Automorphism groups of edge-transitive maps (SCDO 2016)

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Recent history

In 1997 Graver and Watkins showed how edge-transitive maps $\mathcal{M}$ can be partitioned into 14 classes T. These are distinguished by the isomorphism class $\mathcal{N}(T)$ of the one-edge map $\mathcal{M}/\mathrm{Aut}\,\mathcal{M}$.

In 2001 Širáň, Tucker and Watkins showed that for each $n \geq 11$ with $n \equiv 3$ or 11 mod (12), there are finite, orientable, edge-transitive maps $\mathcal{M}$ in each class T with $\mathrm{Aut}\,\mathcal{M} \cong S_n$.

In 2011 Orbanič, Pellicer, Pisanski and Tucker classified the edge-transitive maps of low genus, together with those on $\mathbb{E}^2$.

Karabáš and Nedela (work in progress) have introduced a similar partition of oriented edge-transitive maps, based on $\mathcal{M}/\mathrm{Aut}^+\mathcal{M}$. This is very convenient for computational purposes, and in some cases allows them to extend the classifications to higher genus.

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I shall consider what groups $\mathrm{Aut}\,\mathcal{M}$ of symmetries, finite or infinite, the discrete objects $\mathcal{M}$ in these various classes T can have.

Maps

A map $\mathcal{M}$ is an embedding of a graph $\mathcal{G}$ in a surface $\mathcal{S}$, such that the faces (connected components of $\mathcal{S} \setminus \mathcal{G}$) are simply connected, i.e. homeomorphic to an open disc. The regular (or Platonic) solids are typical examples.

I shall assume that $\mathcal{S}$ and $\mathcal{G}$ are connected; $\mathcal{S}$ may be orientable or not, compact or not, with or without boundary (generally without).

The graph $\mathcal{G}$ may have multiple edges and loops (though not usually in the most symmetric cases which I will concentrate on).

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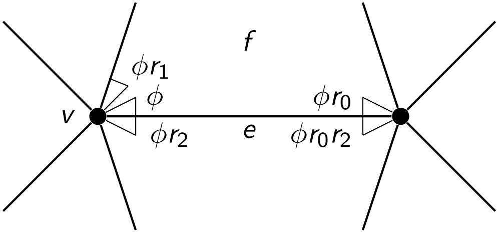
Problem Which groups arise as the automorphism groups of highly symmetric maps?

Maps and permutations



The monodromy group

$$G = \langle r_0, r_1, r_2 \mid r_i^2 = (r_0 r_2)^2 = 1, \ldots \rangle$$

of a map $\mathcal{M}$ acts transitively on the set Φ of flags $\phi = (v, e, f)$ of $\mathcal{M}$, with r_i changing the i-dimensional component of each ϕ while preserving the other two. Vertices, edges and faces correspond to orbits of $\langle r_1, r_2 \rangle$, $\langle r_0, r_2 \rangle$ ($\cong V_4$) and $\langle r_0, r_1 \rangle$ on Φ.

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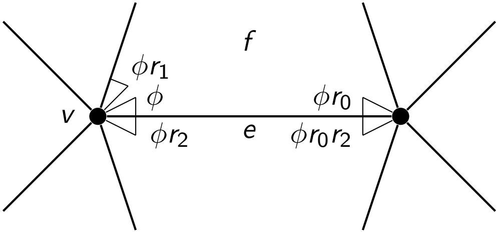

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The automorphism group $A = \mathrm{Aut}\,\mathcal{M}$ of $\mathcal{M}$ is the centraliser of G in the symmetric group $\mathrm{Sym}\,\Phi$, acting semiregularly on Φ.

