $\overline{U^\Gamma_\alpha}$ is the geometric realisation of a simplicial complex of dimension 2 as $\overline{U^e_\alpha}$ is homeomorphic to a 2-simplex.

Finally if Γ is finite then any countable set $C \subset X_\Gamma$ lies in a set of the form $\overline{U^\Gamma_\alpha}$ which is compact so that C has compact closure, which is to say that X_Γ is ω-bounded. $\square$

Lemma 3.2. $\pi_n(X_\Gamma) = 0$ *for all* n *and all connected* Γ.

Proof. If $f : \mathbb{S}^n \to X_\Gamma$ is continuous, then $f(\mathbb{S}^n)$ is compact. Hence, $f(\mathbb{S}^n) \subset U^\Gamma_\alpha$ for some α, and U^Γ_α can be contracted to a point in X_Γ. $\square$

Before giving the proof of Theorem 1.5 we give a corollary.

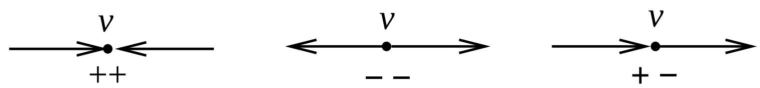
Corollary 3.3. *Suppose that* G *is a countable group. Then* G *is the mapping class group of some* X_Γ*, and thus of a space which is locally a simplicial complex and has trivial* π_n *for all* n*. Moreover, if* G *is finite,* X_Γ *is* ω*-bounded.*

Proof. Given any such group G, it is shown in [8] in the finite case and in Babai in [1, Main Theorem] in the infinite case that there is a directed graph Γ such that $G = \mathrm{Aut}(\Gamma)$. Moreover,

Γ is finite whenever G is finite. The corollary is therefore a consequence of Theorem 1.5 and Lemmas 3.1 and 3.2. $\square$

Proof of Theorem 1.5. We assume that $\Gamma = (V, E)$ is countable. We shall use the tools developed in the previous section. We first adapt the definition of the direction matrix in our settings.

So, let $\Gamma = (V, E)$ be a digraph. The picture below defines the *type* of a vertex of valence 2 in Γ.



A *translate* of $\mathbb{O}(0)$ in $\mathbb{O}$ is a horizontal $\mathbb{L}_+ \times \{y\} \cap \mathbb{O}$.

Lemma 3.4. *If $h : X_\Gamma \to X_\Gamma$ is a homeomorphism, then for all $v \in V$ of valence 1 or ≥ 3, there is a $v' \in V$ of the same valence such that $h(\mathbb{L}_v) = \mathbb{L}_{v'}$. If v is of valence 2, there is $v' \in V$ of the same valence and type satisfying the following:*

- *type $+-$: there is an α such that $h(\mathbb{L}_v) \subset \mathbb{L}_{v'}$ outside of U_α^Γ,*
- *type $++$: $h(\mathbb{L}_v) \cap \mathbb{L}_{v'}$ is closed and unbounded,*
- *type $--$: for some edge $e = (v', w) \in E$, there is an α such that $h(\mathbb{L}_v)$ is included in a translate of $\mathbb{L}_{v'}$ in $\mathbb{O}_e$ outside of U_α^Γ.*

Proof. If the valence of v is 1 or at least 3, the result is clear. For valence 2, the first two cases were dealt with in Section 2.

In the last case, denote the two octants of X_Γ meeting at v by $\mathbb{O}_1$ and $\mathbb{O}_2$: we will assume that $v' = v$. If h does not map all but a bounded interval of $\mathbb{O}_2(0)$ to itself then either h or h^{-1} will map all but a bounded subset of $\mathbb{O}_2(0)$ to a subset of $\mathbb{O}_1$ of the form $[c, \omega_1) \times \{d\}$ and the same one of h and h^{-1} will map a subset of $\mathbb{O}_2$ of the form $[a, \omega_1) \times \{b\}$ to $\mathbb{O}_1(1)$. Then this homeomorphism will take a subset of $\mathbb{O}_2$ of the form $[a, \omega_1) \times [0, b]$ to a subset of $\mathbb{O}_1$ of the form $\{(x, y) \in \mathbb{O}_1 \ / \ x \geq c \text{ and } y \geq d\}$. However this is impossible as these two sets are not homeomorphic. $\square$

