2 SEPARATION AXIOMS

Definition 2.1 *A space X is a* T_0 space *iff it satisfies the* T_0 axiom*, i.e. for each $x, y \in X$ such that $x \neq y$ there is an open set $U \subset X$ so that U contains one of x and y but not the other.*

Obviously the property T_0 is a topological property. An arbitrary product of T_0 spaces is T_0. Discrete spaces are T_0 but indiscrete spaces of more than one point are not T_0.

Definition 2.2 *A space X is a* T_1 space *or* Frechet space *iff it satisfies the* T_1 axiom*, i.e. for each $x, y \in X$ such that $x \neq y$ there is an open set $U \subset X$ so that $x \in U$ but $y \notin U$.*

T_1 is obviously a topological property and is product preserving. Every T_1 space is T_0.

Example 2.3 *The set $\{0,1\}$ furnished with the topology $\{\varnothing, \{0\}, \{0,1\}\}$ is called* Sierpinski space. *It is T_0 but not T_1.*

Proposition 2.4 *X is a T_1 space iff for each $x \in X$, the singleton set $\{x\}$ is closed.*

Proof. Easy. ∎

Definition 2.5 *A space X is a* T_2 space *or* Hausdorff space *iff it satisfies the* T_2 axiom*, i.e. for each $x, y \in X$ such that $x \neq y$ there are open sets $U, V \subset X$ so that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.*

T_2 is a product preserving topological property. Every T_2 space is T_1.

Example 2.6 *Recall the cofinite topology on a set X defined in Section 1, Exercise 3. If X is finite it is merely the discrete topology. In any case X is T_1, but if X is infinite then the cofinite topology is not T_2.*

Proposition 2.7 *Let $f, g : X \to Y$ be maps with Y Hausdorff. Then $\{x \in X \ / \ f(x) = g(x)\}$ is closed.*

Proof. Let $A = \{x \in X \ / \ f(x) \neq g(x)\}$, and suppose $x \in A$. Since $f(x) \neq g(x)$, there are open sets $U, V \subset Y$ so that $f(x) \in U$, $g(x) \in V$ and $U \cap V = \varnothing$. Let $W = f^{-1}(U) \cap g^{-1}(V)$. Then W is open and $x \in W$. Moreover, $W \subset A$. Thus A is open, so $\{x \in X \ / \ f(x) = g(x)\}$ is closed. ∎

In particular if X is T_2 and $f : X \to X$ is a map then the fixed-point set of f [i.e. the set of points x for which $f(x) = x$] is closed.

Definition 2.8 *A space X is* regular *iff for each $x \in X$ and each closed $C \subset X$ such that $x \notin C$ there are open sets $U, V \subset X$ so that $x \in U$, $C \subset V$ and $U \cap V = \varnothing$. A regular T_1 space is called a* T_3 space.

The properties T_3 and regular are both topological and product preserving. Every T_3 space is T_2.

Example 2.9 *The* slit disc topology *on $\mathbb{R}^2$ is T_2 but not regular, hence not T_3.*

Take $X = \mathbb{R}^2$ and let $P = \{(x, y) \in \mathbb{R}^2 \ / \ x \neq 0\}$ and $L = \{(0, y) \in \mathbb{R}^2\}$. Topologise X as follows (c.f. Theorem 1.12):

- if $z \in P$, let the open discs in $\mathbb{R}^2$ centred at z form a basis of neighbourhoods of z;

- if $z \in L$ let the sets of the form $\{z\} \cup (P \cap D)$ form a basis of neighbourhoods of z where D is an open disc in the plane centred at z.

Clearly X is T_2. However, note that L is a closed subset of X and that as a subspace, L is discrete. Thus any subset of L is closed in X; in particular, $L - \{(0,0)\}$ is closed and does not contain $(0,0)$, although every open set containing $(0,0)$ meets every open set containing $L - \{(0,0)\}$. Thus X is not regular and hence not T_3.

Example 2.10 *Every indiscrete space is vacuously regular but no such space (of more than 1 point!) is T_0 and hence also no such space is T_2.*

Theorem 2.11 *A space X is regular iff for each $x \in X$, the closed neighbourhoods of x form a basis of neighbourhoods of x.*

Proof. $\Rightarrow$: given $x \in X$, and a neighbourhood N of x, there is an open set $O \subset X$ such that $x \in O \subset N$. Consider the point x and the closed set $X - O$, which does not contain x. By regularity, there are open sets U and V such that $x \in U$, $X - O \subset V$ and $U \cap V = \varnothing$. Thus $x \in U \subset X - V \subset O \subset N$, so $X - V$ is a closed neighbourhood of X contained in the given neighbourhood N of x.

$\Leftarrow$: given $x \in X$ and the closed set $C \subset X - \{x\}$, since $X - C$ is open and contains x, there is a closed neighbourhood N of x so that $N \subset X - C$. Let $V = X - N$. Then V is open and $C \subset V$. Since N is a neighbourhood of x, there is an open set U such that $x \in U \subset N$. Then $U \cap V \subset N \cap (X - N) = \varnothing$, so X is regular. ∎

Definition 2.12 *A space X is* normal *iff for each pair A, B of disjoint closed subsets of X, there is a pair U, V of disjoint open subsets of X so that $A \subset U$, $B \subset V$ and $U \cap V = \varnothing$. A normal T_1 space is called a* T_4 space.