Map subgroups

Maps $\mathcal{M}$ correspond to transitive permutation representations of

$$\Gamma := \langle R_0, R_1, R_2 \mid R_i^2 = (R_0 R_2)^2 = 1 \rangle,$$

via epimorphisms

$$\Gamma \to G, \quad R_i \mapsto r_i \quad (i = 0, 1, 2),$$

and hence to conjugacy classes of map subgroups

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Easy arguments show that

1. $\mathrm{Aut}\,\mathcal{M} \cong N_\Gamma(M)/M$,
2. $\mathrm{Aut}\,\mathcal{M}$ acts transitively on Φ if and only if M is normal in Γ, in which case

$$\mathrm{Aut}\,\mathcal{M} \cong \Gamma/M \cong G,$$

all acting regularly on Φ. Such maps $\mathcal{M}$ are called regular.

Regular maps and their groups

Regular maps are the most symmetric, the most studied, and the most important of all maps. For example, every map is the quotient of a regular map by some group of automorphisms.

For a given group G, the regular maps $\mathcal{M}$ with $\mathrm{Aut}\,\mathcal{M} \cong G$ correspond to the normal subgroups M of Γ with $\Gamma/M \cong G$. If G is finite, the number of them is

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Problem Which groups G are automorphism groups of regular maps? Equivalently, which groups G are quotients of

$$\Gamma = \langle R_0, R_1, R_2 \mid R_i^2 = (R_0R_2)^2 = 1\rangle\,?$$

Note that

$$\Gamma = \langle R_0, R_2\rangle * \langle R_1\rangle \cong V_4 * C_2,$$

the free product of a Klein four-group and a cyclic group of order 2.

Edge-transitive maps

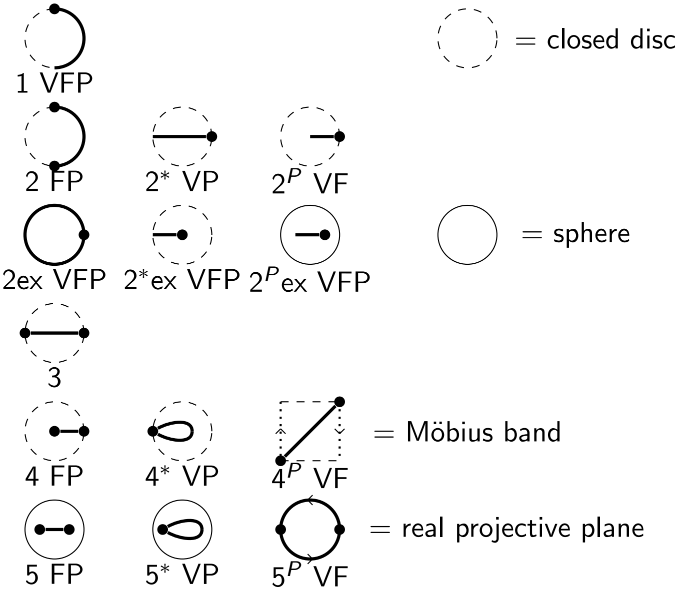
Much is known about automorphism groups of regular maps (more on that later), but what about a wider set of highly symmetric maps, namely edge-transitive maps? The following is easy to prove:

Lemma

$\mathrm{Aut}\,\mathcal{M}$ *acts transitively on the edges of* $\mathcal{M}$ *if and only if* $\Gamma = NE$, *where* $N := N_\Gamma(M)$ *and* $E := \langle R_0, R_2 \rangle \cong V_4$.

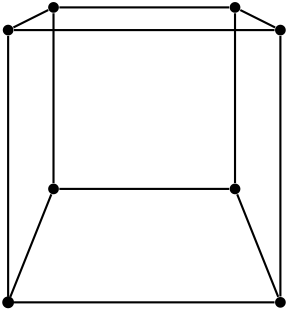
Since $|E| = 4$ this implies that $|\Gamma : N| \leq 4$. By inspection there are just 14 conjugacy classes of subgroups $N \leq \Gamma$ satisfying $\Gamma = NE$. They correspond to the 14 possible maps $\mathcal{M}/\mathrm{Aut}\,\mathcal{M}$ with one edge, and to the 14 classes of edge-transitive maps $\mathcal{M}$ described by Graver and Watkins in 1997 (Mem. Amer. Math. Soc. 601).