Let $f : X_{\Gamma_1} \to X_{\Gamma_2}$ be continuous, and $e_1 \in E_1$, $e_2 \in E_2$ be edges. We say that $\mathbb{O}_{e_1}$ *covers* $\mathbb{O}_{e_2}$ *under* f if there is some $z \in \mathbb{L}_+$ such that for all $y \geq z$,

$$f(\mathbb{O}_{e_1}) \cap [y, \omega_1) \times \{y\} \subset \mathbb{O}_{e_2} \text{ is unbounded.} \tag{1}$$

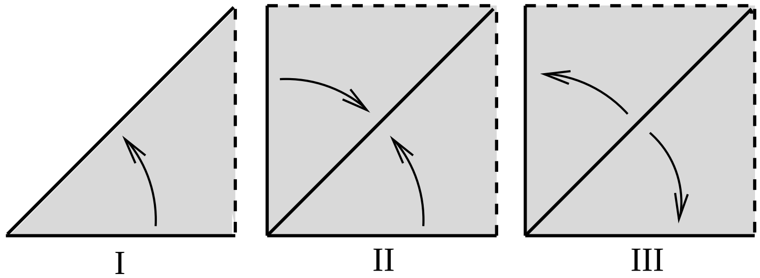
Lemma 3.5. *Given $f : X_{\Gamma_1} \to X_{\Gamma_2}$ continuous, for all $e_1 \in E_1$ there is at most one edge $e_2 \in E_2$ such that $\mathbb{O}_{e_1}$ covers $\mathbb{O}_{e_2}$ under f. If $h : X_\Gamma \to X_\Gamma$ is a homeomorphism, for each edge e there is exactly one edge e' such that $\mathbb{O}_e$ covers $\mathbb{O}_{e'}$ under h.*

Proof. This is essentially contained in [5, Corollary 3.9]. $\square$

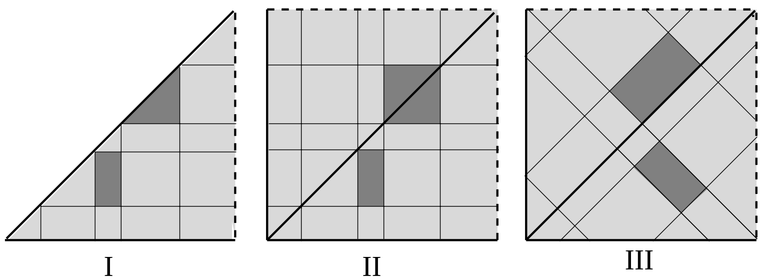
Given a homeomorphism $h : X_\Gamma \to X_\Gamma$, we let $D_h : \Gamma \to \Gamma$ be the automorphism defined by $D_h(v) = v'$ where v' is given by Lemma 3.4. (We then have $D_h(e) = e'$, where e' is given by Lemma 3.5.) D_h is evidently a morphism, and it is easy to see that $D_h \circ D_{h^{-1}} = D_{h^{-1}} \circ D_h = \mathrm{Id}$. Given an automorphism $\pi : \Gamma \to \Gamma$, there is a canonical homeomorphism $\widehat{\pi} : X_\Gamma \to X_\Gamma$ given by sending (pointwise) $\mathbb{L}_v$ and $\mathbb{O}_e$ respectively to $\mathbb{L}_{\pi(v)}$ and $\mathbb{O}_{\pi(e)}$. It is clear that $D_{\widehat{\pi}} = \pi$. Thus Theorem 1.5 follows from the following proposition.

