Polyhedra, Polytopes and Beyond

Asia Ivić Weiss*

York University - Canada

Polyhedra, Polytopes and Beyond

(With symmetry as the central theme)

Asia Ivić Weiss*

York University - Canada

Polyhedra, Polytopes and Beyond

(With symmetry as the central theme)

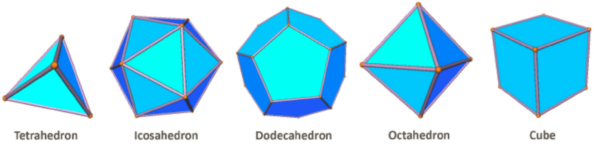
Asia Ivić Weiss*

York University - Canada

* With a lot of help from my friends!

Regular polyhedra with convex faces

FINITE:

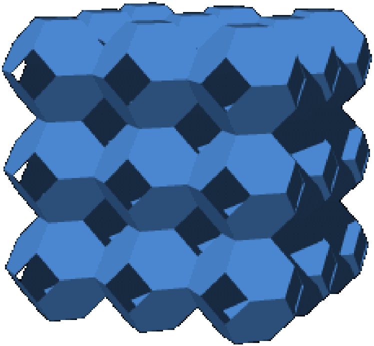


$\{3,3\}$ $\{3,5\}$ $\{5,3\}$ $\{3,4\}$ $\{4,3\}$

Regular polyhedra with convex faces

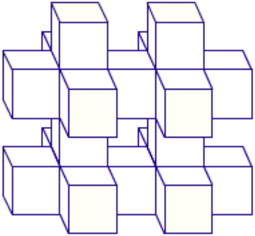
INFINITE:

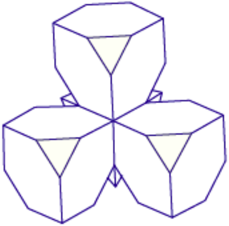
$\{6, 4|4\}$



Regular polyhedra with convex faces

$\{4, 6|4\}$ $\{6, 6|3\}$

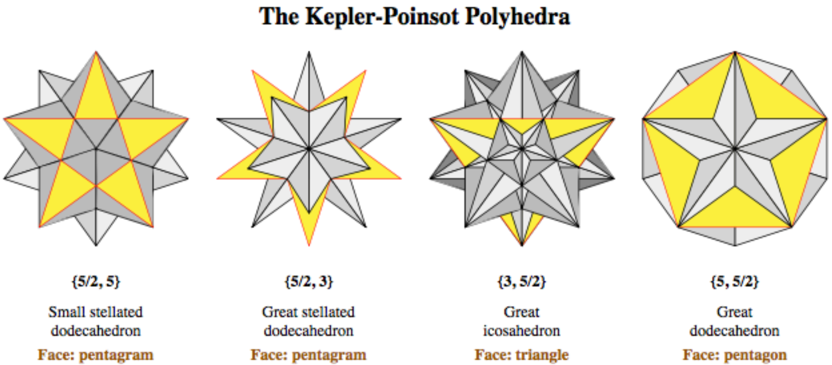




Regular polyhedra with non-convex faces or vertex-figures

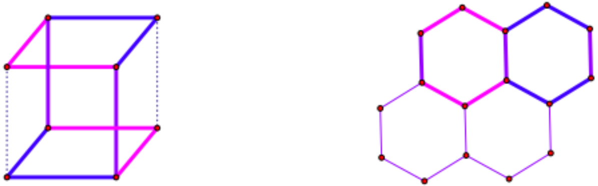
FINITE

(with planar faces)



Regular polyhedra with non-planar (finite) faces

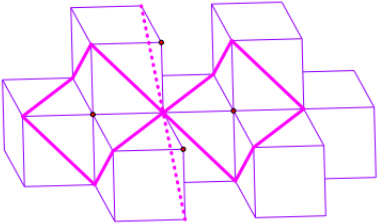
FINITE



$$\pi\{4,3\}_6 = \{6,3\}_4$$

Regular polyhedra with non-planar (finite) faces

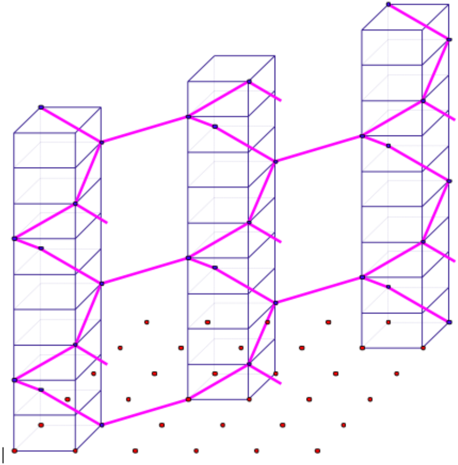
INFINITE



$\{6,6\}_4$ = one half of the vertex figures of $\{4,6|4\}$

Regular polyhedra with infinite faces

Grünbaum-Dress polyhedron $\{\infty, 3\}_{[4]}$



Abstract Polytopes

An *abstract polytope P of rank n, or an n-polytope* is a poset, whose elements are called *faces*, with strictly monotone rank function with range $\{-1, 0, 1, \ldots, n\}$ satisfying the following properties.

Abstract Polytopes

An *abstract polytope P of rank n, or an n-polytope* is a poset, whose elements are called *faces*, with strictly monotone rank function with range $\{-1, 0, 1, \ldots, n\}$ satisfying the following properties.

- P has a unique minimal face F_{-1} and a unique maximal face F_n.

Abstract Polytopes

An *abstract polytope P of rank n, or an n-polytope* is a poset, whose elements are called *faces*, with strictly monotone rank function with range $\{-1, 0, 1, \ldots, n\}$ satisfying the following properties.

- P has a unique minimal face F_{-1} and a unique maximal face F_n.
- The maximal chains, called *flags*, of P contain exactly $n + 2$ faces.

Abstract Polytopes

An *abstract polytope P of rank n, or an n-polytope* is a poset, whose elements are called *faces*, with strictly monotone rank function with range $\{-1, 0, 1, \ldots, n\}$ satisfying the following properties.

- P has a unique minimal face F_{-1} and a unique maximal face F_n.
- The maximal chains, called *flags*, of P contain exactly $n + 2$ faces.
- P is strongly flag-connected.

Abstract Polytopes

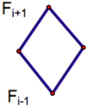
An *abstract polytope P of rank n, or an n-polytope* is a poset, whose elements are called *faces*, with strictly monotone rank function with range $\{-1, 0, 1, \ldots, n\}$ satisfying the following properties.

- P has a unique minimal face F_{-1} and a unique maximal face F_n.
- The maximal chains, called *flags*, of P contain exactly $n + 2$ faces.
- P is strongly flag-connected.
- P satisfies a homogeneity property.

Abstract Polytopes

An *abstract polytope P of rank n, or an n-polytope* is a poset, whose elements are called *faces*, with strictly monotone rank function with range $\{-1, 0, 1, \ldots, n\}$ satisfying the following properties.

- P has a unique minimal face F_{-1} and a unique maximal face F_n.
- The maximal chains, called *flags*, of P contain exactly $n + 2$ faces.
- P is strongly flag-connected.
- P satisfies a homogeneity property.



Regular Abstract Polytopes

Abstract polytope P is said to be *regular* if its group of automorphisms $Aut(P)$ is transitive on the flags of P.

Regular Abstract Polytopes

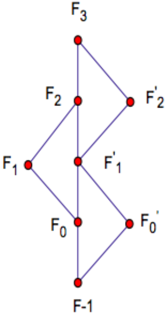
Abstract polytope P is said to be *regular* if its group of automorphisms $Aut(P)$ is transitive on the flags of P.