The properties T_4 and normal are both topological properties but, perhaps surprisingly, are not product preserving. Every T_4 space is clearly a T_3 space, but it should not be surprising that normal spaces need not be regular.

Example 2.13 *Sierpinski's space is vacuously normal but is not regular since 0 and $\{1\}$ cannot be separated.*

Example 2.14 *The* tangent-discs topology *on $\mathbb{R}^2$ is T_3 but not normal, hence not T_4.*

Let X, P and L be as in Example 2.9 but topologise X as follows:

- if $z \in P$, let the open discs in $\mathbb{R}^2$ centred at z form a basis of neighbourhoods of z;
- if $z \in L$ let the sets of the form $\{z\} \cup D$ form a basis of neighbourhoods of z where D is the union of two open discs in P tangent to L at z, one to the left of L and the other to the right of L.

Clearly X is T_1.

X is regular, hence T_3. Indeed, suppose $z \in X$ and $N \in \mathcal{N}(z)$. If $z \in P$, then there is an open disc D centred at z so that $z \in D \subset N$. The closed disc of half the radius also centred at z is a closed neighbourhood of z and lies in N. If $z \in L$, then there is D as in the definition of the neighbourhood basis of z so that $\{z\} \cup D \subset N$. The union of the closed discs of half the radii of those of D also tangent to L at z is a closed neighbourhood of z contained in N. Thus in either case, closed neighbourhoods form a neighbourhood basis so by Theorem 2.11, X is regular.

Suppose that X is normal. Let $M = \{(x, y) \ / \ x, y \in \mathbb{Q}\}$. As in Example 2.9, every subset of L is a closed subset of X. Thus to each $A \subset L$, we can assign open sets U_A and V_A of X so that $A \subset U_A$, $L - A \subset V_A$ and $U_A \cap V_A = \varnothing$. Let $M_A = M \cap U_A$. We show that the function sending A to M_A is injective. This will lead to a contradiction because L is uncountable whereas M is countable, so it is impossible to find an injective function from the power set of L to that of M. Loosely speaking, if X were normal then there would need to be more open subsets of X than there are.

To show that the function constructed in the last paragraph really is injective, let $A, B \subset L$ be such that $A \neq B$: to show that $M_A \neq M_B$. Either $A - B \neq \varnothing$ or $B - A \neq \varnothing$: assume the former. We have $\varnothing \neq A - B = A \cap (L - B) \subset U_A \cap V_B$. Since U_B and V_B are disjoint and open, we must have $\overline{U_B} \cap V_B = \varnothing$, so $V_B \subset X - \overline{U_B}$, and hence $\varnothing \neq U_A \cap V_B \subset U_A - \overline{U_B}$. Every non-empty open subset of X meets M so, since $U_A - \overline{U_B}$ is open, we have: $\varnothing \neq M \cap (U_A - \overline{U_B}) \subset M \cap (U_A - U_B) = M_A - M_B$. Thus if $A - B \neq \varnothing$ then $M_A - M_B \neq \varnothing$. Similarly, if $B - A \neq \varnothing$ then $M_B - M_A \neq \varnothing$, so the function is injective as promised. ∎

Example 2.15 *Let S be the real line with the right half-open interval topology of Example 1.4. Then S is normal but $S \times S$ is not normal. The space $S \times S$ is sometimes called* Sorgenfrey's square*.*

Suppose that A and B are disjoint subsets of S. Since B is closed, for each $a \in A$, there is $a' > a$ so that $[a, a') \cap B = \varnothing$. Let $U = \cup_{a \in A}[a, a')$. Then U is an open set containing A. Similarly define $V = \cup_{b \in B}[b, b')$, where for each $b \in B$, $b' > b$ is chosen so that $[b, b') \cap A = \varnothing$. V too is open and contains B. It is claimed that $U \cap V = \varnothing$, for suppose not. Then there are $a \in A$, $b \in B$ such that $[a, a') \cap [b, b') \neq \varnothing$. Thus either $b \in [a, a')$ or $a \in [b, b')$, so either $[a, a') \cap B \neq \varnothing$ or $[b, b') \cap A \neq \varnothing$, contrary to the choice of a' and b'. Hence $U \cap V = \varnothing$, so S is normal.

To show that Sorgenfrey's square is not normal, let $X = S \times S$ and set $L = \{(x, y) \in X \ / \ x + y = 0\}$ and $M = \{(x, y) \ / \ x, y \in \mathbb{Q}\}$.