Proposition 3.6. *If $h : X_\Gamma \to X_\Gamma$ is an homeomorphism, h is isotopic to $\widehat{D_h}$.*

Proof. By precomposing h with $\widehat{D_h^{-1}}$ if necessary, we can assume that $D_h = \mathrm{Id}$. A consequence of Lemmas 3.4 and 3.5 is that there is a partition of X_Γ into pieces P_j (j in some index set J) which are unions of at most two $\mathbb{O}_e$ of the three types I, II, III depicted below, such that outside some U_α^Γ, h maps P_i homeomorphically to itself. By an isotopy in U_α^Γ, we can assume that α is in fact 0. Moreover, since Γ is countable, so is J.



(Notice that a piece of type II is homeomorphic to $\mathbb{L}_+^2$.) Applying an obvious analogue of Lemma 2.5 to each of these pieces and using Lemma 1.6, we obtain a closed, unbounded set $S \subset \omega_1$ such that for each $\alpha_1, \alpha_2 \in S$, the subsets of P_i depicted below are invariant under h (we have coloured in dark grey the subsets given by two choices of α_1, α_2 in the three types):



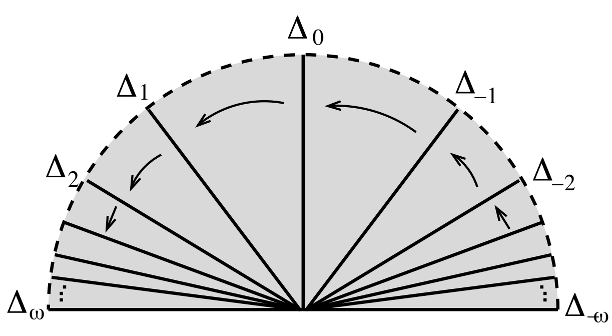
Using isotopies in *each* P_j as in Proposition 2.10 and Corollaries 2.11 and 2.12, we obtain an isotopy between h and the identity, proving the theorem. ☐

4. A surface with uncountable mapping class group

In the previous section we restricted ourselves to countable groups because of Lemma 1.6. Also, an uncountable Γ yields a X_Γ with a quite complicated local topology. However, it is easy to obtain uncountable mapping class groups.

Theorem 4.1. *There is an ω-bounded surface with mapping class group isomorphic to $\mathbb{Z}^{\mathbb{Z}}$.*

Proof. Start with copies $\mathbb{O}_i$ of $\mathbb{O}$ for each $i \in \mathbb{Z}$ and copies $\Delta_\omega, \Delta_{-\omega}$ of $\mathbb{L}_+$, and arrange them as on the figure below, such that the Δ_i $i \geq 0$ and the Δ_j $j \leq 0$ respectively accumulate to Δ_ω and $\Delta_{-\omega}$. Call this manifold $M^{\mathbb{Z}}$.



Build then $M^{\mathbb{Z},\mathbb{Z}}$ in the same way, using copies $M_i^{\mathbb{Z}}$ for $i \in \mathbb{Z}$ instead of $\mathbb{O}_i$, and two copies $\Delta_{\omega,\omega}$, $\Delta_{-\omega,-\omega}$ of $\mathbb{L}_+$. Then, glue together $\Delta_{\omega,\omega}$ and $\Delta_{-\omega,-\omega}$ to obtain a surface without boundary. We claim that the mapping class group of $M^{\mathbb{Z},\mathbb{Z}}$ is $\mathbb{Z} \times \mathbb{Z}^{\mathbb{Z}} \sim \mathbb{Z}^{\mathbb{Z}}$. Indeed, for $j \in \mathbb{Z}$, define $\varphi_j : M^{\mathbb{Z}} \to M^{\mathbb{Z}}$ as the "rotation" that takes $\mathbb{O}_i$ to $\mathbb{O}_{i+j}$. Now, for $(j,s) \in \mathbb{Z} \times \mathbb{Z}^{\mathbb{Z}}$, define $\phi_{(j,s)} : M^{\mathbb{Z},\mathbb{Z}} \to M^{\mathbb{Z},\mathbb{Z}}$ as the 'double rotation' that sends $M_i^{\mathbb{Z}}$ to $M_{i+j}^{\mathbb{Z}}$ precomposed with the $\varphi_{s(i)} : M_i^{\mathbb{Z}} \to M_i^{\mathbb{Z}}$. Then, if $(j,s) \neq (j',s')$, $\phi_{(j,s)}$ and $\phi_{(j',s')}$ are non-homotopic. By applying the same technique as in Section 2–3, we see that any homeomorphism $h : M^{\mathbb{Z},\mathbb{Z}} \to M^{\mathbb{Z},\mathbb{Z}}$ is isotopic to some $\phi_{(j,s)}$. □