$\implies Aut(P)$ is generated by involutions

Regular Abstract Polytopes

Abstract polytope P is said to be *regular* if its group of automorphisms $Aut(P)$ is transitive on the flags of P.

$\Longrightarrow$ $Aut(P)$ is generated by involutions (determined by the "base" flag).



ρ_{n-1}

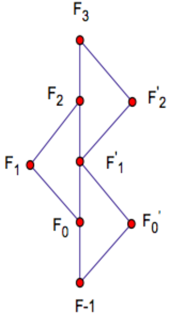
ρ_1

ρ_0

Regular Abstract Polytopes

Abstract polytope P is said to be *regular* if its group of automorphisms $Aut(P)$ is transitive on the flags of P.

$\Longrightarrow$ $Aut(P)$ is generated by involutions (determined by the "base" flag).

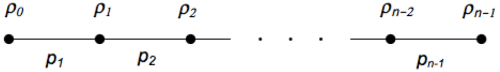


ρ_{n-1}

ρ_1

ρ_0

$C-$diagram:



Regular Abstract Polytopes

Given that P is a regular $n-$polytope and Φ one of its flags, $Aut(P)$ is generated by the distinguished generators ρ_i, $i = 0, \ldots, n-1$, that interchange Φ with its $i-$adjecent flag Φ^i and satisfy the relations implicit in the string Coxeter graph associated with the string Coxeter group $[p_1, \ldots, p_{n-1}]$.

Regular Abstract Polytopes

Given that P is a regular $n-$polytope and Φ one of its flags, $Aut(P)$ is generated by the distinguished generators ρ_i, $i = 0, \ldots, n-1$, that interchange Φ with its $i-$adjecent flag Φ^i and satisfy the relations implicit in the string Coxeter graph associated with the string Coxeter group $[p_1, \ldots, p_{n-1}]$.

$\Rightarrow$ Regular polytopes can be assigned a Schläfli type $\{p_1, \ldots, p_{n-1}\}$.

Regular Abstract Polytopes

Given that P is a regular $n-$polytope and Φ one of its flags, $Aut(P)$ is generated by the distinguished generators ρ_i, $i = 0, \ldots, n-1$, that interchange Φ with its $i-$adjecent flag Φ^i and satisfy the relations implicit in the string Coxeter graph associated with the string Coxeter group $[p_1, \ldots, p_{n-1}]$.

$\Rightarrow$ Regular polytopes can be assigned a Schläfli type $\{p_1, \ldots, p_{n-1}\}$.

The generators of the automorphism group of an abstract polytope satisfy an intersection property IP:

$$\langle \rho_i \mid i \in I \rangle \cap \langle \rho_i \mid i \in J \rangle = \langle \rho_i \mid i \in I \cap J \rangle, \qquad \forall I, J \subseteq \{0, \ldots, n-1\}.$$

Characterization of Groups of Regular Abstract Polytopes

A quotient of a string Coxeter group $[p_1, \ldots, p_{n-1}]$ with generators that satisfy the intersection property IP is called a $C-$group.

Characterization of Groups of Regular Abstract Polytopes

A quotient of a string Coxeter group $[p_1, \ldots, p_{n-1}]$ with generators that satisfy the intersection property IP is called a $C-$group.

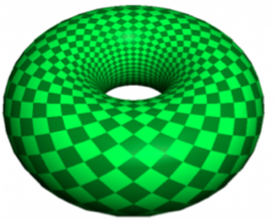
Theorem (Schulte, 1982): Given a $C-$group one can construct a regular polytope having this group as its automorphism group.

Characterization of Groups of Regular Abstract Polytopes

A quotient of a string Coxeter group $[p_1, \ldots, p_{n-1}]$ with generators that satisfy the intersection property IP is called a $C-$group.

Theorem (Schulte, 1982): Given a $C-$group one can construct a regular polytope having this group as its automorphism group.

Example: From a quotient of the Coxeter group $[4, 4]$ by a translation subgroup one can construct regular polytope of rank 3 (a regular map on torus).



Regular Honeycombs and Chirality

"I call any geometrical figure, or group of points, chiral, and say that it has chirality, if its image in a plane mirror, ideally realized, cannot be brought to coincide with itself." William Thomson (Lord Kelvin), Baltimore Lectures, John Hopkins University, 1884.

Regular Honeycombs and Chirality

"I call any geometrical figure, or group of points, chiral, and say that it has chirality, if its image in a plane mirror, ideally realized, cannot be brought to coincide with itself." William Thomson (Lord Kelvin), Baltimore Lectures, John Hopkins University, 1884.

Introduced by Sommerville in 1929, a homogeneous honeycomb is a structure in euclidean space consisting of polyhedral cells, all alike, such that each rotation that is the symmetry operation of a cell is also a symmetry operation of the whole configuration.

Regular Honeycombs and Chirality

"I call any geometrical figure, or group of points, chiral, and say that it has chirality, if its image in a plane mirror, ideally realized, cannot be brought to coincide with itself." William Thomson (Lord Kelvin), Baltimore Lectures, John Hopkins University, 1884.

Introduced by Sommerville in 1929, a homogeneous honeycomb is a structure in euclidean space consisting of polyhedral cells, all alike, such that each rotation that is the symmetry operation of a cell is also a symmetry operation of the whole configuration.

Coxeter defines regularity in maps following Sommerville's ideas, and gives the classification of reflexible and irreflexible maps on torus in 1948.

Regular Honeycombs and Chirality

"I call any geometrical figure, or group of points, chiral, and say that it has chirality, if its image in a plane mirror, ideally realized, cannot be brought to coincide with itself." William Thomson (Lord Kelvin), Baltimore Lectures, John Hopkins University, 1884.

Introduced by Sommerville in 1929, a homogeneous honeycomb is a structure in euclidean space consisting of polyhedral cells, all alike, such that each rotation that is the symmetry operation of a cell is also a symmetry operation of the whole configuration.

Coxeter defines regularity in maps following Sommerville's ideas, and gives the classification of reflexible and irreflexible maps on torus in 1948. In 1970 he attempts to generalize the idea to higher dimensions and defines a twisted honeycomb as a combinatorial structure derived from a 3−dimensional honeycomb by preserving all rotations of its polyhedral cells but abandoning its reflectional symmetries.

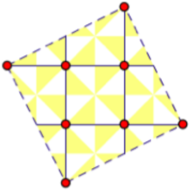
Chiral Abstract Polytopes

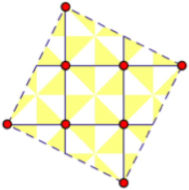
Abstract polytope P is said to be *chiral* if its group of automorphisms $Aut(P)$ has exactly two orbits on the flags, with adjacent flags in distinct orbits.

Chiral Abstract Polytopes

Abstract polytope P is said to be *chiral* if its group of automorphisms $Aut(P)$ has exactly two orbits on the flags, with adjacent flags in distinct orbits.

Example: A chiral rank 3 toroidal polytope with Schläfly type $\{4,4\}$:





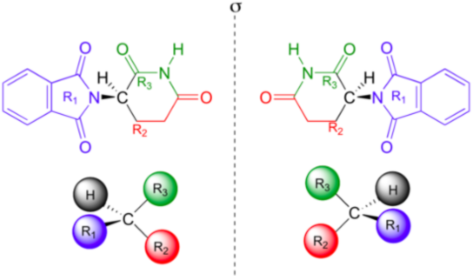
Chirality in Chemistry

Thalidomide drug was discovered in 1953 and used to treat "morning sickness" in pregnant women; birth defects detected in 1960.