Without a single change in notation, we can apply the proof of non-normality of the tangent-discs topology in Example 2.14 to the present X, L and M to show that Sorgenfrey's square is not normal, hence not T_4. ∎

Although the definition of normality does not seem to be very different from the previous definitions, we have already seen in Example 2.15 that there is a rather significant difference between this property and the previous separation properties. There are other important distinctions too in that normality is equivalent to some seemingly very different conditions, making the property very rich and important. We now consider some such properties. The first characterisation is not very different but the remaining ones, Urysohn's lemma, Tietze's extension theorem, the cover shrinking theorem and the partition of unity theorem are all so different from the definition (and even from each other!!) that this is the source of their importance in topology and also in other branches of mathematics.

Proposition 2.16 *The space X is normal iff for each $A, U \subset X$ with A closed, U open and $A \subset U$, there is an open set V such that $A \subset V \subset \bar{V} \subset U$.*

Proof. Easy. ∎

Definition 2.17 *Say that two subsets A and B of X are* completely separated *in X iff there is a continuous function $f : X \to [0, 1]$ with $f(A) = 0$ and $f(B) = 1$. $[0, 1]$ may be replaced by $[a, b]$ with $a < b$. Let X be a space. By a* Urysohn family *in X we mean a family $\{U_r \ / \ r \in D\}$ of open subsets of X satisfying:*

(i) $\bar{D} = \mathbb{R}$;

(ii) $\cup_{r\in D} U_r = X$ *and* $\cap_{r\in D} U_r = \varnothing$;

(iii) *if* $r < s$ *then* $\overline{U_r} \subset U_s$.

Lemma 2.18 *Let* $\mathcal{U} = \{U_r \ / \ r \in D\}$ *be a Urysohn family in the space* X *and define a function* $\varphi_{\mathcal{U}} : X \to \mathbb{R}$ *by* $\varphi_{\mathcal{U}}(x) = inf\{r \in D \ / \ x \in U_r\}$. *Then* $\varphi_{\mathcal{U}}$ *is continuous.*

Proof. Note that $\varphi_{\mathcal{U}}$ is well-defined because by (ii), for any x, $\{r \in D \ / \ x \in U_r\}$ is non-empty and bounded below.

Let $\varphi = \varphi_{\mathcal{U}}$ and let $(a, b) \subset \mathbb{R}$. By Theorem 1.19(iii) it is enough to show that $\varphi^{-1}((a, b))$ is open. Let $x \in \varphi^{-1}((a, b))$. Then $a < \varphi(x) < b$ so by (i), there are $r, s, t \in D$ so that $a < r < t < \varphi(x) < s < b$. Then $x \notin U_t$ so by (iii), $x \notin \overline{U_r}$; also $x \in U_s$. Thus $U_s - \overline{U_r}$ is an open neighbourhood of x: note that $U_s - \overline{U_r} \subset \varphi^{-1}(a, b)$, so φ is continuous. ∎

Lemma 2.19 *Let* A *and* B *be subsets of a space* X. *Then* A *and* B *are completely separated in* X *iff there is a Urysohn family* $\{U_r \ / \ r \in D\}$ *with* $A \subset U_0$ *and* $B \subset X - U_1$.

Proof. $\Rightarrow$: Let $f : X \to [0, 1]$ be continuous with $f(A) = 0$ and $f(B) = 1$. For each $r \in \mathbb{R}$ let $U_r = \{x \in X \ / \ f(x) < \frac{r+1}{2}\}$. Then $\{U_r \ / \ r \in \mathbb{R}\}$ is a Urysohn family of the desired form.

$\Leftarrow$: Let $\mathcal{U} = \{U_r \ / \ r \in D\}$ be a Urysohn family with $A \subset U_0$ and $B \subset X - U_1$, and let $\varphi = \varphi_{\mathcal{U}} : X \to \mathbb{R}$ be as in Lemma 2.18. Define $f : X \to [0, 1]$ by

$$f(x) = \begin{cases} 0 & : \text{ if } \varphi(x) \le 0 \\ \varphi(x) & : \text{ if } \varphi(x) \in [0, 1] \\ 1 & : \text{ if } \varphi(x) \ge 1. \end{cases}$$

By theorem 1.22 f is continuous. Furthermore, if $x \in A$ then for each $r > 0$, $x \in U_r$; so $\varphi(x) \le 0$ and hence $f(x) = 0$. On the other hand, if $x \in B$ then for each $r < 1$, $x \notin U_r$; so $\varphi(x) \ge 1$ and hence $f(x) = 1$. ∎

Theorem 2.20 (Urysohn's lemma) *A space* X *is normal iff any two disjoint closed subsets are completely separated in* X.

Proof. $\Leftarrow$: easy.

$\Rightarrow$: Let A and B be two disjoint closed subsets of X. We find a Urysohn family as in Lemma 2.19.

For each non-negative integer n, let

$$D_n = \{0, 1\} \cup \left\{ \frac{m}{2^n} \ / \ m \text{ is a positive integer and } m \le 2n \right\},$$

and set $D = (-\infty, 0) \cup (1, \infty) \cup [\cup_{n\ge 0} D_n]$. Then $\bar{D} = \mathbb{R}$.

