Cubic arc-transitive k-circulants

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Circulants

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k-circulants

A permutation is called semiregular if all of its cycles have the same length.

We call a graph a k-circulant if it has a semiregular automorphism with k cycles.

bicirculants, tricirculants, tetracirculants, . . .

Polycirculant Conjecture

Polycirculant Conjecture: Every vertex-transitive digraph has a nontrivial semiregular automorphism.

Proved for cubic graphs by Marušič and Scapellato (1998).

Cubic arc-transitive bicirculants

Frucht-Graver-Watkins, Marušič, Marušič-Pisanski

A cubic arc-transitive bicirculant is one of:

- K_4, $K_{3,3}$;
- one of seven arc-transitive Generalised Petersen graphs;
- Heawood graph
- $Cay(D_{2n}, \{b, ba, ba^{r+1}\})$, with $n \geq 11$ odd and $r^2 + r + 1 \equiv 0 \pmod{n}$.

Cubic arc-transitive tricirculants, . . .

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Frelih-Kutnar (2013): A cubic arc-transitive pentacirculant is either F050A or F150A.

Cubic arc-transitive k-circulants

For which values of k are there finitely/infinitely many cubic arc-transitive k-circulants?

Even k

Theorem: For every even positive integer k, there are infinitely many cubic arc-transitive k-circulants.

Construction

- $k = 2m$ and p a prime with $p \equiv 1 \pmod 3$ and $p \nmid m$ (infinitely many such p).
- $G = \left\langle \begin{array}{l|l} u, v, w, x & u^m, v^m, w^p, x^2, [u,v], [u,w], \\ & [v,w], u^x u, v^x v, w^x w \end{array} \right\rangle$

  a generalised dihedral group on $\mathbb{Z}_m^2 \times \mathbb{Z}_p$.
- $S = \{s, s^y, s^{y^2}\}$, $R = \langle S \rangle$ and $\Gamma = \mathrm{Cay}(R, S)$.
- $R = G$ if $(3, m) = 1$ and has index 3 otherwise.
- $C = \begin{cases} \langle u^3 w \rangle & \text{if 3 divides } m \\ \langle uw \rangle & \text{otherwise} \end{cases}$

C is semiregular and has k orbits on the vertices of Γ.

Odd k

Theorem: If k is a square-free positive integer coprime to 6 and Γ is a cubic arc-transitive k-circulant then $|V\Gamma| \leqslant 6k^2$.

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Theorem: For every odd positive integer k there is a cubic arc-transitive k-circulant with $|V\Gamma| = 6k^2$.

Take a particular Cayley graph for $\mathbb{Z}_k^2 \rtimes \mathrm{Sym}(3)$.

Some key observations/lemmas

Let Γ be a cubic $(t+1)$-arc-regular graph with a semiregular cyclic group of automorphisms C with k orbits for $(6, k) = 1$

- $|G| = |G_v||C|k = 3.2^t|C|k$.

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- Each Sylow p-subgroup for each odd p has an index p cyclic subgroup.
- C has even order and its unique involution flips an edge.

Soluble case

- Each Sylow p-subgroup for $p \geq 5$ has order at most p^2 and p divides k.
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Chillag-Soon: $G = N \rtimes A$ where A is a Hall $\{2,3\}$-subgroup and N has a normal series

$$1 = N_0 \lhd N_1 \lhd \cdots \lhd N_n = N$$

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- Each N_i is semiregular and Γ_{N_i} is a k'-circulant with k' dividing k.

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- G/S is almost simple with a cyclic subgroup of even order and index dividing $3.2^t k$.
- $G^{(\infty)}$ is quasisimple, and is either semiregular or locally transitive.