Final Report to the Marsden Fund Committee
Manifolds Near the Limit of Metrisability
UOA 611

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1 Introduction

The goals of the research programme were to investigate conditions equivalent to metrisability of topological manifolds and to investigate the topology of non-metrisable manifolds. We planned to extend the range of investigations already started, including the relationship between microbundles and non-metrisable manifolds, a study of resolutions which may lead to applications to the general problem of metrisability of manifolds, and a study of the group of homeomorphisms of non-metrisable manifolds to include applications of algebraic topology.

2 The Research Objectives

Two particular approaches were taken in attempting to solve problems: examples were constructed both to get insight into the problem and in some cases to answer the problem; and proofs were constructed. As indicated below, there are several lines of attack and these are described in more detail.

2.1 Criteria for the metrisability of manifolds

Before the research began there were many known criteria for metrisability of a manifold. This research has uncovered many new criteria. In various publications arising from this research it is shown that a topological manifold M is metrisable if and only if any one of the following equivalent conditions holds:

1. there is an open cover $\mathcal{U}$ of M such that for each $x \in M$ the set $st(x,\mathcal{U})$ is homeomorphic to an open subset of $\mathbb{R}^m$;

2. there is a point-star-open cover $\mathcal{U}$ of M such that for each $x \in M$ the set $st(x,\mathcal{U})$ is Lindelöf;

3. there is a point-star-open cover $\mathcal{U}$ of M such that for each $x \in M$ the set $st(x,\mathcal{U})$ is metrisable;

4. the tangent microbundle to M is equivalent to a fibre bundle;

5. M is perfectly normal and there is a sequence $\langle \mathcal{U}_n \rangle_{n\in\omega}$ of families of open sets such that $\cap_{n\in C(x)} \overline{st(x,\mathcal{U}_n)} = \{x\}$ for each $x \in M$, where
$$C(x) = \{n \in \omega \ / \ \exists U \in \mathcal{U}_n \text{ with } x \in U\};$$

6. M is separable and there is a sequence $\langle \mathcal{C}_n \rangle_{n\in\omega}$ of point-star-open covers such that $\cap_n \overline{st(x,\mathcal{C}_n)} = \{x\}$ for each $x \in M$ and for each $x, y \in M$ and each $n \in \omega$ we have $y \in \overline{st(x,\mathcal{C}_n)}$ if and only if $x \in \overline{st(y,\mathcal{C}_n)}$;

7. M is separable and there is a sequence $\langle \mathcal{C}_n \rangle_{n\in\omega}$ of point-star-open covers such that $\cap_n \overline{st(x,\mathcal{C}_n)} = \{x\}$ for each $x \in M$ and for each $x \in M$ and each $n \in \omega$, $\mathrm{ord}(x,\mathcal{C}_n)$ is finite;

8. M is separable and hereditarily normal and there is a sequence $\langle \mathcal{C}_n \rangle_{n\in\omega}$ of point-star-open covers such that $\cap_n \overline{st(x,\mathcal{C}_n)} = \{x\}$ for each $x \in M$;

9. M is separable and there is a sequence $\langle \mathcal{U}_n \rangle_{n\in\omega}$ of families of open sets such that $\cap_{n\in C(x)} \overline{st(x,\mathcal{U}_n)} = \{x\}$ for each $x \in M$, and each $n \in \omega$, $\mathrm{ord}(x,\mathcal{C}_n)$ is countable;

10. M^2 has a countable sequence $\langle U_n : n \in \mathbb{N} \rangle$ of open subsets in M^2, such that for all $(x, y) \notin \Delta$, there is $n \in \mathbb{N}$ such that $(x, x) \in U_n$ but $(x, y) \notin \overline{U_n}$;

11. For every subset $A \subset M$ there is a 1-1 mapping $f : M \to Y$, where Y is a metrisable space, such that $f(A) \cap f(M - A) = \varnothing$;

