Skew-morphisms of Groups and Regular Cayley maps

Jun-Yang Zhang

School of mathematics and statistics, Minnan Normal University

Queenstown, New Zealand, Feb. 18, 2016
Outline

Skew-morphism

Regular Cayely map

Skew-morphisms of dihedral groups
Skew-morphism

**Skew-morphism:** A *skew-morphism* \( \varphi \) of a finite group \( G \) is a permutation on \( G \) such that \( \varphi(1) = 1 \) and \( \varphi(gh) = \varphi(g)\varphi^{\pi(g)}(h) \) for all \( g, h \in G \), where \( \pi \) is a function from \( G \) to the cyclic group \( \mathbb{Z}_{|\varphi|} \), called the *power function* of \( \varphi \). (R. Jajcay and J. Širáň, 2002)
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**Kernel of a skew-morphism:** The set \( \text{Ker} \varphi = \{ g \in G \mid \pi(g) = 1 \} \) is a subgroup of \( G \), called the *kernel* of \( \varphi \). (R. Jajcay and J. Širáň, 2002)
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**Skew-type:** A skew-morphism $\varphi$ is said to be of *skew-type* $k$ provided its power function $\pi$ takes on exactly $k$ values in $\mathbb{Z}_{|\varphi|}$. (J. Y. Zhang, 2014)
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**Core of a skew-morphism:** The set

\[
\text{Core} \varphi := \{ x \in G \mid \pi(\varphi^i(x)) = 1, i = 0, 1, 2, \ldots \}
\]

is a normal subgroup of \( G \), called the *core* of \( \varphi \). (J. Y. Zhang, 2015)
Important formulas

\[ \text{Aut}(G) \subseteq \text{Skew}(G) \subseteq \langle \text{Skew}(G) \rangle \subseteq \text{Sym}(G) \]

Unlike \( \text{Aut}(G) \), \( \text{Skew}(G) \) is not necessarily a subgroup of \( \text{Sym}(G) \).
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\varphi^k(gh) = \varphi^k(g) \sum_{i=0}^{k-1} \pi(\varphi^i(g)) (h)
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For a \( \varphi \)-orbit \( X \) satisfying \( X = X^{-1} \), let \( \chi(x) \) be the smallest nonnegative integer such that \( \varphi^{\chi(x)}(x) = x^{-1} \). Then

\[ \pi(x) \equiv \chi(\varphi(x)) - \chi(x) + 1 \pmod{|X|} \quad \text{for all } x \in X. \]

(R. Jajcay and J. Širáň, 2002)
**Skew-product**

**Skew-product:** Let $L_G := \{L_g \mid g \in G\}$ be the left regular representation of a finite group $G$ and let $\varphi$ be a permutation on $G$. Then $\varphi$ is a skew-morphism of $G$ if and only if $L_G\langle \varphi \rangle$ is a subgroup of $\text{Sym}(G)$. For a skew-morphism $\varphi$ of $G$ with power function $\pi$, we have

$$\varphi L_g = L_{\varphi(g)} \varphi^{\pi(g)} \text{ for any } g \in G.$$ 

The group $L_G\langle \varphi \rangle$ is called the skew-product of $L_G$ by $\varphi$. (M. Conder, R. Jajcay, T. Tucker, 2007)
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**Proposition (J.Y Zhang & S. F. Du, 2015).**

Set $T = L_G\langle \varphi \rangle$ and write $L_{\text{Core } \varphi} := \{L_x \mid x \in \text{Core } \varphi\}$. Then

$$\text{Core}_T(L_G) = L_{\text{Core } \varphi}.$$
Some general results


Suppose that \( G = \langle x_i \mid 1 \leq i \leq t \rangle \), \( \varphi \) is a skew-morphism of \( G \) with the power \( \pi \). Then: (i) \( |\varphi| = \text{lcm}\{ |O_{x_1}|, |O_{x_2}|, \ldots, |O_{x_t}| \} \); (ii) for any \( c \in G \), \( c \in \text{Ker} \varphi \) if and only if \( \pi(c) \equiv 1 \pmod{|O_{x_i}|} \) for all \( i = 1, 2, \ldots, t \).
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Suppose that $A$ is a cyclic group with two subgroups $K$ and $M$ such that $A = \langle K, M \rangle$. Let $\varphi$ be a skew-morphism of $A$ preserving both $K$ and $M$. If $\varphi|_K \in \text{Aut}(K)$ and $\varphi|_M \in \text{Aut}(M)$, then $\varphi \in \text{Aut}(A)$. 
Some general results