Chirality in Chemistry

Thalidomide drug was discovered in 1953 and used to treat "morning sickness" in pregnant women; birth defects detected in 1960.

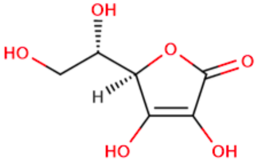
S- and R- isomeric forms of thalidomide molecules:



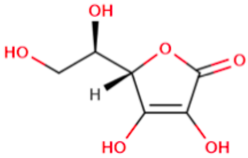
One of the isomers is an effective medication, the other caused the side effects. Both isomeric forms have the same molecular formula and the same atom-to-atom connectivity. Where they differ is in the arrangement in three-dimensional space about one tetrahedral, sp3-hybridized carbon.

Ascorbic acid comes in L-and D-isomeric forms:

L-ascorbate



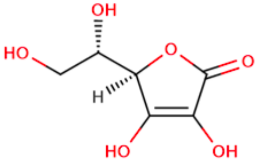
D-ascorbate



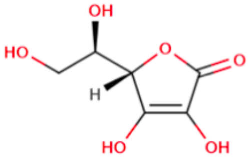
Chirality in Chemistry

Ascorbic acid comes in L-and D-isomeric forms:

L-ascorbate



D-ascorbate



The name vitamin C always refers to the L-enantiomer of ascorbic acid (and of its oxidized forms). The D-enantiomer (called D-ascorbate) is not found in nature. It has equal antioxidant power; however, when synthesized and given to animals that require vitamin C in their diets, it has been found to have far less vitamin activity than the L-enantiomer.

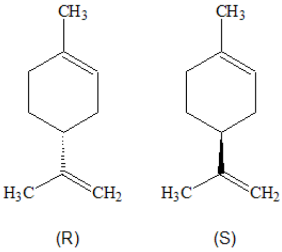
Chirality in Chemistry

Chirality of smell: The nerve-ending receptors in nose absorb molecules and send an impulse to brain. The brain then interprets it as the smell. Molecules with different shapes fit into different receptors (a receptor shaped in a "right-handed" chiral form would interact only with a "right-handed molecule").

Chirality in Chemistry

Chirality of smell: The nerve-ending receptors in nose absorb molecules and send an impulse to brain. The brain then interprets it as the smell. Molecules with different shapes fit into different receptors (a receptor shaped in a "right-handed" chiral form would interact only with a "right-handed molecule").

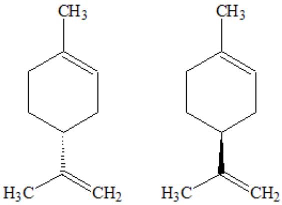
Limonene is the molecule producing smell in orange and lemon peel.



Chirality in Chemistry

Chirality of smell: The nerve-ending receptors in nose absorb molecules and send an impulse to brain. The brain then interprets it as the smell. Molecules with different shapes fit into different receptors (a receptor shaped in a "right-handed" chiral form would interact only with a "right-handed molecule").

Limonene is the molecule producing smell in orange and lemon peel.



(R) (S)

When oxidized the molecule of limonene produces carvone, the two versions of which give smells to spearmint and caraway.

Abstract Chirality - A Historical Note

Chiral polytopes are "maximally symmetric" by rotations in the sense that rotations of each rank 2 section of a polytope extend to a (rotational) symmetry of the polytope.

Abstract Chirality - A Historical Note

Chiral polytopes are "maximally symmetric" by rotations in the sense that rotations of each rank 2 section of a polytope extend to a (rotational) symmetry of the polytope.

Coxeter (1948) Classified regular and chairs maps on torus.
Sherk (1962) Constructed a family of chiral maps of genus 7.
Garbe (1969) There are no chiral maps on surfaces of genus 3, 4, 5, or 6.

Abstract Chirality - A Historical Note

Chiral polytopes are "maximally symmetric" by rotations in the sense that rotations of each rank 2 section of a polytope extend to a (rotational) symmetry of the polytope.

Coxeter (1948) Classified regular and chairs maps on torus.
Sherk (1962) Constructed a family of chiral maps of genus 7.
Garbe (1969) There are no chiral maps on surfaces of genus 3, 4, 5, or 6.

... Edmonds, Wilson, Jones, Conder, Nedela, Širáň, Schulte, Monson, Pisanski, Hubard, Pellicer, Leemans, ...

Abstract Chirality - A Historical Note

Chiral polytopes are "maximally symmetric" by rotations in the sense that rotations of each rank 2 section of a polytope extend to a (rotational) symmetry of the polytope.

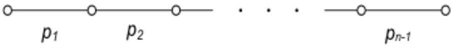
Coxeter (1948) Classified regular and chairs maps on torus.
Sherk (1962) Constructed a family of chiral maps of genus 7.
Garbe (1969) There are no chiral maps on surfaces of genus 3, 4, 5, or 6.

... Edmonds, Wilson, Jones, Conder, Nedela, Širáň, Schulte, Monson, Pisanski, Hubard, Pellicer, Leemans, ...

Zhang (2015) The smallest chiral 6-polytopes have 18432 flags. In fact, there are just two of them (of types $\{3, 3, 4, 6, 3\}$ and $\{3, 6, 4, 3, 3\}$).

Characterization of Groups of Chiral Abstract Polytopes

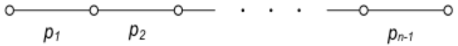
Groups of chiral abstract polytopes can be represented by the diagram



where edges represent the generating rotations $\sigma_1, \ldots, \sigma_{n-1}$ which cyclically permute the faces of a rank 2 sections determined by a base flag.

Characterization of Groups of Chiral Abstract Polytopes

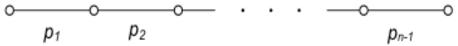
Groups of chiral abstract polytopes can be represented by the diagram



where edges represent the generating rotations $\sigma_1, \ldots, \sigma_{n-1}$ which cyclically permute the faces of a rank 2 sections determined by a base flag. More precisely, σ_i maps the base flag Φ to $(\Phi^i)^{i+1}$.

Characterization of Groups of Chiral Abstract Polytopes

Groups of chiral abstract polytopes can be represented by the diagram

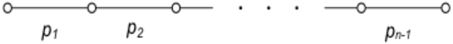


where edges represent the generating rotations $\sigma_1, \ldots, \sigma_{n-1}$ which cyclically permute the faces of a rank 2 sections determined by a base flag. More precisely, σ_i maps the base flag Φ to $(\Phi^i)^{i+1}$.

When so oriented, the product of two or more consecutive such rotations is an involution.

Characterization of Groups of Chiral Abstract Polytopes

Groups of chiral abstract polytopes can be represented by the diagram



where edges represent the generating rotations $\sigma_1, \ldots, \sigma_{n-1}$ which cyclically permute the faces of a rank 2 sections determined by a base flag. More precisely, σ_i maps the base flag Φ to $(\Phi^i)^{i+1}$.

When so oriented, the product of two or more consecutive such rotations is an involution.

The generators satisfy an *intersection property IP^+*

$$\langle \sigma_i \mid i \in I \rangle \cap \langle \sigma_i \mid i \in J \rangle = \langle \sigma_i \mid i \in I \cap J \rangle, \quad \forall I, J \subseteq \{1, \ldots, n-1\}.$$

Characterization of Groups of Chiral Abstract Polytopes

A group generated by rotations with a string diagram satisfying the intersection condition IP^+ is called C^+- group.