12. For every subset $A \subset M$ there is a mapping $f : M \to Y$, where Y is a space with a quasi–regular-G_δ-diagonal, such that $f(A) \cap f(M - A) = \varnothing$;

13. M is weakly normal with a G_δ^*-diagonal;

14. M has a quasi-G_δ^*-diagonal and for every closed subset $A \subset M$ there is a countable family $\mathcal{G}$ of open subsets such that, for every $x \in A$ and $y \in X - A$, there is a $G \in \mathcal{G}$ with $x \in G, y \notin \overline{G}$;

15. M is nearly metaLindelöf;

16. M is ω_1-Lindelöf;

17. M is ω_1-metaLindelöf;

18. M is nearly ω_1-metaLindelöf;

19. M is finitistic;

20. M is strongly finitistic;

21. M is star finitistic;

22. M has property $\omega_1$pp;

23. every open cover of M of cardinality ω_1 has an open refinement $\mathcal{V}$ such that for every choice function $f : \mathcal{V} \to M$ the set $f(\mathcal{V})$ is closed in M;

24. every open cover of M of cardinality ω_1 has an open refinement $\mathcal{V}$ such that for every choice function $f : \mathcal{V} \to M$ the set $f(\mathcal{V})$ is discrete in M;

25. M is a point-countable union of open subspaces each of which is metrisable.

In addition in Set Theories satisfying $(MA(\omega_1))$ it has been shown that a manifold is metrisable if and only if it is perfect and weakly normal.

Publications relevant to these criteria include [3], [7], [9], [10], [11], [12], [13], [15], [16], [18], [19], [25].

As indicated in the application such properties as quasi-semi-developability, strong quasi-developability, quasi-WΔ-spaces have been addressed in this context by Dr Mohamad, [20]. Variants of paracompactness were also addressed, especially conditions which are significantly weaker than paracompactness for general spaces but assume the same status as paracompactness for a manifold: the latest results in this direction are in the paper [15], work which actually followed the expiry of the contract. Another class of variants discussed here is the class involving metrically significant properties of the diagonal, again by Dr Mohamad, [21], [22]. As a spin-off from work on microbundles (see section 2.3 below) we also have the criteria contained in Dr Greenwood's thesis, see [16] and [13]. An eminent algebraic topologist from India, Professor Satya Deo, visited Auckland for several months supported in part by the grant and together we found some new characterisations of metrisability along the lines of bounds on the number of sets from a cover containing a particular point: this work is described in [3].

2.2 Construction of Non-metrisable Manifolds

In her thesis, [16], Dr Greenwood established new ways to construct non-metrisable Type I manifolds by use of what is known as the Υ-tree. This enabled her to determine the number of pairwise non-homeomorphic Type I manifolds having the same Υ-tree. She was able to show the existence of an uncountable branch in the tree of a Type I manifold containing a copy of ω_1, partially solved the problem of associating an ω_1-compact manifold to an arbitrary tree, and gave constructions of new classes of Type II manifolds.

In his thesis [18] and in the joint paper [8] Dr Mohamad has shown how to construct some exciting new manifolds. In some cases these manifolds are constructible in all models of Set Theory but in others forcing has been necessary to construct the manifolds so as a result the construction is valid only in some models Set Theories.

2.3 Resolutions and metrisability

Most of the work undertaken here was by Kerry Richardson and Brian Van Dam. As a result Mr Richardson is in the final stages of writing up his PhD thesis.

Mr Richardson has found necessary and sufficient conditions for a space to have a metrisable resolution and an example of a non-normal space which is the resolution of metrisable spaces. He has also generalised the notion of a resolution and with it he has been able to handle connected spaces, which special resolutions could not.

One major gap in the theory of resolutions is the sporadic and incomplete understanding of the technique. Early work by Mr van Dam on resolutions concentrated on filling this gap and providing useful 'primary' results for the construction of interesting examples and for use in 'secondary' results. In particular he has been working on the problem of characterising those topological properties that are preserved under resolutions, [30]. Another problem which he has been addressing is that of establishing conditions under which resolutions are manifolds.

