Manufacturing Permutation Representations of Monodromy Groups of Polytopes

Barry Monson, University of New Brunswick

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Outline: there's no time for an outline!

But

L.Berman, D.Oliveros, and G.Williams

are part of the project. Thanks as well to

D. Pellicer and M. Mixer.

Abstract Polytopes

Thinking combinatorially (abstractly), an n-polytope $\mathcal{P}$ is a poset with properties modelled on those of the face lattice of a convex n-polytope.

Symmetry is described by $\mathrm{Aut}(\mathcal{P})$, the group of all
automorphisms = order-preserving bijections on $\mathcal{P}$.

An n-polytope $\mathcal{P}$ is regular if $\mathrm{Aut}(\mathcal{P})$ is transitive on flags.

(But most polytopes of rank $n \geq 3$ are not regular.)

If $\mathcal{P}$ is regular, then $\mathrm{Aut}(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ is a string C-group. From such a group (with specified generators) we can reconstruct $\mathcal{P}$ as a coset geometry (using a combinatorial Wythoff's construction).

But now let us disassemble $\mathcal{P}$.

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$\mathrm{Mon}(\mathcal{P})$ scrambles the flags of an n-polytope $\mathcal{P}$

The *diamond condition* on the n-polytope $\mathcal{P}$ amounts to this:

for each flag Φ and proper rank j $(0 \leq j \leq n-1)$ there exists a unique flag Φ^j which is j-adjacent to Φ.

So $r_j : \Phi \mapsto \Phi^j$ defines a fixed-point-free involution on the flag set $\mathcal{F}(\mathcal{P})$.

Defn. The *monodromy group* $\mathrm{Mon}(\mathcal{P}) := \langle r_0, \ldots, r_{n-1} \rangle$

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More on $\mathrm{Mon}(\mathcal{P}) = \langle r_0, \ldots, r_{n-1} \rangle$

- $\mathrm{Mon}(\mathcal{P})$ encodes combinatorial essence of $\mathcal{P}$:

  eg. flag connectedness of $\mathcal{P} \Rightarrow \mathrm{Mon}(\mathcal{P})$ transitive on $\mathcal{F}(\mathcal{P})$

- $\mathrm{Mon}(\mathcal{P})$ says a lot about how $\mathcal{P}$ can be covered by an abstract regular n-polytope $\mathcal{R}$

- $\mathrm{Mon}(\mathcal{P})$ is an sggi (= string group generated by involutions):

  $$r_j \text{ and } r_k \text{ commute if } |j-k| > 1$$

- Th actions of $\mathrm{Mon}(\mathcal{P})$ and $\mathrm{Aut}(\mathcal{P})$ on $\mathcal{F}(\mathcal{P})$ commute: for $g \in \mathrm{Mon}(\mathcal{P})$, $\alpha \in \mathrm{Aut}(\mathcal{P})$, flag $\Phi \in \mathcal{F}(\mathcal{P})$

  $$(\Phi\alpha)^g = (\Phi^g)\alpha$$

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$\mathrm{Mon}(\mathcal{P})$ can be a beast to compute ...

If $\mathcal{P}$ is regular then $\mathrm{Mon}(\mathcal{P}) \simeq \mathrm{Aut}(\mathcal{P})$ (as sggi's).

But typically $\mathrm{Mon}(\mathcal{P})$ is far larger than $\mathrm{Aut}(\mathcal{P})$ and is obscurely structured.

Our main result here is a simple way to build manageable and (one hopes) useful permutation representations of $\mathrm{Mon}(\mathcal{P})$.

Permutation representations of $\mathrm{Mon}(\mathcal{P})$

Theorem [B.M. et al, 2015]. Say G any subgroup of $\mathrm{Aut}(\mathcal{P})$. Choose any base flag $\Psi \in \mathcal{F}(\mathcal{P})$ and let $\mathcal{O}$ be the G-orbit of Ψ in $\mathcal{F}(\mathcal{P})$. Then

(a) For each $g \in \mathrm{Mon}(\mathcal{P})$, the set $\mathcal{O}^g$ is the G-orbit of the flag Ψ^g.