Example Class 1 consists of the regular maps, those with $N = \Gamma$. These include the Platonic solids, the antipodal quotients of the cube, octahedron, dodecahedron and icosahedron, and many more.



Basic maps $\mathcal{N}(T) = \mathcal{M}/\mathrm{Aut}\,\mathcal{M}$ for the edge-transitive classes T.

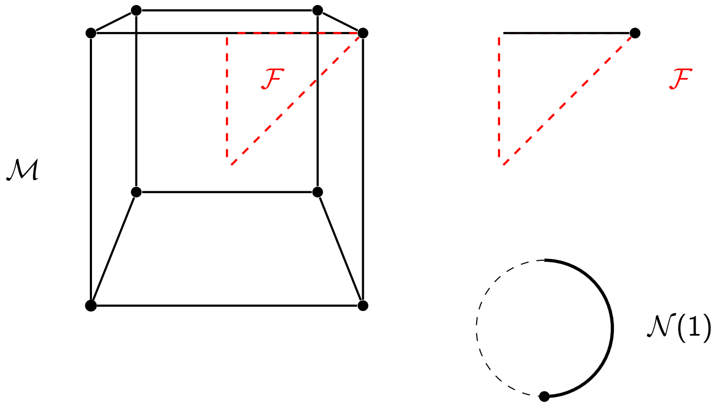
Example: the cube



The cube, as a map $\mathcal{M}$ on the sphere, has

$$\mathrm{Aut}\,\mathcal{M} \cong S_4 \times C_2.$$

It is regular, hence vertex-, edge-, and face-transitive.

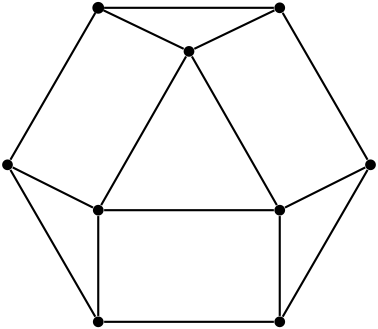


The cube $\mathcal{M}$ satisfies

$$\mathcal{M}/\mathrm{Aut}\,\mathcal{M} \cong \mathcal{F} \cong \mathcal{N}(1),$$

where $\mathcal{F}$ is a fundamental region for $\mathrm{Aut}\,\mathcal{M}$, so $\mathcal{M}$ is in class 1.

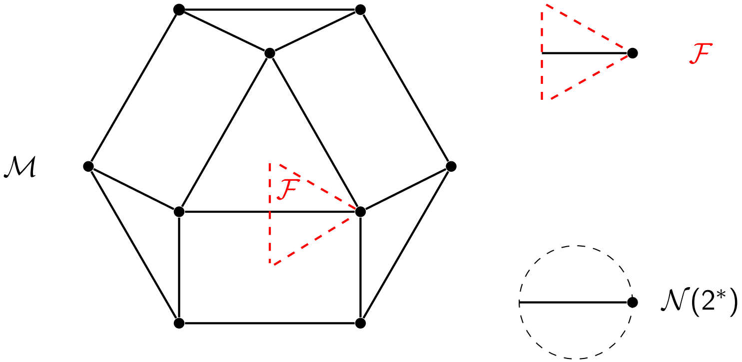
Example: the cuboctahedron



The cuboctahedron, as a map $\mathcal{M}$ on the sphere, also has

$$\mathrm{Aut}\,\mathcal{M} \cong S_4 \times C_2.$$

It is edge- and vertex-transitive, but not face-transitive.



The cuboctahedron $\mathcal{M}$ satisfies

$$\mathcal{M}/\mathrm{Aut}\,\mathcal{M} \cong \mathcal{F} \cong \mathcal{N}(2^*),$$

where $\mathcal{F}$ is a fundamental region for $\mathrm{Aut}\,\mathcal{M}$, so $\mathcal{M}$ is in class 2^*.

Regular maps and Mazurov's question

In 1980 Mazurov asked in the Kourovka Notebook (Problem 7.30): which finite simple groups are generated by three involutions, two of them commuting, i.e. which of them are quotients of Γ?

