Affine flag-transitive biplanes with a prime number of points

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February 17, 2016, SCDO

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Example

The complement of the Fano plane is a flag-transitive $(7, 4, 2)$ biplane with $Aut(D) = PSL_2(7)$.

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Theorem (O'Reilly-Regueiro, 2005)

If $D = (P, B)$ is a non-trivial biplane with a primitive, flag-transitive automorphism group G, then one of the following holds:

(1) *D has parameters $(16, 6, 2)$.*

(2) *$G \leq A\Gamma L_1(q)$, for some odd prime power q.*

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(3) *G is of almost simple type.*

(O'Reilly-Regueiro, 2005, 2007, 2008.) The only biplanes with a primitive and flag-transitive automorphism group of almost simple type are the Fano complement with parameters $(7, 4, 2)$ and the unique Hadamard design of order 3 with parameters $(11, 5, 2)$.

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- Since G is transitive in P, $G = T \rtimes G_0$, where $G_0 \leq \langle \hat{g} \rangle = GL_1(p)$ is the point stabilizer of 0 and $\hat{g}$ denotes multiplication by g.

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- Since G is transitive in P, $G = T \rtimes G_0$, where $G_0 \leq \langle \hat{g} \rangle = GL_1(p)$ is the point stabilizer of 0 and $\hat{g}$ denotes multiplication by g.
- The $(37, 9, 2)$ biplane with $G = \mathbb{Z}_{37} \rtimes \mathbb{Z}_9$ is the only known example.

Conjecture

Let D be a non-trivial biplane that admits a primitive flag-transitive automorphism group G such that $G \leq AGL_1(p)$, where p is prime. Then D is the unique flag-transitive $(37, 9, 2)$ biplane.

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This is true when $p < 10^7$

Lemma

If D is a biplane with a flag-transitive group $G = T \rtimes G_0 \leq AGL_1(p)$, *then* G_0 *also stabilizes a block* b *not incident with* 0 *and the points of* b *form a* G_0*-orbit.*

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Lemma
If G is the automorphism group of a biplane $D = (P, D)$ and $G \leq AGL_1(p)$ is flag-transitive, then G is flag-regular.

Special pairs

A pair (p, n) is special if $p = nk + 1$ is a prime such that $D_n = \{x^n | x \in \mathbb{F}_p^\times\}$ is a $(p, k, (k-1)/n)$-difference set of $\mathbb{F}_p$. That is, every element of $\mathbb{F}_p \backslash 0$ can be represented as the difference of two elements of D_n and the number of different representations is $(k-1)/n$.

Theorem (K. Thas, D. Zagier, 2008)

If D is a (p, k, λ)-symmetric design with a flag-regular automorphism group, then $k = (p-1)/n$, $\lambda = (k-1)/n$ and (p, n) is a special pair.

Theorem (K. Thas, D. Zagier, 2008)

Let p be a prime and $n|(p-1)$. Then (p,n) is a special pair in each in the following five cases:

(a) *$n=1$, p arbitrary.*

(b) *$n=2$, $p \equiv 3 \pmod 4$.*

(c) *$n=4$, $p = 4b^2+1$ with b odd.*

(d) *$n=8$, $p = 64b^2+9 = 8d^2+1$ with b and d integers.*

(e) *$n=p-1$, p arbitrary.*

It is conjectured that the only special pairs are the ones listed in the previous theorem. In the same paper they found that for $p < 10^7$ this are the only special pairs. This was done through some computations that check that the number of distinct representations of every element $\alpha \in \mathbb{F}_p \backslash 0$ is constant, regardless of the choice of α.

Summing up

Suppose that D is a $(p, k, 2)$ biplane and that $Aut(D) \leq AGL_1(p)$ and that p is a prime less than 10^7.

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- $\Rightarrow$ $Aut(D)$ is flag-regular.
- $\Rightarrow$ we have that $p = nk + 1$ and (p, n) is a special pair with $2 = (k-1)/n$, so $p = 2n^2 + n + 1$.

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- $\Rightarrow$ D is a $(2n^2 + n + 1, 2n + 1, 2)$ biplane and since $p < 10^7$ then n must be $1, 2, 4, 8$ or $p - 1$.

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If $n = 1$ then $p = 4$ that is not prime.
If $n = 2$ then $p = 11$ and $k = 5$, but the only $(11, 5, 2)$ biplane is the Hadamard design of order 3 with $Aut(D) = PSL_2(11)$ that is not of affine type.

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Suppose that D is a $(p, k, 2)$ biplane and that $Aut(D) \leq AGL_1(p)$ and that p is a prime less than 10^7.

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If $n = 8$ then $p = 137$ and $k = 17$, but the $p = 64b^2 + 9 = 8d^2 + 1$ condition is not satisfied.
if $n = p - 1$ then $n + 1 = 2n^2 + n + 1$, a contradiction.

Thank you!