(b) $B := \{\mathcal{O}^g : g \in \mathrm{Mon}(\mathcal{P})\}$ is a partition of the flag set $\mathcal{F}(\mathcal{P})$.

(c) We get a permutation representation in B:

$$\begin{array}{rcl} f : \mathrm{Mon}(\mathcal{P}) & \to & \mathrm{Sym}(B) \\ h & \mapsto & \pi_h \end{array}$$

where $(\mathcal{O}^g)\pi_h = \mathcal{O}^{gh}$.

(d) If G is core-free in $\mathrm{Aut}(\mathcal{P})$, then f is injective.

Test Case 1: the truncated icosahedron.

In 2010 M. Hartley & G. Williams computed the monodromy group for each Archimedean polyhedron. Some surface topology motivated a complicated presentation, which was then analyzed in *GAP*.

Challenge: have a somewhat limited human do the *truncated icosahedron* $\mathcal{P}$ by hand.

(From H-W above, the order of the monodromy group for this polyhedron was known to be 2592000.)

▸ A chiral example

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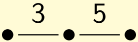
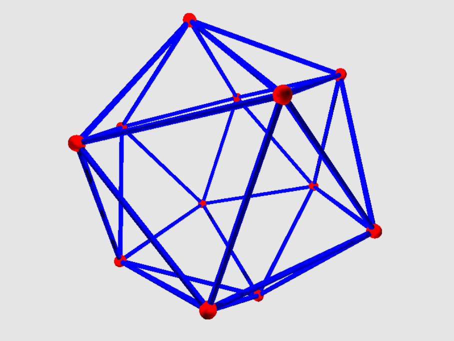
First the regular icosahedron $\{3, 5\}$

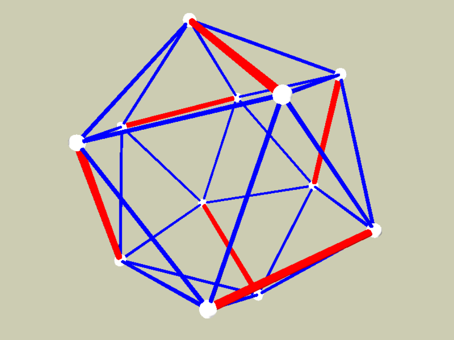
Its automorphism group is the Coxeter group $H_3 = \langle \rho_0, \rho_1, \rho_2 \rangle$ with diagram





A subgroup $G < H_3$

Notice that $H_3 \simeq A_5 \times C_2$ has a subgroup $G \simeq A_4$. Indeed, G is the group of rotations preserving 3 mutually orthogonal golden rectangles inscribed in $\{3, 5\}$:



It is easy to check that

$$G = \langle \rho_0\rho_1, (\rho_0\rho_2)^{\rho_1\rho_2} \rangle$$

is (to conjugacy) the *largest core-free* subgroup of H_3.