It is now known from work of Nuzhin and others that all non-abelian finite simple groups have such generators, except:

- $L_3(q)$ ($:= PSL_3(q)$) and $U_3(q)$ for all prime powers q,
- $L_4(q)$ and $U_4(q)$ for $q = 2^e$,
- A_6, A_7, M_{11}, M_{22}, M_{23} and McL.

Note that these exceptions include $L_2(7) \cong L_3(2)$, $L_2(9) \cong A_6$ and $A_8 \cong L_4(2)$. (See surveys by Mazurov or Širáň for references.)

Thus, apart from these exceptions, every non-abelian finite simple group is the automorphism group of a regular map. Indeed, for some groups one can count, and even classify, the associated maps.

Example 1: $G = A_5$

Look for epimorphisms $\Gamma = V_4 * C_2 \to G$. The factors V_4 and C_2 must be embedded in G. There are 15 involutions in G, each commuting with two others, so there are 30 embeddings $V_4 \to G$.

There are three involutions in any subgroup $V \cong V_4$, leaving 12 involutions outside it.

The only maximal subgroup containing V is its normaliser, a subgroup $A \cong A_4$, which contains no further involutions.

Hence any of the remaining 12 involutions, together with V, generates G, so there are $30.12 = 360$ epimorphisms $\Gamma \to G$.

$\mathrm{Aut}\, G = S_5$ permutes these epimorphisms regularly, so there are $360/5! = 3$ normal subgroups $N \triangleleft \Gamma$ with $\Gamma/N \cong G$.

Thus there are three regular maps $\mathcal{M}$ with $\mathrm{Aut}\,\mathcal{M} \cong A_5$.

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Thus there are three regular maps $\mathcal{M}$ with $\mathrm{Aut}\,\mathcal{M} \cong A_5$.

They are the antipodal quotients of the icosahedron, dodecahedron and great dodecahedron, non-orientable maps of genus $1, 1$ and 5.

Example 2: $G = L_3(2)$ ($\cong L_2(7)$)

There are two conjugacy classes of seven subgroups $V \cong V_4$ in G, each fixing three points or one; they are transposed by $\mathrm{Out}\, G$.

Hence there are $14.3! = 84$ embeddings $V_4 \to G$. Without loss we may assume that the image V fixes three points, forming a line L.

There are 21 involutions in G, leaving 18 outside V.

The stabiliser of a point $p \in L$ is a subgroup $G_p \cong S_4$ containing V; it contains 9 involutions, 6 of them outside V.

If p and q are distinct points in L then $G_p \cap G_q = V$, so the three subgroups G_p $(p \in L)$ contain all 18 involutions outside V.

Thus no involution, together with V, generates G, so there are no epimorphisms $\Gamma \to G$.

Hence there are no regular maps $\mathcal{M}$ with $\mathrm{Aut}\, \mathcal{M} \cong L_3(2)$.

Orientably regular chiral maps

Class $2^P\mathrm{ex}$ (blame Jack Graver and Mark Watkins for the notation!) consists of those maps for which N is the even subgroup

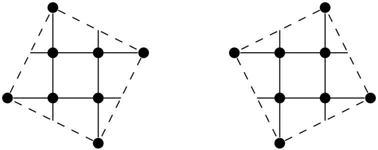
$$\Gamma^+ = \langle X = R_1R_2, Y = R_2R_0 \mid Y^2 = 1\rangle \cong C_\infty * C_2$$

of index 2 in Γ, consisting of the words of even length in the R_i.

These maps $\mathcal{M}$ are orientable and without boundary.

They are orientably regular, meaning that $\mathrm{Aut}\,\mathcal{M}$ is transitive on directed edges, and chiral, meaning that $\mathcal{M}$ is not isomorphic to its mirror image $\overline{\mathcal{M}}$, so they occur in chiral pairs.

Example In this chiral pair of maps, opposite sides of the outer squares are identified to form a torus, with $\mathrm{Aut}\,\mathcal{M} \cong AGL_1(5)$.