2.4 Bundles

In this context the big result discovered as part of this project is that the tangent microbundle over a manifold is equivalent to a fibre bundle if and only if the manifold is metrisable, [13]. Certainly this was not our original expectation. As a result of this discovery a number of new characterisations of when a manifold is metrisable were discovered, see also [16]. On the other hand, because of the equivalence of the two conditions this work did not readily lead to the discovery of new non-metrisable manifolds. Some time has also been spent on determining the total number of microbundles, [14].

2.5 Homeomorphisms of non-metrisable manifolds

Progress was made under this heading in two directions. During a visit to Oxford in 1997, Dr Mohamad worked on one aspect of the homeomorphism group of a manifold. In [4] the authors show that if the homeomorphism groups of two manifolds are isomorphic then there is a homeomorphism between the manifolds which realises a given isomorphism.

In 1998, a visitor from India, Professor Satya Deo of RD University, Jabalpur, came to Auckland supported in part by the grant. It was particularly rewarding to have the opportunity to combine his expertise in Algebraic Topology with Professor Gauld's knowledge in Set Theoretic Topology and Geometric Topology: there are very few applications which combine both Algebraic and Set Theoretic Topology. The researchers were able to identify the torsion in the group of homeomorphisms of low powers of the long line, finding in [2] that the only torsion is that dictated by the very compact geometry of the situation.

3 Additional Related Work Done

The proof of the consistency of the Normal Moore space conjecture when $\mathfrak{c} = \omega_2$ involves coming up with a partial order which has the right cardinal arithmetic when iterated with the right combinatorics, namely an endowment or a lynx. Mr Richardson has shown that there can be no "general scheme" for such a partial order, because in general even when "nicely" endowed posets are iterated the iteration itself may not be endowed. The same can be said of Linked posets. So if there is a forcing to do the job it must be exhibited directly. This work is in progress with a focus on the Sacks Poset - a poset of subtrees of the binary tree of countable height.

Dr Gartside and Dr Mohamad studied the group topology on the unit interval and on continua, showing in particular under a set theoretic hypothesis that there are a continuum K and a compact, connected, locally connected, non-metrisable space L such that the groups of homeomorphisms of K and L are isomorphic but not homeomorphic. While this space L cannot be a manifold, it does raise the interesting challenge of establishing a suitable example in the context of manifolds. Some of this work has been extended to the context of Peano continua, [5].

During a visit to Galway (Ireland) Dr Mohamad considered the class of p–adic analytic manifolds. In [6] the authors show that the wide variety of non–metrisable p–adic analytic manifolds contrasts with the scarcity of metrisable p–adic analytic manifolds. The topology of p–adic analytic manifolds is compared with that of real (analytic) manifolds.

During a recent visit to Toronto, Mr Van Dam obtained characterisations of normality and hereditary normality for resolutions, [30]. While these properties may seem to be part of the series of general results of the form when is a particular property preserved under resolutions which he has been obtaining, they are rather closely related to the question of metrisability as they involve important preconditions for metrisability.

4 Future Directions

Thanks to the Marsden Fund grant, over the past three years a strong research group in Set Theoretic Topology has been built up at the University of Auckland. Part of the strength of this group is the links which have been established with many other Set Theoretic Topologists in other parts of the world, especially Britain, Canada and the United States of America. These links will enable us to tackle further and deeper problems such as the following.

1. Is every hereditarily normal, locally compact, locally connected space collectionwise Hausdorff? We will investigate variants of this problem obtained by weakening some of the hypotheses and strengthening others: for example hereditary normality may be weakened to just normality while local compactness and local connectedness may be strengthened to being a manifold.

2. Is it consistent with the axioms of Set Theory that every normal tree or every collectionwise Hausdorff tree is monotone normal? Many variants of this problem will be explored, including questions which may even have positive answers in all Set Theories.