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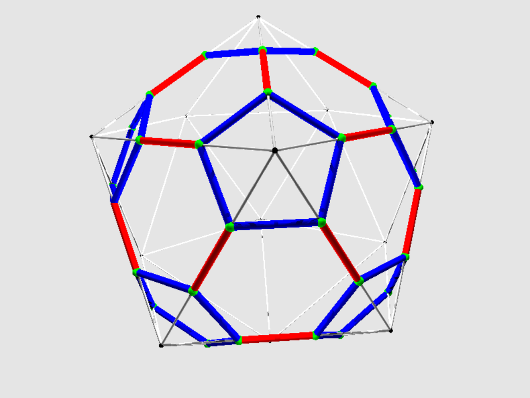
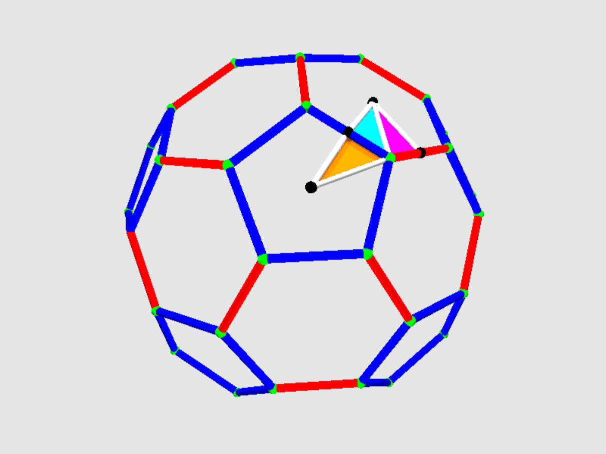
A fragment of the truncated icosahedron $\mathcal{P}$ (with some icosahedral scaffolding)

Still $\mathrm{Aut}(\mathcal{P}) = H_3$:



The truncated icosahedron $\mathcal{P}$

has three symmetry classes of flags, hence 360 flags.
Here are three base flags:



Type 1 - orange (pent. to hexa.)
Type 2 - cyan (hexa. to penta.)
Type 3 - magenta (hexa. to hexa.)

Apply the Theorem.

- we found G largest core-free subgroup, order 12.
- get a faithful representation of degree $30 = 360/12$.
- let $\gamma = \rho_0\rho_1\rho_2$, a *Coxeter element*; order 10.
- so powers γ^j, taking $j \pmod{10}$, give a transversal to G in H_3.
- **Upshot**: $\mathrm{Mon}(\mathcal{P})$ faithfully represented on $\{1, 2, \ldots, 30\}$.
  For $1 \leq i \leq 3$, $1 \leq j \leq 10$,
  the number $10(i-1)+j$ represents

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Using just a model of the icosahedron

(and several patient minutes) we get that $\mathrm{Mon}(\mathcal{P}) \simeq \langle r_0, r_1, r_2 \rangle$, where

$$\begin{aligned} r_0 &= (1,4)(2,7)(3,10)(5,8)(6,9)(11,14)(12,17)(13,20) \\ &\quad (15,18)(16,19)(21,26)(22,29)(23,30)(24,27)(25,28) \end{aligned}$$

$$\begin{aligned} r_1 &= (1,6)(2,3)(4,5)(7,8)(9,10)(11,21)(12,22)(13,23) \\ &\quad (14,24)(15,25)(16,26)(17,27)(18,28)(19,29)(20,30) \end{aligned}$$

$$\begin{aligned} r_2 &= (1,11)(2,12)(3,13)(4,14)(5,15)(6,16)(7,17)(8,18) \\ &\quad (9,19)(10,20)(21,26)(22,23)(24,25)(27,28)(29,30) \end{aligned}$$

Experiment a bit ...

We know from general theory that $\mathrm{Mon}(\mathcal{P})$ is a string C-group of Schläfli type $\{30, 3\}$. The '30' prompts a look at the cycle structure of

$$\begin{aligned} r_0 r_1 &= (1,5,7,3,9)(2,8,4,6,10)(11,24,17,22,19,26) \\ &\quad (12,27,14,21,16,29)(13,30)(15,28)(18,25)(20,23), \end{aligned}$$

so that

$$(r_0 r_1)^6 = (1,5,7,3,9)(2,8,4,6,10),$$

a 'parallel product' of 5-cycles supported only by type 1 flag blocks.