Automorphism groups of orientably regular chiral maps

The automorphism groups $G = \mathrm{Aut}\,\mathcal{M}$ of the maps $\mathcal{M}$ in class $2^P\mathrm{ex}$ are the quotients of $\Gamma^+ = \langle X, Y \mid Y^2 = 1\rangle$ by subgroups M which are normal in Γ^+ but not in Γ. This is equivalent to

1. $G = \langle x, y \mid y^2 = 1, \ldots\rangle$, and
2. no automorphism of G inverts x and fixes y.

It is known that every finite simple group has a generating pair satisfying (1), but what about (2)? An observation of Singerman, building on work of Macbeath, shows that:

Proposition

Every generating pair for $L_2(q)$ are simultaneously inverted by some automorphism.

Thus no orientably regular chiral map $\mathcal{M}$ has $\mathrm{Aut}\,\mathcal{M} \cong L_2(q)$.

Are any other non-abelian finite simple groups excluded?

Theorem

There is a map $\mathcal{M} \in 2^P\mathrm{ex}$ (i.e. orientably regular and chiral) with $\mathrm{Aut}\,\mathcal{M} \cong A_n$ if and only if $n \geq 8$.

Proof We need to determine when $A_n = \langle x, y \rangle$ with $y^2 = 1$ and no automorphism inverting x and y.

$\Rightarrow$ If $n \leq 6$ then $A_n \cong L_2(q)$ for some q, so Singerman's observation applies. If $n = 7$ then any pair generating a transitive group are either inverted or generate a proper subgroup $L_3(2)$.

$\Leftarrow$ For $n \geq 8$ we give explicit generators x and y, using:

Theorem (Jordan, 1871–3; Wielandt, FPG, Theorem 13.9)

If G is a primitive group of degree n containing a cycle of (prime) length $m \leq n - 3$ then $G \geq A_n$ (so $G = A_n$ or S_n).

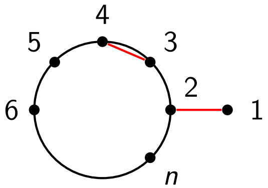
[By using the classification of finite simple groups, the primality condition can be removed (J, 2014).]

Proof

For even $n \geq 8$ let

$$x = (2, 3, \ldots, n) \quad \text{and} \quad y = (1,2)(3,4)$$

in A_n, so $G := \langle x, y \rangle$ is 2-transitive and hence primitive. Now $[y, x] = (1, 2, 3, 5, 4)$, so by Jordan's Theorem $G = A_n$.

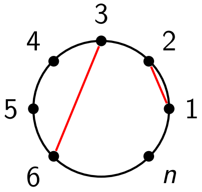


$\mathrm{Aut}\, A_n = S_n$, acting by conjugation, since $n \neq 6$; no permutation inverts x and y, so there are no forbidden automorphisms.
(The map $\mathcal{M}$ has type $\{n-1, n-1\}$ and genus $g \sim n!/8$.)

For odd $n \geq 9$, let

$$x = (1, 2, \ldots, n) \quad \text{and} \quad y = (1, 2)(3, 6),$$

An easy argument with congruences shows that G is primitive.
Now $[y, x^2] = (1, 2)(3, 6, 4)(5, 8)$, so $[y, x^2]^2 = (3, 4, 6)$ and hence $G = A_n$ by Jordan's Theorem.



As before, asymmetry of the diagram implies that no permutation in S_n inverts x and y, so there are no forbidden automorphisms. $\square$

Back to edge-transitive maps

The 14 Graver-Watkins classes T correspond to the 14 conjugacy classes of subgroups $N(T) \leq \Gamma \cong V_4 * C_2$.

