The real genus of the symmetric groups

(A result from Carmen Cano in her Ph. D. Thesis, 2011.)

by

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\[ o(\text{Aut}(X)) \leq 12(g(X) - 1) \]

\[ G = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^2 = (ac)^3 = 1 \rangle \]

\[ X = ac \]
\[ A = ab \]
\[ B = a \]

\[ G = \langle X, A, B \mid X^3 = A^2 = B^2 = (XB)^2 = (AB)^2 = 1 \rangle \]
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\[ X = [XA, B]^2 \]
\[ A = X^2(XA) \]

\[ \Downarrow \]

\[ G = \langle XA, B \rangle \]
Proposition: Let $g$ and $h$ be permutations which generate a transitive subgroup of $S_n$. Suppose that $g$ contains a cycle of prime length $p$ such that:

a) $p < n - 2$.

b) $p$ divides the length of no other cycle of $g$.

c) The $p$-cycle contains either a fixed point of $h$, or the points from a cycle of $h$.

Then, the subgroup generated by $g$ and $h$ is either $A_n$ or $S_n$. 

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Initial diagram

(A result from Carmen Cano in her Ph. D. The real genus of the symmetric groups)
Insider and Outsider blocks

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**Proposition:** Let $m$ be a prime number. For each prime $s$ such that

$$2(t - t_0) + m \leq s \leq 4(t - t_0) + m,$$

the system

$$s = m + 4x + 2y$$

$$t - t_0 = x + y$$

has solution in $\mathbb{Z}^+$. 

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$S_{13}$ and $S_{19}$

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$S_{14}$
$S_{15}$
$S_{10}$
(A result from Carmen Cano in her Ph. D. The real genus of the symmetric groups)
A result from Carmen Cano in her Ph. D. The real genus of the symmetric groups
$S_7$ and $S_9$

(A result from Carmen Cano in her Ph.D.)
$a = (2, 4)$

$b = (1, 5)$

$c = (1, 4)(2, 3)$

$a^2 = b^2 = c^2 = (ab)^2 = (ac)^4 = 1$

$o(cab) = 5$

$\langle a, b, c \rangle = S_5$
\[ a = (1, 4)(2, 3) \]
\[ b = (5, 6) \]
\[ c = (1, 6)(2, 4) \]

\[ a^2 = b^2 = c^2 = (ab)^2 = (ac)^5 = 1 \]

\[ acb = (1, 2, 3, 4, 5, 6) \]
\[ \langle a, b, c \rangle = S_6 \]
\[ a = (1, 2)(3, 4)(5, 6)(7, 8) \]
\[ b = (1, 2)(3, 5)(4, 6) \]
\[ c = (1, 3)(5, 7)(6, 8) \]

\[ a^2 = b^2 = c^2 = (ab)^2 = (ac)^4 = 1 \]

\[ \langle a, b, c \rangle = S_8 \]
Theorem: Let $n \geq 4$. The symmetric group $S_n$ has real genus

$$\rho(S_n) = \frac{n!}{12} + 1$$

for all $n \notin \{5, 6, 8\}$, and

$$\rho(S_5) = 16$$
$$\rho(S_6) = 109$$
$$\rho(S_8) = 5041$$

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