For $r < 0$ set $U_r = \varnothing$ and for $r >1$ set $U_r = X$. We define U_r for $r \in D \cap [0, 1]$ inductively as follows: induction is on n with U_r defined for $r \in D_n$. By Proposition 2.16, there is an open set U_0 such that $A \subset U_0 \subset \overline{U_0} \subset X - B$ and an open set U_1 such that $\overline{U_0} \subset U_1 \subset \overline{U_1} \subset X - B$. Now suppose n is such that U_r has been defined for $r \in D_n$ so that if $r, s \in D_n$ are such that $r < s$, then $\overline{U_r} \subset U_s$. Let $t \in D_{n+1} - D_n$, and let $r = \max\{p \in D_n \ / \ p < t\}$ and $s = \min\{p \in D_n \ / \ p > t\}$. Then we have $\overline{U_r} \subset U_s$ so by Proposition 2.16, there is an open set U_t such that $\overline{U_r} \subset U_t \subset \overline{U_t} \subset U_s$. Thus the induction continues and hence we obtain a Urysohn family as needed by Lemma 2.19. ∎

Theorem 2.21 (Tietze's extension theorem) *A space X is normal iff every bounded continuous function $f : C \to \mathbb{R}$, where C is a closed subset of X, has a bounded continuous extension to X, i.e. there is a continuous function $\hat{f} : X \to \mathbb{R}$ so that for each $x \in C$, $\hat{f}(x) = f(x)$.*

Proof. $\Rightarrow$: Given f, let β be an upper bound for $|f(C)|$, and for each $n \in \mathbb{N}$ let $r_n = \beta \left(\frac{2}{3}\right)^{n-1}$. We construct a sequence $\langle g_n : X \to \mathbb{R}\rangle$ of continuous functions so that for each n, $|g_n(X)| \leq r_n$ and $|f - \sum_{i=1}^{n}(g_i|C)| \leq 2r_n$.

Define $g_1 : X \to \mathbb{R}$ to be the function sending X to 0. Now suppose $g_1, \ldots, g_{n-1}$ have been constructed. The function $f - \sum_{i=1}^{n-1}(g_i|C) : C \to [-2r_{n-1}, 2r_{n-1}] = [-3r_n, 3r_n]$ is continuous, so the sets $A = \{x \in C \ / \ f(x) - \sum_{i=1}^{n-1}(g_i(x)) \leq -r_n\}$ and $B = \{x \in C \ / \ f(x) - \sum_{i=1}^{n-1}(g_i(x)) \geq r_n\}$ are closed and disjoint. Hence there is a continuous function $g_n : X \to [-r_n, r_n] \subset \mathbb{R}$ with $g_n(A) = -r_n$ and $g_n(B) = r_n$.

Suppose that $x \in C$: we show that $|f(x) - \sum_{i=1}^{n}(g_i(x))| \leq 2r_n$. There are three possibilities: either $x \in A$ or $x \in B$ or $x \in C - (A \cup B)$. If $x \in A$ then $g_n(x) = -r_n$ and $-3r_n \leq f(x) - \sum_{i=1}^{n-1}(g_i(x)) \leq -r_n$, so $-2r_n \leq f(x) - \sum_{i=1}^{n}(g_i(x)) \leq 0$. Similarly if $x \in B$ then $0 \leq f(x) - \sum_{i=1}^{n}(g_i(x)) \leq 2r_n$. If $x \in C - (A \cup B)$ then both of $f(x) - \sum_{i=1}^{n-1}(g_i(x))$ and $g_n(x)$ lie between $-r_n$ and r_n so that they are within $2r_n$ of each other, i.e. $-2r_n \leq f(x) - \sum_{i=1}^{n}(g_i(x)) \leq 2r_n$. Thus the induction proceeds.

We now have a sequence $\langle g_n\rangle$, with its corresponding series $\sum_{n=1}^{\infty} g_n$. This series is uniformly convergent since $\sum_{n=1}^{\infty} r_n = \beta \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^{n-1} = 3\beta$, and so, setting $\hat{f} = \sum_{n=1}^{\infty} g_n$, it is seen that $\hat{f}$ is continuous and for each $x \in X$, $|\hat{f}(x)| \leq 3\beta$. If $x \in C$ then for each n, $|f(x) - \sum_{i=1}^{n} g_i| \leq 2r_n$, so, letting $n \to \infty$, we obtain $|f(x) - \hat{f}(x)| \leq 2.\lim_{n\to\infty} r_n = 0$, i.e., $\hat{f}(x) = f(x)$.

$\Leftarrow$: Suppose that A and B are disjoint closed subsets of X. Define $f : A \cup B \to [0, 1]$ by $f(A) = 0$ and $f(B) = 1$. By Theorem 1.22(ii), f is continuous so has an extension $\hat{f}$. Then $\hat{f}^{-1}((-\infty, \frac{1}{2}))$ and $\hat{f}^{-1}((\frac{1}{2}, \infty))$ are disjoint open sets containing A and B. Thus X is normal. ■

Corollary 2.22 *There is a continuous surjection* $[0, 1] \to [0, 1] \times [0, 1]$.