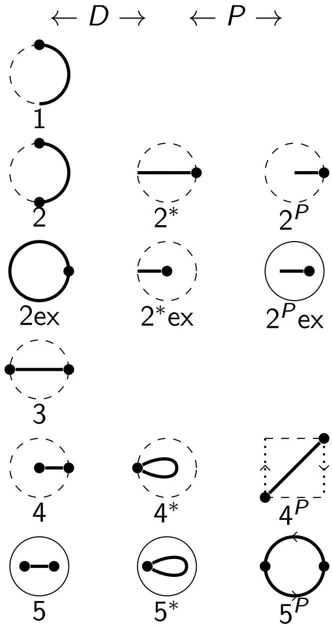
The maps $\mathcal{M}$ in each class T are regular covers, by groups $G = \mathrm{Aut}\,\mathcal{M}$, of the corresponding basic maps $\mathcal{N}(T)$.

The outer automorphism group

$$\mathrm{Out}\,\Gamma \cong \mathrm{Aut}\,V_4 \cong S_3$$

of Γ corresponds to Wilson's group $\langle D, P \rangle$ of map operations, where D = duality and P = Petrie duality.

It has six orbits on these conjugacy classes, and also on the basic maps, so it is sufficient to consider one representative of each orbit.



The six rows are the orbits of $\mathrm{Out}\,\Gamma = \langle D, P\rangle \cong S_3$ on the 14 basic maps $\mathcal{N}(T)$.

Back to edge-transitive maps

The 14 GW-classes T correspond to the 14 conjugacy classes of subgroups $N(T) \leq \Gamma \cong V_4 * C_2$. The outer automorphism group

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of Γ, corresponding to Wilson's group $\langle D, P\rangle$ of map operations, has six orbits on these conjugacy classes, and it is sufficient to consider one representative of each orbit. We can take

- $N(1) = \Gamma \cong V_4 * C_2$ (regular maps, already considered);
- $N(2^P\mathrm{ex}) = \Gamma^+ \cong C_\infty * C_2$ (chiral maps, already considered);
- $N(2) \cong C_2 * C_2 * C_2$;
- $N(3) \cong C_2 * C_2 * C_2 * C_2$ (just-edge-transitive maps);
- $N(4) \cong C_\infty * C_2 * C_2$;
- $N(5) \cong C_\infty * C_\infty \cong F_2$.

Realising automorphism groups

To realise a given group G as $\mathrm{Aut}\,\mathcal{M}$ for a map $\mathcal{M}$ in a class T, we need $G \cong N(T)/M$, where $N(T) = N_\Gamma(M)$, that is, M is normal in $N(T)$ but not normal in any $N(T') > N(T)$.

This means that G must not have any 'forbidden automorphisms' arising from conjugation in $N(T')$. These are as follows (with $x, y, z, \ldots$ generating successive cyclic factors of $N(T)$):

- $N(1) = \Gamma \cong V_4 * C_2$: no forbidden automorphisms;
- $N(2^P\mathrm{ex}) = \Gamma^+ \cong C_\infty * C_2$: $x \mapsto x^{-1}$, $y \mapsto y$;
- $N(2) \cong C_2 * C_2 * C_2$: $x \leftrightarrow y, z \mapsto z$;
- $N(3) \cong C_2 * C_2 * C_2 * C_2$: all three double transpositions;
- $N(4) \cong C_\infty * C_2 * C_2$: $x \mapsto x^{-1}$, $y \leftrightarrow z$;
- $N(5) \cong C_\infty * C_\infty$: transposing and/or inverting x and y.

Reducing the problem

One can choose epimorphisms onto $N(1) = \Gamma \cong V_4 * C_2$ from

$$N(2) \cong C_2 * C_2 * C_2, \ N(3) \cong C_2 * C_2 * C_2 * C_2, \ N(4) \cong C_\infty * C_2 * C_2,$$

which ensure (by composition) that if G is a quotient of Γ then it is a quotient, without forbidden automorphisms, of these three groups, so G is also realised for these three classes.

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Similarly, an epimorphism from $N(5) \cong C_\infty * C_\infty$ onto $N(2^P\mathrm{ex}) = \Gamma^+ \cong C_\infty * C_2$ ensures that any G realised for class $2^P\mathrm{ex}$ is also realised for class 5.