3. One of our overseas colleagues has given an example of a tree with no uncountable chains, which is not quasi-metrisable: the proof requires the extension of a model of Set Theory by forcing but the result applies in any Set Theory. This technique will be applied to other properties of trees and the tree itself may be the key to finding in all Set Theories a countably metacompact tree with no uncountable chains that is not special.

4. Drawing to some extent on work of the members of the group in Auckland, the same overseas colleague has proved under certain set theoretic assumptions that every hereditarily normal, collectionwise Hausdorff manifold of dimension greater than 1 is metrisable. The need for the assumptions involving large cardinal numbers will be investigated.

5. Extending Marsden-funded work started by Professors Satya Deo and Gauld, we will try to determine the torsion of homeomorphisms of higher powers of the long line and then attempt to extend the ideas to other manifolds. It is unclear how far this exciting possibility of combining Algebraic and Set Theoretic Topology can be pushed.

6. Exploring whether techniques developed here can be carried over to Banach spaces, for example exploring whether a locally completely metrisable space can be given a Banach analytic structure.

7. Seeking characterisations of metrisability and manifolds in resolutions, and studying the behaviour of resolutions under algebraic and set theoretic conditions.

5 Participants

The following members participated in the project: Professor David Gauld (principal investigator), Sina Greenwood (postgraduate student), Abdul Mohamad (postgraduate student), Kerry Richardson (postgraduate student), Brian Van Dam (postgraduate student), Professor Satya Deo (visitor), Dr Paul Gartside (visitor), Professor Peter Nyikos (visitor) and Professor Richard Wilson (visitor).

6 Financial Review

All the money allocated to the project was spent as in the budget submitted. Approximately half of the money allocated to the project after GST was for graduate student support. Most of this money was divided between Abdul Mohamad and Kerry Richardson, both PhD students at the University of Auckland. The flexibility allowed by this award enabled both of these students to travel (also supported by the award) overseas for extended periods to work with overseas experts in the area. The small amount of the remaining postgraduate student money was allocated to Dr Greenwood when she was a PhD student.

The other half of the money was spent in part to enable those three named above, together with the fourth PhD student, Brian Van Dam, and Professor Gauld to attend overseas conferences. The remainder was spent to bring in overseas experts for visits to Auckland.

References

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[3] Satya Deo and David Gauld, *Boundedly Metacompact or Finitistic Spaces and the Star Order of Covers*, The University of Auckland Dept of Mathematics Report 412, (submitted).

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[7] P.M. Gartside and A.M. Mohamad, *Cleavability of Manifolds*, Topology Proceedings (to appear).

[8] P.M. Gartside, C. Good, R.W. Knight and A.M. Mohamad, *Quasi-Developable Manifolds*, Topology and its Applications (to appear).

[9] David Gauld, *Metrisability of Manifolds*, preliminary version at http://matu1.math.auckland.ac.nz/∼gauld/research/metrisability.ps.

[10] David Gauld, *Manifolds At and Beyond the Limit of Metrisability*, Proceedings of the Second Galway Colloquium, Paul Gartside ed., Topology Atlas, http://at.yorku.ca/p/p/a/d/19.htm (1998), 8pp.

[11] David Gauld, *Covering Properties and Metrisation of Manifolds*, Topology Proceedings, (to appear).

[12] David Gauld, *Manifolds at and beyond the limit of metrisability*, Geometry and Topology (to appear).

[13] David Gauld and Sina Greenwood, *Microbundles, Manifolds and Metrisability*, Proc. Amer. Math. Soc., (to appear).

[14] David Gauld and Sina Greenwood, *How many Microbundles are there?*, (to appear).

[15] David Gauld and M K Vamanamurthy, *Weak Covering Conditions Guaranteeing Metrisability for a Manifold* (in preparation).

[16] Sina Greenwood, *Nonmetrisable Manifolds*, PhD Thesis, The University of Auckland (1999), 92pp.

[17] Sina Greenwood, *Constructing Nonmetrisable Manifolds with given Υ-Trees*, (to appear)

[18] Abdul Mohamad, *Metrization and Manifolds*, PhD Thesis, The University of Auckland (1999), 134pp.