Proof. Let C be Cantor's ternary set as defined in Exercise 12 of Section 1. Define $f : C \to [0, 1] \times [0, 1]$ as follows. Let $x \in C$. Then there is a sequence $\langle a_n\rangle$ of 0's and 2's so that $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$. If we set $f(x) = (\sum_{n=1}^{\infty} \frac{a_{2n-1}}{2^{n+1}}, \sum_{n=1}^{\infty} \frac{a_{2n}}{2^{n+1}})$, then f is continuous and surjective. By Tietze's extension theorem, f extends to a continuous function $[0, 1] \to [0, 1] \times [0, 1]$ which must be a surjection. ■

Definition 2.23 *Let X be a space. A* cover *of X is a family of subsets whose union is all of X. An* (open) [closed] cover *is a cover in which all subsets are (open) [closed]. A family of subsets is* (point-finite) [locally finite] *iff each point of X (lies in) [has a neighbourhood which meets] only finitely many members of the family. By a* shrinkage *of a cover $\{U_\alpha \ / \ \alpha \in A\}$ of the space X is meant another cover $\{V_\alpha \ / \ \alpha \in A\}$ so that for each $\alpha \in A$, $\overline{V_\alpha} \subset U_\alpha$.*

Theorem 2.24 *Let X be a space. Then the following three conditions are equivalent:*

(i) X is normal;

(ii) every point-finite open cover of X has an open shrinkage;

(iii) every locally finite open cover of X has an open shrinkage.

Proof. (i)$\Rightarrow$(ii): let $\{U_\alpha \ / \ \alpha \in A\}$ be a point-finite open cover of X, and let $\mathcal{S}$ be the set of all functions $V : B \to \mathcal{T}$, where $\mathcal{T}$ is the topology of X, satisfying:

(a) $B \subset A$;

(b) for each $\alpha \in B$, $\overline{V(\alpha)} \subset U_\alpha$;

(c) $X = (\cup_{\alpha \in B} V(\alpha)) \cup (\cup_{\alpha \in A-B} U_\alpha)$.

Partially order $\mathcal{S}$ by extension. The trivial function ($B = \varnothing$) belongs to $\mathcal{S}$, so $\mathcal{S} \neq \varnothing$.

Suppose $\mathcal{R} = \{V_\lambda : B_\lambda \to \mathcal{T} \ / \ \lambda \in \Lambda\}$ is a totally ordered set of elements of $\mathcal{S}$. Then $\mathcal{R}$ has an upper bound in $\mathcal{S}$, for let $B = \cup_\lambda B_\lambda$ and define $V : B \to \mathcal{T}$ by $V(\alpha) = V_\lambda(\alpha)$ when $\alpha \in B_\lambda$. The total ordering of $\mathcal{R}$ implies well-definition of V. Clearly V satisfies (a) and (b). Suppose $x \in X - (\cup_{\alpha \in A-B} U_\alpha)$. Point-finiteness of $\{U_\alpha\}$ implies that the set $\{\alpha \in A \ / \ x \in U_\alpha\}$, which is a subset of B, is finite so there are finitely many elements λ_i of Λ such that $\{\alpha \in A \ / \ x \in U_\alpha\} \subset \cup_{i=1}^n B_{\lambda_i}$. Thus since $\{B_\lambda \ / \ \lambda \in \Lambda\}$ is totally ordered by inclusion, there is $\lambda \in \Lambda$ so that if $x \in U_\alpha$ then $\alpha \in B_\lambda$. Hence $x \notin \cup_{\alpha \in A-B_\lambda} U_\alpha$, so by (c) for the function V_λ, $x \in \cup_{\alpha \in B_\lambda} V_\lambda(\alpha) \subset \cup_{\alpha \in B} V(\alpha)$. Thus (c) is also satisfied by the function V, which, therefore, is an upper bound for $\mathcal{R}$.

Applying Zorn's lemma, $\mathcal{S}$ has a maximal element, call it $V : B \to \mathcal{T}$. It suffices to show that $B = A$, for then conditions (b) and (c) tell us that the family $\{V(\alpha) \ / \ \alpha \in A\}$ is a shrinkage of $\{U_\alpha \ / \ \alpha \in A\}$.

Suppose $B \neq A$, say $\beta \in A - B$. By (c), $X - [(\cup_{\alpha \in B} V(\alpha)) \cup (\cup_{\alpha \in A-B-\{\beta\}} U_\alpha)] \subset U_\beta$, so by normality and Proposition 2.16, there is an open set $V(\beta)$ so that

$$X - [(\cup_{\alpha \in B} V(\alpha)) \cup (\cup_{\alpha \in A-B-\{\beta\}} U_\alpha)] \subset V(\beta) \subset \overline{V(\beta)} \subset U_\beta.$$

This extends V to a function from $B \cup \{\beta\} \to \mathcal{T}$ which is also an element of $\mathcal{S}$, contradicting the maximality of V. Thus $B = A$ and the proof is complete.