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$$N(2) \cong C_2 * C_2 * C_2,\ N(3) \cong C_2 * C_2 * C_2 * C_2,\ N(4) \cong C_\infty * C_2 * C_2,$$

which ensure (by composition) that if G is a quotient of Γ then it is a quotient, without forbidden automorphisms, of these three groups, so G is also realised for these three classes.

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This focuses attention on classes 1 and $2^P\mathrm{ex}$ (the regular and chiral maps, those of most interest combinatorially).

However, some groups G can be realised for other classes but not for 1 or $2^P\mathrm{ex}$, so these other four classes cannot be ignored.

Realising finite simple groups

It is interesting to ask which non-abelian finite simple groups G can arise for each class. Ignoring forbidden automorphisms, we know that every such G is a quotient of each $N(T)$, except:

- $L_3(q)$, $U_3(q)$, $L_4(2^e)$, $U_4(2^e)$, A_6, A_7, M_{11}, M_{22}, M_{23}, McL for $N(1) = \Gamma \cong V_4 * C_2$ (Nuzhin et al.);
- $U_3(3)$ for $N(2) \cong C_2 * C_2 * C_2$ (Malle, Saxl and Weigel).

Problem Which G are quotients with no forbidden automorphisms?

For example, $L_2(q)$ cannot be realised for classes $2^P\mathrm{ex}$ or 5, where $N(T) \cong C_\infty * C_2$ or $C_\infty * C_\infty$, since a forbidden automorphism always appears, inverting both generators.

A_n revisited

Using the reduction, and treating small cases individually, gives:

Theorem

$A_n \cong \mathrm{Aut}\,\mathcal{M}$ for some map $\mathcal{M}$ in class T if and only if:

- *$T = 1$ and $n = 5$ or $n \geq 9$ (Nuzhin);*
- *$T = 2^P\mathrm{ex}$ and $n \geq 8$;*
- *$T = 2$ and $n \geq 5$;*
- *$T = 3$ and $n \geq 5$;*
- *$T = 4$ and $n \geq 5$;*
- *$T = 5$ and $n \geq 7$.*

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- *$T = 3$ and $n \geq 5$;*
- *$T = 4$ and $n \geq 5$;*
- *$T = 5$ and $n \geq 7$.*

Similar methods can be applied to the symmetric groups::

Theorem

- *Classes $T = 1, 2, 3$ and 4 realise S_n if and only if $n \geq 3$.*
- *Classes $T = 2^\alpha\,\mathrm{ex}$ and 5 realise S_n if and only if $n \geq 6$.*

Other finite simple groups

Similar methods also give:

Theorem

- *if* $q \neq 7$ *or* 9 *then* $L_2(q)$ *is realised by each class* $T = 1$ *(Nuzhin),* 2, 3 *or* 4*;*
- *no group* $L_2(q)$ *is realised by* $T = 2^P\mathrm{ex}$ *or* 5*.*
- *The Suzuki groups* $Sz(2^e)$ *and 'small' Ree groups* $R(3^e)$ *are realised by all classes* T*.*

Problem Which other non-abelian finite simple groups are realised by the various classes $T \neq 1$?

A conjecture

It seems likely that, for each of the 14 classes T, 'almost all' non-abelian finite simple groups G are realised as automorphism groups. Indeed, if G is 'large enough', then randomly chosen elements of suitable orders will generate G as a quotient of $N(T)$, without forbidden automorphisms, with probability close to 1.

Example If G is O'Nan's sporadic simple group O'N, of order

$$460,815,505,920 = 2^9.3^4.5.7^3.11.19.31,$$

then a randomly-chosen pair $x, y \in G$ of orders 31 and 2 generate G as a quotient of $N(2^P\mathrm{ex}) = \Gamma^+ \cong C_\infty * C_2$, without forbidden automorphisms, with probability greater than 0.98. (Elements of order 31 are inverted by outer automorphisms.)

Such pairs correspond to about $150,000$ non-isomorphic 31-valent orientably regular chiral maps with automorphism group $G \cong O'N$.