5. Classifying maps up to homotopy

The goal of this section is to classify the maps $f : X_{\Gamma_1} \to X_{\Gamma_2}$ up to *homotopy*, i.e., to compute $[X_{\Gamma_1}, X_{\Gamma_2}]$. We omit most of the proofs, since they are mere adaptations of the arguments in the preceding sections and of [4]. Let LC be the category of long complexes with at most countable Γ, with morphisms the continuous maps. We will define a category of digraphs DGCP and a covariant functor $D : \mathsf{LC} \rightsquigarrow \mathsf{DGCP}$ such that the following holds:

Theorem 5.1. *$f, g : X_{\Gamma_1} \to X_{\Gamma_2}$ are homotopic if and only if $D(f) = D(g)$, and $[X_{\Gamma_1}, X_{\Gamma_2}] \simeq \mathrm{Mor}(D(X_{\Gamma_1}), D(X_{\Gamma_2}))$.*

In this section, we will *not* make the restriction that there is at most one directed edge from v to w (v, w vertices). Therefore, if $\Gamma = (V, K, E)$, then X_Γ is $\bigsqcup_{v\in V} \mathbb{L}_v \bigsqcup_{k\in K} \mathbb{O}_k / \sim$, where $\sim$ is defined by the identifications of $\mathbb{L}_v$ and $\mathbb{L}_{v'}$ with $\mathbb{O}_k(0)$ and $\mathbb{O}_k(1)$ respectively whenever $E(k) = (v, v')$ as in Section 3.

We begin with the following lemmas.

Lemma 5.2. *Let $\Gamma = (V, K, E)$ be a digraph. If $f : \mathbb{L}_+ \to X_\Gamma$ is unbounded, then outside some U^Γ_α, either $f(\mathbb{L}_+) \subset \mathbb{L}_v$ for some $v \in V(\Gamma)$, or $f(\mathbb{L}_+) \subset \mathbb{O}_k$ for some $k \in K$, or $f(\mathbb{L}_+) \cap \mathbb{L}_v$ is closed and unbounded for some unique $v \in V$.*

Proof. By considering the different ways in which octants can be glued. □

Corollary 5.3. *If $f : \mathbb{L}_+ \to X_\Gamma$ is unbounded, there is a unique $v \in V$ for which $f(\mathbb{L}_+)$ and $\mathbb{L}_v$ are homotopic in X_Γ.*

Proof. Immediate by Lemma 5.2. □

We now introduce the category DGCP of *bicoloured digraphs with smaller point.* The objects of DGCP are tuples $\Gamma = (V, K, E, C, *)$, where

- V is a set of vertices containing $*$, K is an index set,
- $E = (E_1, E_2)$ is a pair of maps $K \to V$, such that for all $v \in V$, there is a $k \in K$ with $(*, v) = E(k)$,
- $C : K \to \{0, 1\}$ is a colouring map.

A morphism $\varphi : \Gamma = (V, K, E, C, *) \to \Gamma' = (V', K', E', C', *')$ is a pair of maps $\varphi_V : V \to V'$, $\varphi_E : K \to K'$, such that

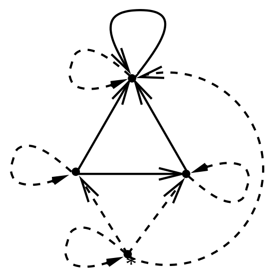
- $\varphi_V(*) = *'$,
- for all $k \in K$, $E'^i(\varphi_E(k)) = \varphi_V(E^i(k))$ for $i = 1, 2$, and
- $C(k) = C'(\varphi_E(k))$.