(ii)⇒(iii): trivial, because every locally finite family is point-finite.

(iii)⇒(i): Suppose that A and B are disjoint closed subsets of X. Then $\{X - A, X - B\}$ is a locally finite (!) open cover of X. Let $\{U, V\}$ be an open shrinkage. Since $\bar{U} \subset X - A$, $X - \bar{U}$ is an open set containing A while $X - \bar{V}$ is an open set containing B. Further, $X = U \cup V = \bar{U} \cup \bar{V}$, so $(X - \bar{U}) \cap (X - \bar{V}) = \varnothing$. ∎

Definition 2.25 *Let X be a space. For any continuous function $f : X \to \mathbb{R}$, the* support *of f is the closure of the set $\{x \in X \ / \ f(x) \neq 0\}$. A family of maps $\{\kappa_\alpha : X \to [0,1] \ / \ \alpha \in A\}$ is a* partition of unity *on X iff*

(i) the supports of the maps κ_α form a locally finite cover of X;

(ii) $\sum_{\alpha \in A} \kappa_\alpha(x) = 1$, for each $x \in X$.

Note that the sum in (ii) is really only finite for each x. Let $\{U_\alpha \ / \ \alpha \in A\}$ be an open cover of X. A partition of unity $\{\kappa_\alpha : X \to [0,1] \ / \ \alpha \in A\}$ is subordinate *to $\{U_\alpha\}$ iff the support of each κ_α lies in U_α.*

Theorem 2.26 *A space X is normal iff every locally finite open cover of X has a subordinate partition of unity.*

Proof. ⇒: Let $\{U_\alpha \ / \ \alpha \in A\}$ be a locally finite open cover of X. Then by Theorem 2.24, $\{U_\alpha\}$ has a shrinkage, say $\{V_\alpha \ / \ \alpha \in A\}$. This cover is also locally finite, so it must also have a shrinkage, say $\{W_\alpha \ / \ \alpha \in A\}$. Since $\bar{W}_\alpha \subset V_\alpha$, the two closed sets $\bar{W}_\alpha$ and $X - V_\alpha$ are disjoint, so by Urysohn's lemma, there is a map $\lambda_\alpha : X \to [0,1]$ with $\lambda_\alpha(\bar{W}_\alpha) = 1$ and $\lambda_\alpha(X - V_\alpha) = 0$. Note that the support of λ_α lies in $\bar{V}_\alpha \subset U_\alpha$.

Consider $\sum_{\alpha \in A} \lambda_\alpha$: by local finiteness of $\{U_\alpha\}$, in a neighbourhood of each point this sum is finite, hence continuous. Thus if we define $\kappa_\alpha : X \to [0,1]$ by

$$\kappa_\alpha(x) = \frac{\lambda_\alpha(x)}{\sum_{\beta \in A} \lambda_\beta(x)}$$

then κ_α, the quotient of a continuous function by a positive continuous function, is continuous.

The support of κ_α is the same as that of λ_α, so lies inside U_α and hence the supports of the maps κ_α are locally finite. Since the support of κ_α contains W_α, and $\{W_\alpha \ / \ \alpha \in A\}$ is a cover of X, condition (i) for a partition of unity is satisfied. Condition (ii) is obviously satisfied as is the subordinacy requirement.

$\Leftarrow$: Let A and B be disjoint closed subsets of X. Then $\{X - A, X - B\}$ is a locally finite open cover of X. Let $\kappa, \lambda : X \to [0,1]$ be continuous so that $\kappa + \lambda = 1$ and the supports of κ and λ lie, respectively, in $X - A$ and $X - B$. Then for each $x \in A$, $\kappa(x) = 0$ and for each $x \in B$, $\lambda(x) = 0$, so that for each $x \in B$, $\kappa(x) = 1$. Let $U = \kappa^{-1}([0, \frac{1}{2}))$ and $V = \kappa^{-1}((\frac{1}{2}, 1])$. Then U and V are disjoint open sets containing, respectively, A and B. ∎

Urysohn's lemma suggests a strengthening of the definition of regular spaces, giving a condition which is distinct from that of regularity.

Definition 2.27 *A space X is* completely regular *iff for each $x \in X$ and each closed set $C \subset X$ so that $x \notin C$, there is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(C) = 1$. Completely regular T_1 spaces are called* Tychonoff or $T_{3\frac{1}{2}}$ *spaces.*

Clearly completely regular spaces are regular, and Urysohn's lemma tells us that T_4 spaces are $T_{3\frac{1}{2}}$.

Sierpinski's space is normal but not completely regular. It may be verified that any product of completely regular spaces is completely regular.

Let S be the real line with the right half-open interval topology, discussed in Example 1.4. Then, as noted in 2.15, S is normal but $S \times S$ is not normal. Clearly S is also T_1 , so S is T_4 and hence Tychonoff, so $S \times S$ is Tychonoff. Hence $S \times S$ is a space which is Tychonoff but not normal.

There are examples of T_3 spaces in which every real-valued map is constant, eg Hewitt's condensed corkscrew in Steen & Seebach "Counterexamples in topology". Such a space will not in general be completely regular.