Characterization of Groups of Chiral Abstract Polytopes

A group generated by rotations with a string diagram satisfying the intersection condition IP^+ is called C^+- group.

Theorem (Schulte, Ivić Weiss 1991): Given a C^+- group one can construct a regular or a chiral polytope having this group as its automorphism group. The polytope is chiral if and only if there is no (involutory) automorphism which extends this group to the "corresponding" $C-$group.

Geometric Polyhedra

A geometric polyhedron is a discrete faithful realization of an abstract rank 3 polytope in E^3.

Geometric Polyhedra

A geometric polyhedron is a discrete faithful realization of an abstract rank 3 polytope in E^3.

A polyhedron in E^3 is said to be geometrically regular if its symmetry group (the group of isometries keeping the polyhedron invariant) is transitive on the set of its flags.

Geometric Polyhedra

A geometric polyhedron is a discrete faithful realization of an abstract rank 3 polytope in E^3.

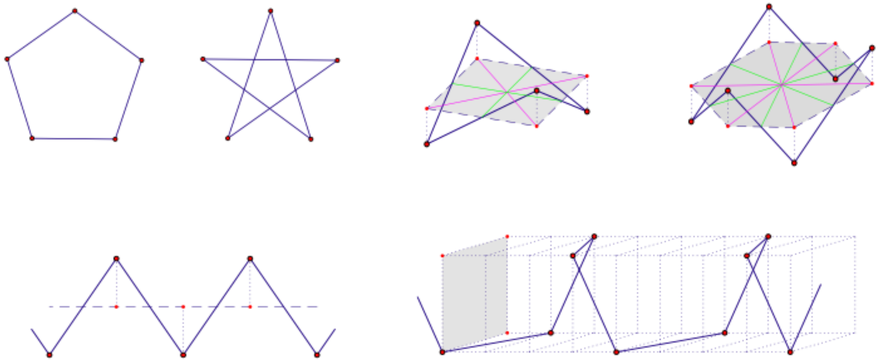
A polyhedron in E^3 is said to be geometrically regular if its symmetry group (the group of isometries keeping the polyhedron invariant) is transitive on the set of its flags.

A polyhedron in E^3 is said to be geometrically chiral if its symmetry group has two orbits on the flags with the adjacent flags always being in distinct orbits.

Faces of both geometrically regular and chiral polyhedra must be regular polygons.

Geometric Polyhedra

Regular polygons in E^3:



Geometrically Regular Polyhedra

Classification (Grünbaum-Dress 1985):

Platonic solids	$\{3,3\}$ $\{3,4\}$ $\{4,3\}$ $\{3,5\}$ $\{5,3\}$	5
Kepler-Poinsot polyhedra	$\{3,5/2\}$ $\{5/2,3\}$ $\{5,5/2\}$ $\{5/2,5\}$	4
Petrials of these	$\cdots$	9
Regular tessellations of E^2	$\{4,4\}$ $\{3,6\}$ $\{6,3\}$	3
Blends of these with segments	$\cdots$	3
Blends of these with $\{\infty\}$	$\cdots$	3
Petrials of these	$\cdots$	9
Petrie-Coxeter polyhedra	$\{4,6\|4\}$ $\{6,4\|4\}$ $\{6,6\|3\}$	3
Grünbaum-Dress polyhedra		9
		48

18 finite polyhedra
6 planar polyhedra
24 infinite 3-dimensional polyhedra

Geometrically Chiral Polyhedra

A geometrically chiral polyhedron is either a chiral or a regular abstract polyhedron (rank 3 polytope)!

Geometrically Chiral Polyhedra

A geometrically chiral polyhedron is either a chiral or a regular abstract polyhedron (rank 3 polytope)!

$\Longrightarrow$

Theorem: Finite chiral (geometric) polyhedra do not exist.

Geometrically Chiral Polyhedra

A geometrically chiral polyhedron is either a chiral or a regular abstract polyhedron (rank 3 polytope)!

$\Longrightarrow$

Theorem: Finite chiral (geometric) polyhedra do not exist.

Theorem (Schulte 2005): Discrete chiral polyhedra can be classified in the following six families.

Finite faced polyhedra:	$\{6,6\}_{[a,b]}$	$\{4,6\}_{[a,b]}$	$\{6,4\}_{[a,b]}$
Infinite faced polyhedra:	$\{\infty,3\}_{[3]}$	$\{\infty,3\}_{[4]}$	$\{\infty,4\}_{[3]}$

Geometrically Chiral Polyhedra

A geometrically chiral polyhedron is either a chiral or a regular abstract polyhedron (rank 3 polytope)!

$\Longrightarrow$

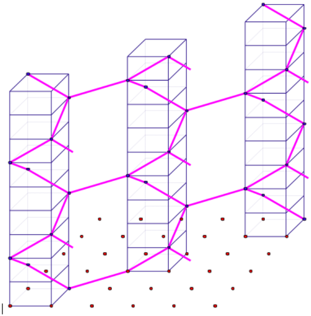
Theorem: Finite chiral (geometric) polyhedra do not exist.

Theorem (Schulte 2005): Discrete chiral polyhedra can be classified in the following six families.

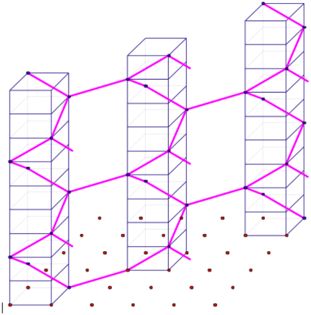
Finite faced polyhedra:	$\{6,6\}_{[a,b]}$	$\{4,6\}_{[a,b]}$	$\{6,4\}_{[a,b]}$
Infinite faced polyhedra:	$\{\infty,3\}_{[3]}$	$\{\infty,3\}_{[4]}$	$\{\infty,4\}_{[3]}$

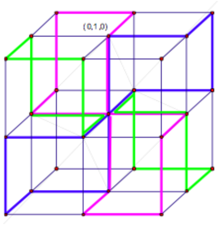
Theorem (Pellicer, Ivić Weiss (2010): Chiral polyhedra with finite faces are abstract chiral polyhedra. The chiral polyhedra with infinite faces are regular abstract polyhedra.

Geometrically Chiral Polyhedra

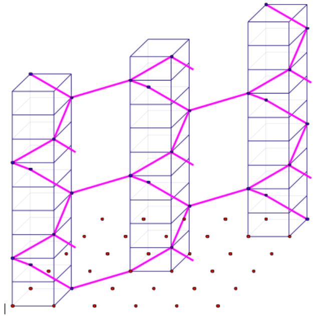


Geometrically Chiral Polyhedra

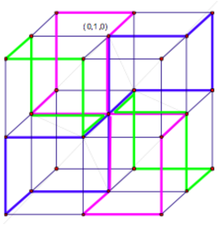




Geometrically Chiral Polyhedra



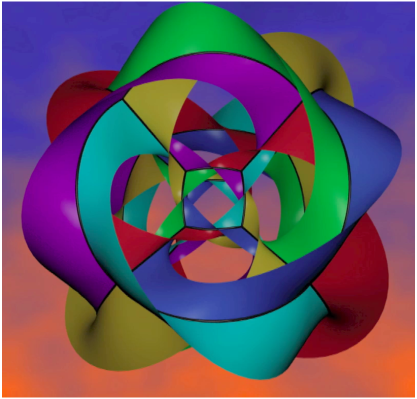
$\{\infty, 3\}_{[4]}$



$\{6, 6\}_{[1,0]}$

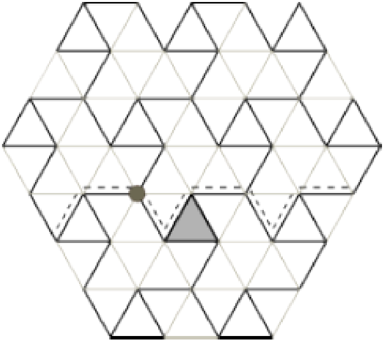
Geometrically Chiral 4-Polytope in E^4

Roli's Cube (Bracho, Hubard, Pellicer 2014) is geometrically chiral, but abstractly regular.