We shall now describe the covariant functor $D : \mathsf{LC} \rightsquigarrow \mathsf{DGCP}$. First, its action on the objects. For a (usual) digraph $\Gamma = (V, K, E)$, we define the coloured digraph with smaller point $\widetilde{\Gamma} = (\widetilde{V}, \widetilde{K}, \widetilde{E}, \widetilde{C}, *)$ by adding a vertex $*$ to Γ, edges from $*$ to v for each vertex v, loops at any vertex, and colouring each 'old' edge with 0 and 'new' edge with 1. Formally, we choose some

$* \notin V$, and set:

$$\begin{aligned}
\widetilde{V} &= V \cup \{*\},\\
\widetilde{K} &= K \times \{0\} \cup V \times \{1\} \cup \widetilde{V} \times \{2\},\\
\widetilde{C}(k,i) &= \begin{cases} 0 & \text{if } i = 0 \\ 1 & \text{otherwise,} \end{cases}\\
\widetilde{E}(k,i) &= \begin{cases} E(k) & \text{if } i = 0,\ k \in K \\ (*,k) & \text{if } i = 1,\ k \in V \\ (k,k) & \text{if } i = 2,\ k \in \widetilde{V}. \end{cases}
\end{aligned}$$

As often with graphs, a little picture says more than the above painstaking formulas. In the figure below, the edges of Γ are in plain lines, while the new edges of $\widetilde{\Gamma}$ (those coloured with 1) are in dashed lines.



Set $D(X_\Gamma) = \widetilde{\Gamma}$. Let us now define the action of D on the morphisms. Let $f : X_{\Gamma_1} \to X_{\Gamma_2}$, with $\Gamma_i = (V_i, K_i, E_i)$, $\widetilde{\Gamma_i} = (\widetilde{V_i}, \widetilde{K_i}, \widetilde{E_i}, \widetilde{C_i}, *_i)$. We define $D(f) : \Gamma_1 \to \widetilde{\Gamma_2}$ as follows. First, set $D(f)(*_1) = *_2$. Recall that the vertices and edges of Γ correspond respectively to the $\mathbb{L}_v$ and $\mathbb{O}_k$ ($v \in V, k \in K$) in X_Γ. For $v \in V_1$, we let $D(f)(v)$ be $*_2$ if $f\big|_{\mathbb{L}_v}$ is bounded, and if $f\big|_{\mathbb{L}_v}$ is unbounded, we let $D(f)(v)$ be the only $w \in V_2$ such that $f(\mathbb{L}_v) \cap \mathbb{L}_w$ is unbounded (we use Corollary 5.3).

We now define $D(f) : \widetilde{K_1} \to \widetilde{K_2}$. Let $k_1 \in \widetilde{K_1}$ with $\widetilde{E_1}(k_1) = (v,w) \in \widetilde{V_1}$. If $k_1 \in K_1 \subset \widetilde{K_1}$ and there is a $k_2 \in K_2 \subset \widetilde{K_2}$ such that $\mathbb{O}_{k_1}$ covers $\mathbb{O}_{k_2} \subset X_{\Gamma_2}$ under f then k_2 is unique by Lemma 3.5, so we may set $D(f)(k_1) = k_2$. Otherwise, define $D(f)(k_1)$ to be the unique $k_2 \in \widetilde{K_2}$ with $C_2(k_2) = 1$ (k_2 is thus a 'new' edge) such that $\widetilde{E_2}(k_2) = (D(f)(v), D(f)(w))$.