Definition 2.28 *If X is a space and A a set then by the* power X^A *we mean the product space $\Pi_\alpha X_\alpha$, where $X_\alpha = X$, for each $\alpha \in A$. Any power of $[0,1]$ is called a* cube. *A map $e : X \to Y$ is an* embedding *iff the map $e : X \to e(X)$ is a homeomorphism. If there is an embedding $e : X \to Y$ then we say that X can be* embedded *in Y.*

Theorem 2.29 (Tychonoff's embedding theorem) *A space is Tychonoff iff it can be embedded in a cube.*

Proof. $\Rightarrow$: Let X be a Tychonoff space and let $A = \{f : X \to [0,1] \ / \ f \text{ is continuous}\}$. Define $e : X \to [0,1]^A$ by $e(x)(f) = f(x)$.

(i) e is injective: if $x, y \in X$ with $x \neq y$, then there is $f \in A$ so that $f(x) = 0$ and $f(y) = 1$. Then $e(x)(f) \neq e(y)(f)$, so $e(x) \neq e(y)$.

(ii) e is continuous: this is immediate from Proposition 1.26 as $\pi_f e = f$.

(iii) $e : X \to e(X)$ carries open sets of X to open subsets of $e(X)$: for let U be open in X and let $x \in U$. Then there is $f \in A$ so that $f(x) = 0$ and $f(X - U) = 1$. Let $V = \pi_f^{-1}([0, 1))$, an open subset of $[0, 1]^A$. Then $e(x) \in V$ and if $y \in X$ is such that $e(y) \in V$, then $e(y)(f) \in [0, 1)$, so $f(y) < 1$ and $y \in U$. Thus $e(x) \in V \cap e(X) \subset e(U)$.

(i), (ii) and (iii) together imply that e is an embedding.

$\Leftarrow$: $[0, 1]$ is clearly Tychonoff so $[0, 1]^A$ is Tychonoff for any A. Any subspace of a Tychonoff space is Tychonoff. Thus if X can be embedded in a cube, then X is homeomorphic to a Tychonoff space and so is itself Tychonoff. ∎

As has been observed Sorgenfrey's square of Example 2.15 is Tychonoff but not normal. In problem 4 of section 4 you are asked to show that every cube is normal. Thus a subspace of a normal space need not be normal.

Here is another embedding theorem. It can be argued from this theorem that in order to study topology in complete generality, one need only study finite spaces, powers of spaces and subspaces. Note that the requirement $\mathcal{U} \cap X = \varnothing$ is not really a restriction; it can even be deleted if we replace X in $\mathcal{T}^{\mathcal{U} \cup X}$ by a set of the same size which is disjoint from $\mathcal{U}$.

Theorem 2.30 *Let $(T, \mathcal{T})$ be the 3-point topological space defined by $T = \{0, 1, 2\}$ and $\mathcal{T} = \{\varnothing, \{0\}, T\}$. Let $(X, \mathcal{U})$ be any topological space and suppose that $\mathcal{U} \cap X = \varnothing$. Then there is an embedding $e : X \to \mathcal{T}^{\mathcal{U} \cup X}$.*

Proof. For each $U \in \mathcal{U}$, define $f_U : X \to T$ by $f_U(y) = 0$ if $y \in U$ and $f_U(y) = 1$ if $y \notin U$. Then f_U is continuous. For each $x \in X$, define $f_x : X \to T$ by $f_x(y) = 2$ if $y = x$ and $f_x(y) = 1$ if $y \neq x$. Then f_x is also continuous.

Define e by $e_i(y) = f_i(y)$ for each $i \in \mathcal{U} \cup X$. Then

(i) e is injective, for if $x, y \in X$ with $x \neq y$ then $e_x(y) = 1$ but $e_x(x) = 2$, so $e_x(x) \neq e_x(y)$ and hence $e(x) \neq e(y)$;

(ii) e is continuous because each f_i is continuous;

(iii) e is open into $e(X)$, for if $U \in \mathcal{U}$ and $x \in U$ then $V = \pi_U^{-1}(0)$ is open in $\mathcal{T}^{\mathcal{U} \cup X}$. Furthermore, $\pi_U e(x) = 0$, so $e(x) \in V$ while if $y \in X$ is such that $e(y) \in V$ then $\pi_U e(y) = 0$ and hence $y \in U$. Thus $V \cap e(X) \subset e(U)$. ∎

Exercises

1. Prove that a space X is T_0 iff for each $x, y \in X$, if $x \neq y$ then $\overline{\{x\}} \neq \overline{\{y\}}$.

2. Prove that a space X is T_2 iff the diagonal $\Delta = \{(x, x) \in X \times X\}$ is closed in $X \times X$.

3. Prove that every space which is T_0 and regular is T_1.

4. Prove that a space X is normal iff for each finite family $\{A_i \ / \ i = 1, \ldots, k\}$ of mutually disjoint closed subsets, there is a family $\{U_i \ / \ i = 1, \ldots, k\}$ of mutually disjoint open sets so that for each i, $A_i \subset U_i$.