Geometrically Chiral 4-Polytope in E^3

$P_{\{\infty,3,4\}}$ has eight infinite facets $\{\infty,3\}_{[3]}$ arranged as images of one of them under the group $[3,4]^+$ of rotations of the octahedron centred at one of its vertices. It is abstractly and geometrically chiral (Pellicer 2015).



Incidence Systems

We next extend the concept of a polytope to a more general structure.

Incidence Systems

We next extend the concept of a polytope to a more general structure.

An incidence system $\Gamma := (X, *, t, I)$ is a 4-tuple such that

- X is a set whose elements are called the elements of Γ;
- I is a finite set whose elements are called the types of Γ;
- $t : X \to I$ is a type function, associating to each element $x \in X$ of Γ a type $t(x) \in I$;
- $*$ is a binary relation on X called incidence, that is reflexive, symmetric and such that for all $x, y \in X$, if $x * y$ and $t(x) = t(y)$ then $x = y$.

Incidence Systems

We next extend the concept of a polytope to a more general structure.

An incidence system $\Gamma := (X, *, t, I)$ is a 4-tuple such that

- X is a set whose elements are called the elements of Γ;
- I is a finite set whose elements are called the types of Γ;
- $t : X \to I$ is a type function, associating to each element $x \in X$ of Γ a type $t(x) \in I$;
- $*$ is a binary relation on X called incidence, that is reflexive, symmetric and such that for all $x, y \in X$, if $x * y$ and $t(x) = t(y)$ then $x = y$.

The incidence graph of Γ is the graph whose vertex set is X and where two vertices are joined if the corresponding elements of Γ are incident.

Incidence Systems

We next extend the concept of a polytope to a more general structure.

An incidence system $\Gamma := (X, *, t, I)$ is a 4-tuple such that

- X is a set whose elements are called the elements of Γ;
- I is a finite set whose elements are called the types of Γ;
- $t : X \rightarrow I$ is a type function, associating to each element $x \in X$ of Γ a type $t(x) \in I$;
- $*$ is a binary relation on X called incidence, that is reflexive, symmetric and such that for all $x, y \in X$, if $x * y$ and $t(x) = t(y)$ then $x = y$.

The incidence graph of Γ is the graph whose vertex set is X and where two vertices are joined if the corresponding elements of Γ are incident.

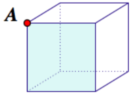
The rank of Γ is the cardinality of I.

Incidence Geometry

A flag is a set of pairwise incident elements of Γ.

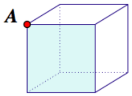
Incidence Geometry

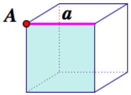
A flag is a set of pairwise incident elements of Γ. An element of Γ is incident to a flag F if it is incident to elements of F.



Incidence Geometry

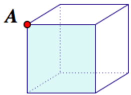
A flag is a set of pairwise incident elements of Γ. An element of Γ is incident to a flag F if it is incident to elements of F.

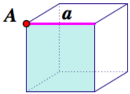




Incidence Geometry

A flag is a set of pairwise incident elements of Γ. An element of Γ is incident to a flag F if it is incident to elements of F.

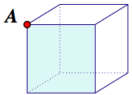


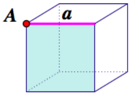


The type of a flag F is $\{t(x) : x \in F\}$. A chamber is a flag of type I.

Incidence Geometry

A flag is a set of pairwise incident elements of Γ. An element of Γ is incident to a flag F if it is incident to elements of F.





The type of a flag F is $\{t(x) : x \in F\}$. A chamber is a flag of type I.

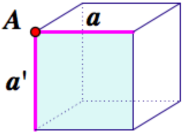
An incidence system Γ is a geometry (or incidence geometry) if every flag of Γ is contained in a chamber.

Thin Geometries

A geometry Γ is called thin if for each $i \in I$ any flag of type $I \setminus \{i\}$ is contained in exactly two chambers.

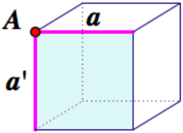
Thin Geometries

A geometry Γ is called thin if for each $i \in I$ any flag of type $I \setminus \{i\}$ is contained in exactly two chambers.



Thin Geometries

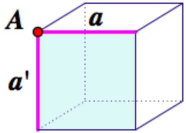
A geometry Γ is called thin if for each $i \in I$ any flag of type $I \setminus \{i\}$ is contained in exactly two chambers.



Equivalently, in the language of incidence geometries, geometry Γ is thin if every residue of rank 1 of Γ contains exactly two elements.

Thin Geometries

A geometry Γ is called thin if for each $i \in I$ any flag of type $I \setminus \{i\}$ is contained in exactly two chambers.



Equivalently, in the language of incidence geometries, geometry Γ is thin if every residue of rank 1 of Γ contains exactly two elements.

The diamond condition in the definition of abstract polytopes guaranties that abstract polytopes are thin geometries.

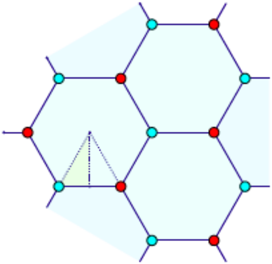
Examples

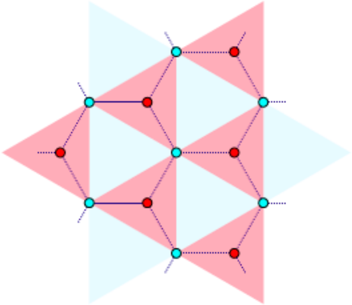
Polytopes and non-degenerate maps and hypermaps are examples of thin geometries.

Examples

Polytopes and non-degenerate maps and hypermaps are examples of thin geometries.

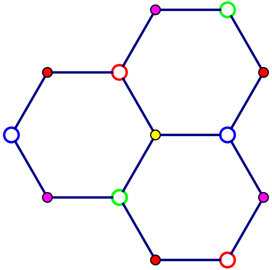
Hypermap $(3,3,3)_{(b,c)}$ on torus has vertices, edges and faces of valency 3:





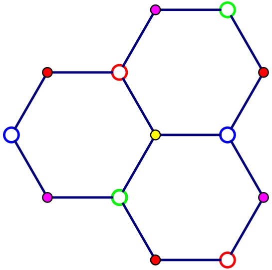
Examples

An example of geometry that is not thin: toroidal hypermap $(3,3,3)_{(1,1)}$.



Examples

An example of geometry that is not thin: toroidal hypermap $(3, 3, 3)_{(1,1)}$.



This hypermap has 3 vertices, 3 edges and 3 faces and its incidence graph is $K_{3,3,3}$.