Uncountably many automorphism groups

Infinite edge-transitive maps and their automorphism groups are also of interest, and the same methods apply to them.

Theorem

Each class T realises $2^{\aleph_0}$ non-isomorphic automorphism groups.

Outline proof In 1937 Bernhard Neumann proved that there are uncountably many 2-generator groups G.

He used epimorphisms $\Delta := C_\infty * C_3 \to A_n$ to construct, for any set S of integers $n \equiv 1 \bmod (4)$, a quotient G of Δ with a normal subgroup $N \cong A_n$ in G if and only if $n \in S$.

One can apply a similar method to our free products $N(T)$, using epimorphisms $N(T) \to A_n$ without forbidden automorphisms. $\square$

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Corollary

Each edge-transitive class T contains $2^{\aleph_0}$ non-isomorphic maps.

Embedding countable groups

Theorem

If C is any countable group, then each class T contains a map $\mathcal{M}$ with C isomorphic to a subgroup of $\mathrm{Aut}\,\mathcal{M}$.

Proof Schupp (1976) proved that if $|A| \geq 3$ and $|B| \geq 2$, than each countable group C can be embedded in a simple quotient S of $A * B$. Apply this to our groups $N(T) = A * B$ to get

$$M_1 \triangleleft N(T) \ \text{ with } \ C \leq S := N(T)/M_1, \ S \ \text{ simple.}$$

Now choose

$$M_2 \triangleleft N(T) \ \text{ with } \ N(T)/M_2 \cong A_n \not\cong S,$$

where A_n has no extra automorphisms. If $M := M_1 \cap M_2$ then

$$C \leq G := N(T)/M \cong N(T)/M_1 \times N(T)/M_2 \cong S \times A_n.$$

Both S and A_n are characteristic subgroups of G, so any forbidden automorphism of G would induce one on A_n, a contradiction. Hence $C \leq \mathrm{Aut}\,\mathcal{M}$ where $\mathcal{M}$, corresponding to M, is in class T. □

Intermediate growth

If a group G has a finite generating set X, let $\gamma_X(n)$ be the number of $g \in G$ of length at most n in the generators in X. The asymptotic behaviour of $\gamma_X(n)$ as $n \to \infty$ is independent of X.

Example Nilpotent-by-finite groups have polynomial growth, whereas non-elementary Fuchsian groups have exponential growth.

Intermediate growth

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Example Nilpotent-by-finite groups have polynomial growth, whereas non-elementary Fuchsian groups have exponential growth.

In 1980 Grigorchuk constructed a group G (followed in 1983 by uncountably many examples) with intermediate growth, strictly between polynomial and exponential. Each is generated by four involutions a, b, c, d satisfying $abc = 1$, so it is a quotient of Γ, and hence there are $2^{\aleph_0}$ regular maps with intermediate growth, in terms of the number of vertices, edges or faces within a given distance of an arbitrary base-point (J, 2011).

Intermediate growth is inherited by subgroups of finite index, so the same applies to the groups $N(T) \leq \Gamma$ and the associated maps.

Related work in progress

- (with Tom Tucker) Which groups are automorphism groups of maps with boundary in the various edge-transitive classes T?

Theorem

Of the 14 *edge-transitive classes,*

- $2ex$, 2^*ex, 2^Pex, 5, 5^* *and* 5^P *contain no such maps,*
- 1, 2, 2^* *and* 2^P *realise only dihedral automorphism groups,*
- 3, 4, 4^* *and* 4^P *realise 'many' automorphism groups.*

What does 'many' mean here?

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What does 'many' mean here?

- Which groups are automorphism groups of orientable maps in the various edge-transitive classes T?

Example There is a regular map $\mathcal{M}$ with $\mathrm{Aut}\,\mathcal{M} \cong S_5$ (N12.3 in Marston's list), but there is no orientable regular map.

To Marston, Richard and Steve:

Ra Whanau ki a Koe!

To Marston, Richard and Steve:

Ra Whanau ki a Koe!

And to the rest of you:

Diolch i chi am wrando!

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