Lemma 5.4. *$D(f)$ is a* DGCP*-morphism, and $D : \mathsf{LC} \rightsquigarrow \mathsf{DGCP}$ is a covariant functor.*

Proof. This follows from the definition. □

Lemma 5.5. *Given any* DGCP*-morphism $\pi : \widetilde{\Gamma_1} \to \widetilde{\Gamma_2}$, there is a (canonical) $\widehat{\pi} : X_{\Gamma_1} \to X_{\Gamma_2}$ with $D(\widehat{\pi}) = \pi$.*

Proof. Let $\Gamma_i = (V_i, K_i, E_i)$, $\widetilde{\Gamma_i} = (\widetilde{V_i}, \widetilde{K_i}, \widetilde{E_i}, \widetilde{C_i}, *_i)$ $i = 1,2$. For $k_1 \in K_1$, we define $\widehat{\pi}\big|_{\mathbb{O}_{k_1}}$ for $k_1 \in K_1$ as follows. If $\pi(k_1) = k_2$ and $\widetilde{C_2}(k_2) = 0$ (thus k_2 is an edge in $\Gamma_2 \subset \widetilde{\Gamma_2}$), $\widehat{\pi}$ sends $\mathbb{O}_{k_1}$ to $\mathbb{O}_{k_2} \subset X_{\Gamma_2}$ by the identity map. If $\pi(k_1) = k_2$, $\widetilde{C_2}(k_2) = 1$ and $\widetilde{E_2}(k_2) = (*_2, v)$ with $v \neq *_2$, we let $\widehat{\pi}\big|_{\mathbb{O}_{k_1}}$ be the map $\mathbb{O}_{k_1} \approx \mathbb{O} \to \mathbb{L}_+ \approx \mathbb{L}_v$ given by $(x,y) \mapsto \min\{x,y\}$. If $\pi(k_1) = k_2$, $\widetilde{C_2}(k_2) = 1$ and $\widetilde{E_2}(k_2) = (v,v)$, we let $\widehat{\pi}\big|_{\mathbb{O}_{k_1}}$ be the map $\mathbb{O}_{k_1} \approx \mathbb{O} \to \mathbb{L}_+ \approx \mathbb{L}_v$ given by $(x,y) \mapsto \max\{x,y\}$. Then, $\widehat{\pi}$ is continuous and $D(\widehat{\pi}) = \pi$. □

Idea of the proof of Theorem 5.1. By the above, it suffices to show that $f : X_{\Gamma_1} \to X_{\Gamma_2}$ is homotopic to $\widehat{D(f)}$. First, we construct a global homotopy such that $f(\mathbb{L}_v) = \mathbb{L}_{D(f)(v)}$ if $D(f)(v) \neq *_2$ and $f(\mathbb{L}_v)$ is constant on 0 if $D(f)(v) = *_2$. We then investigate the action of f on each $\mathbb{O}_k$ for $k \in K_1$. If f is constant on $\mathbb{O}_k$, there is nothing to do. If $\mathbb{O}_k$ does not cover any $\mathbb{O}_{k'}$

under f, after a homotopy that leaves all $\mathbb{L}_v$ untouched we may assume that $f(\mathbb{O}) \subset \mathbb{L}_v$. Then, as shown in [4] (in which the case of maps $\mathbb{L}_+^n \to \mathbb{L}_+$ is investigated), there are two possibilities: either $f\big|_{\mathbb{O}_k(0)}$ and $f\big|_{\mathbb{O}_k(1)}$ are both unbounded, or only $f\big|_{\mathbb{O}_k(1)}$ is unbounded. These two cases yield two different partitions of $\mathbb{O}_k$ which are described in [4, Section 5] (they are similar to those of our Section 2).

If $\mathbb{O}_k$ covers some $\mathbb{O}_{k'}$ under f, after a homotopy again leaving each $\mathbb{L}_v \subset X_{\Gamma_1}$ untouched, we may assume that $f(\mathbb{O}_k) = \mathbb{O}_{k'}$. Then the partition properties of Section 2 apply (the proof which was for homeomorphisms can easily be adapted in these settings). Using Lemma 1.6, it is possible to 'index' the partitions of each $\mathbb{O}_k$ with the same club set $S \subset \omega_1$, and we finish the proof by applying homotopy techniques similar to those of [4, Section 6] and in our Section 2. □

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