Automorphisms of Thin Geometry

An automorphism of $\Gamma := (X, *, t, I)$ is a mapping $\alpha : X \mapsto X$ such that for all $x, y \in X$

- α is a bijection on X (inducing a bijection on I);
- $x * y$ if and only if $\alpha(x) * \alpha(y)$;
- $t(x) = t(y)$ if and only if $t(\alpha(x)) = t(\alpha(y))$.

Automorphisms of Thin Geometry

An automorphism of $\Gamma := (X, *, t, I)$ is a mapping $\alpha : X \mapsto X$ such that for all $x, y \in X$

- α is a bijection on X (inducing a bijection on I);
- $x * y$ if and only if $\alpha(x) * \alpha(y)$;
- $t(x) = t(y)$ if and only if $t(\alpha(x)) = t(\alpha(y))$.

An automorphism is type preserving when for each $x \in X$, $t(\alpha(x)) = t(x)$.

Automorphisms of Thin Geometry

An automorphism of $\Gamma := (X, *, t, I)$ is a mapping $\alpha : X \mapsto X$ such that for all $x, y \in X$

- α is a bijection on X (inducing a bijection on I);
- $x * y$ if and only if $\alpha(x) * \alpha(y)$;
- $t(x) = t(y)$ if and only if $t(\alpha(x)) = t(\alpha(y))$.

An automorphism is type preserving when for each $x \in X$, $t(\alpha(x)) = t(x)$. The set of all type preserving automorphism of Γ is denoted by $Aut_I(\Gamma)$.

Automorphisms of Thin Geometry

An automorphism of $\Gamma := (X, *, t, I)$ is a mapping $\alpha : X \mapsto X$ such that for all $x, y \in X$

- α is a bijection on X (inducing a bijection on I);
- $x * y$ if and only if $\alpha(x) * \alpha(y)$;
- $t(x) = t(y)$ if and only if $t(\alpha(x)) = t(\alpha(y))$.

An automorphism is type preserving when for each $x \in X$, $t(\alpha(x)) = t(x)$. The set of all type preserving automorphism of Γ is denoted by $Aut_I(\Gamma)$.

Γ is chamber transitive if $Aut_I(\Gamma)$ is transitive on the set of chambers of Γ.

Hypertopes

A hypertope is a thin incidence geometry that is strongly chamber connected (SCC).

Hypertopes

A hypertope is a thin incidence geometry that is strongly chamber connected (SCC). (Or, residually connected as commonly used in the terminology of incidence geometries).

Hypertopes

A hypertope is a thin incidence geometry that is strongly chamber connected (SCC). (Or, residually connected as commonly used in the terminology of incidence geometries).

A hypertope Γ is said to be

- regular if $Aut_I(\Gamma)$ has one orbit on the chambers of Γ;

Hypertopes

A hypertope is a thin incidence geometry that is strongly chamber connected (SCC). (Or, residually connected as commonly used in the terminology of incidence geometries).

A hypertope Γ is said to be

- regular if $Aut_I(\Gamma)$ has one orbit on the chambers of Γ;
- chiral has two orbits on the chambers of Γ such that any two adjacent chambers (differing in one element only) lie in distinct orbits.

Groups of Regular Hypertopes

Let Γ be a regular hypertope and Φ one of its chambers. Then for each $i \in I$ there exists and involutory type-preserving automorphism ρ_i that interchanges Φ with its i-adjacent chamber Φ^i.

Groups of Regular Hypertopes

Let Γ be a regular hypertope and Φ one of its chambers. Then for each $i \in I$ there exists and involutory type-preserving automorphism ρ_i that interchanges Φ with its i-adjacent chamber Φ^i.

$Aut_I(\Gamma)$ is generated by the distinguished generators $\{\rho_0, \rho_1, \ldots, \rho_{n-1}\}$, where $n = |I|$, which satisfy

- the relations implicit in the $C-$diagram, the complete graph on n vertices whose vertices are labeled by the generators and the edges between vertices labelled with ρ_i and ρ_j labeled by $o(\rho_i \rho_j)$ (with the usual convention of omitting the edges labeled by 2);

Groups of Regular Hypertopes

Let Γ be a regular hypertope and Φ one of its chambers. Then for each $i \in I$ there exists and involutory type-preserving automorphism ρ_i that interchanges Φ with its i-adjacent chamber Φ^i.

$Aut_I(\Gamma)$ is generated by the distinguished generators $\{\rho_0, \rho_1, \ldots, \rho_{n-1}\}$, where $n = |I|$, which satisfy

- the relations implicit in the $C-$diagram, the complete graph on n vertices whose vertices are labeled by the generators and the edges between vertices labelled with ρ_i and ρ_j labeled by $o(\rho_i \rho_j)$ (with the usual convention of omitting the edges labeled by 2);
- and the intersection property IP

$$\langle \rho_i \mid i \in I \rangle \cap \langle \rho_i \mid i \in J \rangle = \langle \rho_i \mid i \in I \cap J \rangle, \qquad \forall I, J \subseteq \{0, \ldots, n-1\}.$$

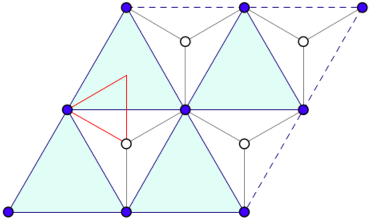
$C-$Groups

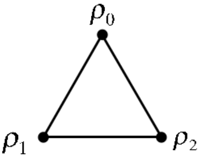
A pair (G, R), where G is a group and $R = \{\rho_0, \ldots, \rho_{n-1}\}$ its generating set of involutions that satisfy the IP, is called a $C-$group.

$C-$Groups

A pair (G, R), where G is a group and $R = \{\rho_0, \ldots, \rho_{n-1}\}$ its generating set of involutions that satisfy the IP, is called a $C-$group.

The group $\langle \rho_0, \rho_1, \rho_2 \mid \rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1\rho_2\rho_1)^2 = 1 \rangle$ with the triangular $C-$diagram is the group of automorphisms of the hypermap $(3, 3, 3)_{(2,0)}$.





Groups of Chiral Hypertopes

Let Γ be a chiral hypertope and Φ one of its chambers. For any pair $i \neq j \in I = \{0, \ldots, n-1\}$, there exists a type-preserving automorphism α_{ij} mapping the chamber Φ to $(\Phi^i)^j$.

Groups of Chiral Hypertopes

Let Γ be a chiral hypertope and Φ one of its chambers. For any pair $i \neq j \in I = \{0, \ldots, n-1\}$, there exists a type-preserving automorphism α_{ij} mapping the chamber Φ to $(\Phi^i)^j$.

- The generators α_{ij} satisfy the intersection property IP^+

$$\langle \alpha_{ij} \mid i,j \in J \rangle \cap \langle \alpha_{ij} \mid i,j \in K \rangle = \langle \alpha_{ij} \mid i,j \in J \cap K \rangle, \quad \forall J, K \subseteq I.$$

Groups of Chiral Hypertopes

Let Γ be a chiral hypertope and Φ one of its chambers. For any pair $i \neq j \in I = \{0, \ldots, n-1\}$, there exists a type-preserving automorphism α_{ij} mapping the chamber Φ to $(\Phi^i)^j$.

- The generators α_{ij} satisfy the intersection property IP^+

$$\langle \alpha_{ij} \mid i,j \in J \rangle \cap \langle \alpha_{ij} \mid i,j \in K \rangle = \langle \alpha_{ij} \mid i,j \in J \cap K \rangle, \quad \forall J, K \subseteq I.$$

- $Aut_I(\Gamma)$ is generated by the distinguished generators

$$\alpha_i := \alpha_{0i} \quad \text{for} \quad i = 1, \ldots, n-1.$$

(Here $\alpha_{ij} = \alpha_i^{-1}\alpha_j$.) The set of generators $R = \{\alpha_1, \ldots, \alpha_{n-1}\}$ is independent, meaning that $\alpha_i \notin \langle \alpha_j \mid j \neq i \rangle$.