Continuing this way we soon find that

- $\mathrm{Mon}(\mathcal{P})$ has a normal subgroup $K \simeq A_5 \times A_5 \times A_5$, of order $60^3 = 216000$. The exponent 3 derives from the three flag classes.
- The centre of $\mathrm{Mon}(\mathcal{P})$ is generated by the involution

$$\begin{aligned} z = (r_0 r_1 r_2)^9 \quad = \quad & (1,6)(2,7)(3,8)(4,9)(5,10)(11,16)(12,17) \\ & (13,18)(14,19)(15,20)(21,26)(22,27) \\ & (23,28)(24,29)(25,30) \end{aligned}$$

- The subgroup $T = \langle z, r_1, r_2 \rangle \simeq C_2 \times S_3$ is of order 12 and is transverse to K.

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Summing up ...

- $\mathrm{Mon}(\mathcal{P}) \simeq (C_2 \times S_3) \ltimes (A_5 \times A_5 \times A_5)$, a semidirect product.
- The minimal regular cover $\mathcal{R}$ of the truncated icosahedron is a map of Schläfli type $\{30, 3\}$ and having 2592000 flags.

Let's get out of this ...

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Test Case 2: the finite chiral 5-Polytope $\mathcal{P}$

of type $\{3, 4, 4, 3\}$ (described by Conder, Hubard, Pisanski in 2008).

Here $\mathrm{Aut}(\mathcal{P}) = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4, \rangle \simeq \mathrm{Sym}_6$ with

$$
\begin{aligned}
\sigma_1 &= (1, 2, 3) \\
\sigma_2 &= (1, 3, 2, 4) \\
\sigma_3 &= (1, 5, 4, 3) \\
\sigma_4 &= (1, 2, 3)(4, 6, 5)
\end{aligned}
$$

$\mathrm{Aut}(\mathcal{P})$ has 2 flag orbits (as in any chiral polytope) so $\mathcal{P}$ has $1440 = 2 \cdot 720$ flags.
But the facet group $G = \langle \sigma_1, \sigma_2, \sigma_3 \rangle = \mathrm{Sym}_5$ is core-free in $\mathrm{Aut}(\mathcal{P})$, so we get a faithful representation of $\mathrm{Mon}(\mathcal{P})$ on $2 \cdot 6 = 12$ blocks (of 120 flags each).

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Test case 2, continued

When you ponder $\mathrm{Aut}(\mathcal{P})$ a bit you get (by hand+brain) this faithful representation of $\mathrm{Mon}(\mathcal{P}) = \langle r_0, r_1, r_2, r_3, r_4 \rangle$ by permutations of degree 12:

$r_0 = (1,7)(2,8)(3,9)(4,10)(5,11)(6,12)$
$r_1 = (1,7)(2,8)(3,10)(4,11)(5,9)(6,12)$
$r_2 = (1,7)(2,11)(3,9)(4,10)(5,8)(6,12)$
$r_3 = (1,8)(2,7)(3,11)(4,10)(5,9)(6,12)$
$r_4 = (1,12)(2,8)(3,10)(4,9)(5,11)(6,7)$

(a considerable improvement over degree 1440).

We find that $\mathrm{Mon}(\mathcal{P})$ has order 518400.

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Test case 2, continued

Furthermore, $\mathrm{Mon}(\mathcal{P})$ has to be a a sggi and must also have type $\{3,4,4,3\}$. However, the intersection condition, which would give a regular polytope from $\mathrm{Mon}(\mathcal{P})$, actually fails in $\mathrm{Mon}(\mathcal{P})$.

To 'resolve this singularity', we use a mixing technique of BM-Schulte to produce a regular polytopal cover $\mathcal{R}$ of $\mathcal{P}$. This $\mathcal{R}$ has Schläfli type $\{3,4,4,6\}$ and 3732480000 flags.

Remarkably, this cover is minimal (among regular covers of $\mathcal{P}$), even though it covers $\mathcal{P}$ in a 2592000 : 1 fashion.

I have no idea how to comprehensively describe the minimal regular covers of $\mathcal{P}$. It is hardly likely that $\mathcal{R}$ is unique.

MANY THANKS TO YOU
AND THE ORGANIZERS!

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