5. Would the statement of problem 4 hold if we merely assumed $\cap A_i = \varnothing$ and required $\cap U_i = \varnothing$ in place of mutual disjointedness?

6. Prove that a space is normal iff every pair of disjoint closed subsets have disjoint closed neighbourhoods.

7. A topological space X is called *perfectly normal* provided that for each pair A and B of disjoint closed subsets there is a continuous function $f : X \to [0,1]$ with $f^{-1}(0) = A$ and $f^{-1}(1) = B$; note the difference between this definition and the definition of completely separated in Definition 2.17. Show that the following four conditions are equivalent:

   (a) X is perfectly normal;

   (b) For every closed subset A of X there is a continuous function $f : X \to [0,1]$ with $f^{-1}(0) = A$;

   (c) X is normal and every closed subset of X is a G_δ set [i.e. is the intersection of countably many open sets];

   (d) Every closed subset of X is a regular G_δ set [i.e. there are countably many open sets containing the given closed set, the intersection of whose closures is the set].

8. Let $\mathcal{P}$ denote any one of the 9 topological properties: T_i ($i = 0, 1, 2, 3, 3\frac{1}{2}, 4$), regular, completely regular, normal. Let $\mathcal{T}$ and $\mathcal{U}$ be two topologies on the same set X and suppose that $(X, \mathcal{T})$ satisfies property $\mathcal{P}$. In each of the two cases $\mathcal{T} \subset \mathcal{U}$ and $\mathcal{U} \subset \mathcal{T}$ decide whether $(X, \mathcal{U})$ must satisfy property $\mathcal{P}$. Justify answers with proofs or counterexamples.

9. A topological property is (weakly) hereditary iff whenever a space possesses the property so do all of its (closed) subspaces. Decide which of the 9 topological properties in problem 8 are (weakly) hereditary.

10. Let X be a set totally ordered by $<$ and containing more than one element. For each $a, b \in X$ with $a < b$, let $(a, b) = \{x \in X \ / \ a < x < b\}$, $L(b) = \{x \in X \ / \ x < b\}$ and $R(a) = \{x \in X \ / \ a < x\}$. Prove that the collection of all subsets of the forms above is a basis for a topology on X; this is called the *order topology* (cf the usual topology on $\mathbb{R}$, which is the order topology induced by the usual order on $\mathbb{R}$). Prove that the order topology is always T_3.

11. Let A be a closed subset of the T_3 space X. Let $\sim$ be the equivalence relation on X defined by $x \sim y$ iff $x = y$ or $\{x, y\} \subset A$. Let X/A denote the quotient space $X/\sim$. Prove that X/A is Hausdorff.

12. Prove that the word "bounded" can be removed from the statement of Tietze's extension theorem.

13. Prove that every metrisable space is T_4: thus none of the examples of topological spaces given in chapter 2 which are not T_4 can be metrisable.

14. Prove that $\mathbb{R}$ with the right half-open interval topology is not metrisable.

15. Decide which of the 9 properties listed in Problem 8 are preserved by continuous functions (i.e. if $\mathcal{P}$ is one of these properties, X is a $\mathcal{P}$ space and $f : X \to Y$ is a continuous surjection, then Y is a $\mathcal{P}$ space).

16. Let A be a subset of a space X, let Y be a Hausdorff space and let $f : A \to Y$ be a continuous function. Prove that there is at most one continuous function $g : \bar{A} \to Y$ extending f. Give examples where:

    (a) this is false if Y is not assumed to be Hausdorff;

    (b) there is no continuous extension to $\bar{A}$.

17. A function $f : X \to Y$ is called *nearly continuous* if for each open set $V \subset Y$, we have $f^{-1}(V) \subset \mathrm{int}\overline{f^{-1}(V)}$; and is *quasicontinuous* if for each open set $V \subset Y$, we have $f^{-1}(V) \subset \overline{\mathrm{int} f^{-1}(V)}$. Prove that:

    (a) if f is continuous then f is both nearly continuous and quasicontinuous;

    (b) if f is both nearly continuous and quasicontinuous and Y is regular then f is continuous.

18. Show that the tangent discs topology of 2.14 is Tychonoff but not normal.

19. Let S be the Sierpinski space and let $(X, \mathcal{T})$ be any T_1 topological space. Prove that there is an embedding $e : X \to S^{\mathcal{T}}$.

20. Let X be as in Problem 18 of Section 1. Prove that X is not T_4 .

21. Prove the following result, known as Jones' Lemma. Suppose that X is normal, that $D \subset X$ is dense and that $F \subset X$ is closed and discrete. Then $2^{|F|} \leq 2^{|D|}$ and hence $|F| < 2^{|D|}$. (Hint: Extract a proof from the proof of the more special case of Jones' Lemma given in Example 2.14.)
    The comment in Example 2.15 raises a general issue across lots of Mathematics: if the proofs of two results are nearly identical then there is surely a more general result which gives the special cases as corollaries.