C^+-Groups and $B-$Diagrams

A pair (G^+, R) with $G^+ = \langle R \rangle$ and $R = \{\alpha_1, \ldots, \alpha_{n-1}\}$ an independent set of generators satisfying IP^+ (with $\alpha_{ij} = \alpha_i^{-1}\alpha_j$) is called a C^+-group.

C^+-Groups and $B-$Diagrams

A pair (G^+, R) with $G^+ = \langle R \rangle$ and $R = \{\alpha_1, \ldots, \alpha_{n-1}\}$ an independent set of generators satisfying IP^+ (with $\alpha_{ij} = \alpha_i^{-1}\alpha_j$) is called a C^+-group.

The B-diagram of a C^+-group (G^+, R) is the graph defined as follows.

- The vertex set of the graph is the set $R \cup \{\alpha_0 := 1_{G^+}\}$.
- The two vertices α_i and α_j of the graph are connected by an edge labeled by $o(\alpha_i^{-1}\alpha_j)$ whenever $o(\alpha_i^{-1}\alpha_j) \neq 2$ (with the usual convention of omitting label 3).

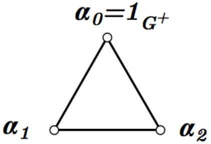
C^+-Groups and $B-$Diagrams

A pair (G^+, R) with $G^+ = \langle R \rangle$ and $R = \{\alpha_1, \ldots, \alpha_{n-1}\}$ an independent set of generators satisfying IP^+ (with $\alpha_{ij} = \alpha_i^{-1}\alpha_j$) is called a C^+-group.

The B-diagram of a C^+-group (G^+, R) is the graph defined as follows.

- The vertex set of the graph is the set $R \cup \{\alpha_0 := 1_{G^+}\}$.
- The two vertices α_i and α_j of the graph are connected by an edge labeled by $o(\alpha_i^{-1}\alpha_j)$ whenever $o(\alpha_i^{-1}\alpha_j) \neq 2$ (with the usual convention of omitting label 3).

Example $B-$diagram for the group of a chiral hypertope $(3,3,3)_{(b,c)}$ with $bc(b-c) \neq 0$:



Coset Geometry

Construction of an incidence geometry from a group (Tits, 1961):

Let G be a group and $(G_i)_{i \in I}$ a finite family of subgroups of G. With X, $*$ and t defined as

- X is the set of all cosets $G_i g$, $g \in G$, $i \in I$;
- $t : X \to I$ defined by $t(G_i g) = i$;
- $G_i g_1 * G_j g_2$ if a and only if $G_i g_1 \cap G_j g_2 \neq \emptyset$;

$\Gamma := (X, *, t, I)$ is an incidence system.

Coset Geometry

Construction of an incidence geometry from a group (Tits, 1961):

Let G be a group and $(G_i)_{i \in I}$ a finite family of subgroups of G. With X, $*$ and t defined as

- X is the set of all cosets $G_i g$, $g \in G$, $i \in I$;
- $t : X \rightarrow I$ defined by $t(G_i g) = i$;
- $G_i g_1 * G_j g_2$ if a and only if $G_i g_1 \cap G_j g_2 \neq \emptyset$;

$\Gamma := (X, *, t, I)$ is an incidence system. When Γ is a geometry, we call it a coset geometry, denote it by $\Gamma(G, (G_i)_{i \in I})$ and call G_i its maximal parabolic subgroups.

Question: When is such an incidence geometry a hypertope?

Regular Hypertopes From Groups

Theorem (Fernandes, Leemans and Ivić Weiss, 2014) Given that $(G, \{\rho_0, \rho_1, \rho_2\})$ is a $C-$group of rank 3, the coset geometry $\Gamma(G, (\langle \rho_1, \rho_2 \rangle, \langle \rho_0, \rho_2 \rangle, \langle \rho_0, \rho_1 \rangle))$ is thin if and only if G acts faithfully on Γ and is transitive on chambers. Moreover, if it is thin it is strongly chamber-connected.

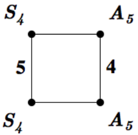
Regular Hypertopes From Groups

Theorem (Fernandes, Leemans and Ivić Weiss, 2014) Given that $(G, \{\rho_0, \rho_1, \rho_2\})$ is a $C-$group of rank 3, the coset geometry $\Gamma(G, (\langle \rho_1, \rho_2 \rangle, \langle \rho_0, \rho_2 \rangle, \langle \rho_0, \rho_1 \rangle))$ is thin if and only if G acts faithfully on Γ and is transitive on chambers. Moreover, if it is thin it is strongly chamber-connected.

The $C-$group with triangular $C-$diagram, seen above as the automorphism group of the hypermap $(3,3,3)_{(1,1)}$, gives a coset geometry that is not thin (it is however strongly chamber-connected).

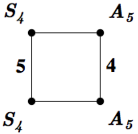
Regular Hypertopes From Groups

Unfortunately in higher ranks thinness need not suffice:



is a $C-$group, but the induced coset geometry is not thin, it is not strongly chamber-connected, nor flag transitive.

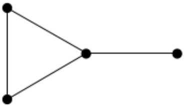
Unfortunately in higher ranks thinness need not suffice:



is a $C-$group, but the induced coset geometry is not thin, it is not strongly chamber-connected, nor flag transitive.

Theorem (Fernandes, Leemans and Ivić Weiss, 2014) Given that $(G, S = \{\rho_0, \rho_1, \ldots, \rho_{n-1}\})$ is a $C-$group of rank n, the coset geometry $\Gamma := \Gamma(G, (G_i)_{i \in I})$ with $G_i := \langle \rho_j \mid \rho_j \in S, j \in I \setminus \{i\} \rangle$ for all $i \in I := \{0, 1, \ldots, n-1\}$, if Γ is flag transitive, then Γ is regular incidence geometry (it is thin, SCC and regular giving a regular hypertope).

Example: A rank 4 hypertope related to the tessellation $\{6, 3, 3\}$ of the hyperbolic space.



Chiral Hypertopes From Groups

Similarly, starting with a group G^+ and a set $R = \{\alpha_1, \ldots, \alpha_{n-1}\}$ of independent generators, we can construct a coset geometry $\Gamma(G^+, R) := \Gamma(G^+, (G_i)_{i \in \{0,\ldots,n-1\}})$ where $G_i := \langle \alpha_j \mid j \neq i \rangle$ for $i = 1, \ldots, n-1$ and $G_0 := \langle \alpha_1^{-1} \alpha_j \rangle$.

Chiral Hypertopes From Groups

Similarly, starting with a group G^+ and a set $R = \{\alpha_1, \ldots, \alpha_{n-1}\}$ of independent generators, we can construct a coset geometry $\Gamma(G^+, R) := \Gamma(G^+, (G_i)_{i \in \{0,\ldots,n-1\}})$ where $G_i := \langle \alpha_j \mid j \neq i \rangle$ for $i = 1, \ldots, n-1$ and $G_0 := \langle \alpha_1^{-1} \alpha_j \rangle$.