[19] A M Mohamad, *Metrization and Semimetrization Theorems with Applications to Manifolds*, Acta Math. Hungar, 83 (4) (1999), 383-394.

[20] A M Mohamad, *Developable Spaces and Problems of Fletcher and Lindgren*, Q and A Gen. Topology (to appear).

[21] A M Mohamad, *Conditions Which Imply Metrizability in Some Generalized Metric Spaces*, (to appear).

[22] A M Mohamad, *Some Results on Quasi-σ and θ–Spaces*, Houston J. of Math. (to appear).

[23] A M Mohamad, *A Result on $\aleph_1$-Compact Spaces*, Acta Math.Hungar. (to appear).

[24] A M Mohamad, *On Spaces with Quasi-Regular-G_δ–Diagonals*, Q and A Gen. Topology (to appear).

[25] A M Mohamad, *Metrizability of Manifolds by Diagonal Properties*, (to appear).

[26] A M Mohamad, *Characterization of Developable and Semi-stratifiable Spaces*, N Z J. Math. (to appear).

[27] A M Mohamad, *Weak Bases and Metrizability*, N Z J. Math. (to appear).

[28] Kerry Richardson and Stephen Watson, *Metrisable and Discrete Special Resolutions*, (to appear).

[29] Kerry Richardson and Stephen Watson, *The Many Faces of Some Hyperresolutions*, (to appear).

[30] Brian van Dam, *Special Resolutions by Multifunctions*, (to appear).

In addition members of the group attended the following conferences at which (in nearly all cases) they gave talks. The topics of these talks have been included in the papers listed above.

August 1997: David Gauld, Sina Greenwood, Abdul Mohamad and Kerry Richardson attended the 12th Summer Conference on Topology and its Applications at Nipissing University near Toronto, Canada.

June 1998: David Gauld and Kerry Richardson attended the 13th Summer Conference on Topology and its Applications at Universidad Nacional Autónoma de México, Mexico City, Mexico.

June-July 1998: Abdul Mohamad attended the Conference on Convergence and Topology in Erice, Italy.

August 1998: Sina Greenwood and Kerry Richardson attended the Colloquium on Topology in Gyula, Hungary.

September 1998: David Gauld, Sina Greenwood and Abdul Mohamad attended the 2nd Galway Topology Colloquium in Oxford, England.

August 1999: David Gauld, Sina Greenwood, Abdul Mohamad, Kerry Richardson and Brian Van Dam attended the 14th Summer Conference on Topology and its Applications at Long Island University, Long Island, New York, USA.

August 1999: Abdul Mohamad attended the International Conference on Topology and its Applications in Yokohama, Japan.

September 1999: Brian Van Dam attended the 3rd Galway Topology Colloquium in Belfast, Northern Ireland.

8 Summary

A manifold is a set which, to a sufficiently myopic observer, looks just like a fixed model such as a line, a plane or the space in which we live, but overall need not be so. For example prior to Columbus most Europeans believed that the earth was flat as locally it looks like a plane. The main study of manifolds has been in the context of those which are small enough to carry the notion of a distance: such manifolds are metrisable. The major aim of this project has been to look at very large manifolds, those which are so large that they are not metrisable. Closely associated with this study is the question of identifying when a manifold is metrisable.

In this project we have discovered a number of new criteria for metrisability of a manifold. We have refined some old tools and developed a range of new tools for the construction of non-metrisable manifolds. Criteria for metrisability include some very weak conditions which, nevertheless, are adequate to guarantee metrisability. Those non-metrisable manifolds which we have constructed require a range of techniques from Set Theory from the relatively simple to the very complex.

In this project we have also developed some novel combined applications of Algebraic Topology and Set Theoretic Topology. Implications for these have still to be worked out.

There is, of course, one regret. We now have a very active group of topologists working at the University of Auckland thanks to a large extent to the Marsden Award. Now that the award has expired and has not been renewed the group will disperse and the gains may well be lost.

David Gauld, 21/10/99