**Theorem A (J.Y Zhang & S. F. Du, 2015).**

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**Corollary (J.Y Zhang & S. F. Du, 2015).**

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**Theorem B (J.Y Zhang & S. F. Du, 2015).**

Let $H$ be a normal subgroup of $G$ and write $\overline{G} = G/H$. Let $\varphi$ be a skew-morphism of $G$. If $\varphi$ preserves $H$, then it induces a permutation $\overline{\varphi} : \overline{G} \to \overline{G}$, $\overline{g} \mapsto \overline{\varphi(g)}$, which defines a skew-morphism of $\overline{G}$.
Regular Cayely maps

**Map:** A *map* is a 2-cell embedding of a connected graph into a closed surface. An *automorphism* of a map is an automorphism of the underlying graph which can be extended to a self-homeomorphism of the supporting surface. For a map on an orientable surface, the group of all its orientation-preserving automorphisms acts always semi-regularly on the set of its arcs. If it acts regularly, then the map is called *orientably-regular* (or regular for simplicity).
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**Cayley map:** A *Cayley map* $CM(G, X, \sigma)$ is a 2-cell embedding of the Cayley graph $C(G, X)$ into an orientable surface with the same local rotation induced by the permutation $\sigma$ at every vertex, where $\sigma$ is a cyclic permutation on $X$. 
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**Proposition (R. Jajcay & J. Širáň, 2002)**

A Cayley map $CM(G, X, \sigma)$ is regular if and only if there exists a skew-morphism $\varphi$ of $G$ such that $\varphi|_X = \sigma$. 
Skew-morphisms of dihedral groups I

Let $D_{2n} := \langle a, b \mid a^n = b^2 = (ab)^2 = 1 \rangle$ and $\varphi \in \text{Skew}(D_{2n})$. 


- If $\varphi$ preserves $\langle a \rangle$ set-wise, then the following hold:
  1. $\varphi$ is an automorphism if the restriction of $\varphi$ to $\langle a \rangle$ is an automorphism;
  2. $a^2 \in \text{Ker} \varphi$ and $\varphi$ is of skew-type 1, 2, or 4;
  3. $\varphi$ is of skew-type 1 or 2 if $\varphi$ fixes an element in the coset $\langle a \rangle b$;
  4. $\varphi$ is of skew-type 1 or 2 if $n$ is a prime power.


$\text{Ker} \varphi < \langle a \rangle$ if and only if $\varphi$ is of skew-type 4 and preserves $\langle a \rangle$.


If $n$ is an odd number not divisible by 3, then $\varphi$ must be an automorphism.
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Skew-morphisms of dihedral groups II

Let $\varphi$ be a skew-morphism of the group $D_{2n}$ not preserving $\langle a \rangle$ and let $X$ be the orbit of $a$ under $\varphi$. Then $X \cap \langle a \rangle b \neq \emptyset$, $X^{-1} = X$, $D_{2n} = \langle X \rangle$ and $\text{CM}(D_{2n}, X, \varphi|_X)$ is a regular Cayley map.
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Such skew-morphisms have been classified under the following conditions:

1. $\varphi$ is $t$-balanced, that is, $\varphi(xg) = \varphi(x) \varphi(tg)$ for any $x \in X$ and any $g \in D_{2n}$; (H. Kwak, Y.S. Kwon, R. Q. Feng, 2006)
2. $n$ is an odd number; (I. Kovács, D. Marušič, M. Muzychuk, 2013)
3. $\varphi$ is of skew-type $3$; (J. Y. Zhang, 2015)
4. $\text{Core}(\varphi) = \{1\}$. (J. Y. Zhang, 2015)
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Problem

Classify all skew-morphisms of dihedral groups.
Thank you very much!