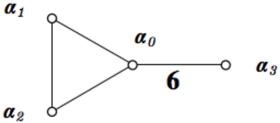
Theorem (Fernandes, Leemans and Ivić Weiss, 2014) Let $\Gamma = \Gamma(G^+, R) := \Gamma(G^+, (G_i)_{i \in \{0,\ldots,n-1\}})$ be a coset geometry constructed from G^+ and a set on independent generators $R = \{\alpha_1, \ldots, \alpha_{n-1}\}$. If Γ is thin and SCC it is chiral if and only if there is no automorphism of G^+ that inverts all elements of R.

Chiral Hypertopes From Groups

Similarly, starting with a group G^+ and a set $R = \{\alpha_1, \dots, \alpha_{n-1}\}$ of independent generators, we can construct a coset geometry $\Gamma(G^+, R) := \Gamma(G^+, (G_i)_{i \in \{0,\dots,n-1\}})$ where $G_i := \langle \alpha_j \mid j \neq i \rangle$ for $i = 1, \dots, n-1$ and $G_0 := \langle \alpha_1^{-1} \alpha_j \rangle$.

Theorem (Fernandes, Leemans and Ivić Weiss, 2014) Let $\Gamma = \Gamma(G^+, R) := \Gamma(G^+, (G_i)_{i \in \{0,\dots,n-1\}})$ be a coset geometry constructed from G^+ and a set on independent generators $R = \{\alpha_1, \dots, \alpha_{n-1}\}$. If Γ is thin and SCC it is chiral if and only if there is no automorphism of G^+ that inverts all elements of R.

Example: $B-$ diagram of a hypertope related to the tessellation $\{6, 3, 6\}$ of H^3.



Toroidal Hypertopes of Rank 3

The toroidal hypertopes of rank 3 are divided into the following families:

toroidal maps $\{3,6\}_{(b,c)}$, $\{6,3\}_{(b,c)}$, $\{4,4\}_{(b,c)}$, and

hypermaps $(3,3,3)_{(b,c)}$ with $(b,c) \neq (1,1)$.

Toroidal Hypertopes of Rank 3

The toroidal hypertopes of rank 3 are divided into the following families:

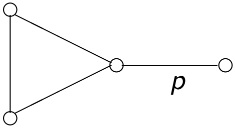
toroidal maps $\{3,6\}_{(b,c)}$, $\{6,3\}_{(b,c)}$, $\{4,4\}_{(b,c)}$, and

hypermaps $(3,3,3)_{(b,c)}$ with $(b,c) \neq (1,1)$.

Note: Hypermap $(3,3,3)_{(b,c)}$ is obtained from the toroidal map $\{6,3\}_{(b,c)}$ by doubling the fundamental region, but in the case $(b,c) = (1,1)$ the corresponding incidence graph is a complete tripartite graph $K_{3,3,3}$ and therefore the geometry is not thin.

Toroidal Hypertopes of Rank 4

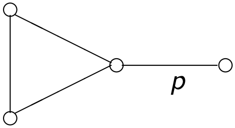
Doubling the fundamental region of rank 4 polytope $\{6, 3, p\}$ which tessellates the hyperbolic 3−space for $p = 3, 4, 5$ we similarly obtain the finite universal locally toroidal hypertopes with diagram



These hypertopes have only one toroidal residue that is the hypermap $(3, 3, 3)_{(b,c)}$, all the remaining residues are spherical.

Toroidal Hypertopes of Rank 4

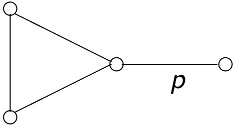
Doubling the fundamental region of rank 4 polytope $\{6, 3, p\}$ which tessellates the hyperbolic 3−space for $p = 3, 4, 5$ we similarly obtain the finite universal locally toroidal hypertopes with diagram



These hypertopes have only one toroidal residue that is the hypermap $(3, 3, 3)_{(b,c)}$, all the remaining residues are spherical. We denote these hypertopes by $(3, 3, 3; p)_{(b,c)}$

Toroidal Hypertopes of Rank 4

Doubling the fundamental region of rank 4 polytope $\{6, 3, p\}$ which tessellates the hyperbolic $3-$space for $p = 3, 4, 5$ we similarly obtain the finite universal locally toroidal hypertopes with diagram



These hypertopes have only one toroidal residue that is the hypermap $(3, 3, 3)_{(b,c)}$, all the remaining residues are spherical. We denote these hypertopes by $(3, 3, 3; p)_{(b,c)}$ and with Fernandes and Leemans show that when $p \in \{3, 4, 5\}$ and $(b, c) \neq (1, 1)$, the hypertope $(3, 3, 3; p)_{(b,c)}$ is finite if and only if the universal polytope $\{\{6, 3\}_{(b,c)}, \{3, p\}\}$ is finite.

Toroidal Hypertopes of Rank 4

The existence of regular universal locally toroidal polytopes of rank 4 is investigated in ARP where McMullen and Schulte give an enumeration of finite such universal polytopes.

Toroidal Hypertopes of Rank 4

The existence of regular universal locally toroidal polytopes of rank 4 is investigated in ARP where McMullen and Schulte give an enumeration of finite such universal polytopes.

In particular, they have complete classification of $\{\{6,3\}_{(b,c)},\{3,p\}\}$ with $p \in \{3,4,5\}$ and $\{\{6,3\}_{(b,c)},\{3,6\}_{(e,f)}\}$ thus enabling the classification of hypertopes $(3,3,3;p)_{(b,c)}$ when $p \in \{3,4,5,6\}$.

Toroidal Hypertopes of Rank 4

The existence of regular universal locally toroidal polytopes of rank 4 is investigated in ARP where McMullen and Schulte give an enumeration of finite such universal polytopes.

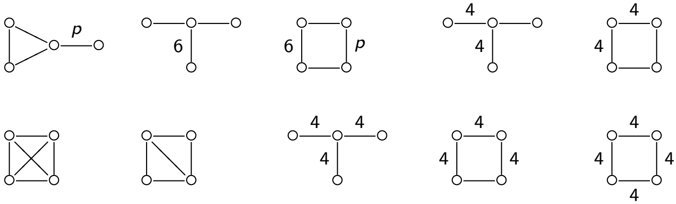
In particular, they have complete classification of $\{\{6,3\}_{(b,c)},\{3,p\}\}$ with $p \in \{3,4,5\}$ and $\{\{6,3\}_{(b,c)},\{3,6\}_{(e,f)}\}$ thus enabling the classification of hypertopes $(3,3,3;p)_{(b,c)}$ when $p \in \{3,4,5,6\}$.

Other toroidal hypertopes ...

Toroidal Hypertopes of Rank 4

Universal locally toroidal non-polytopal hypertopes of rank 4 (all residues of rank 3 are either spherical or toroidal, with at least one being toroidal)

Universal locally toroidal non-polytopal hypertopes of rank 4 (all residues of rank 3 are either spherical or toroidal, with at least one being toroidal)



(Here $p = 3, 4, 5$ or 6).

Some Open Problems

Some Open Problems

Classification of regular toroidal hypertopes in ranks greater than 3.

Some Open Problems

Classification of regular toroidal hypertopes in ranks greater than 3.

Existence of chiral toroidal hypertopes in ranks greater than 3.

Some Open Problems

Classification of regular toroidal hypertopes in ranks greater than 3.

Existence of chiral toroidal hypertopes in ranks greater than 3.

Classification of locally spherical (and locally toroidal) hypertopes.

Some Open Problems

Classification of regular toroidal hypertopes in ranks greater than 3.

Existence of chiral toroidal hypertopes in ranks greater than 3.

Classification of locally spherical (and locally toroidal) hypertopes.

Classification of uniform polyhedra.

Thank You!