The real genus of the symmetric groups

(*A result from Carmen Cano in her Ph. D. Thesis, 2011.*)

by

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$$o(Aut(X)) \leq 12(g(X) - 1)$$

$$G = \langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^2 = (ac)^3 = 1 \rangle$$

$$\begin{aligned} X &= ac \\ A &= ab \\ B &= a \end{aligned}$$

$$G = \langle X, A, B \mid X^3 = A^2 = B^2 = (XB)^2 = (AB)^2 = 1 \rangle$$

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$$\begin{aligned} X &= [XA, B]^2 \\ A &= X^2(XA) \end{aligned}$$

$$\Downarrow$$

$$G = \langle XA, B \rangle$$

Proposition: Let g and h be permutations which generate a transitive subgroup of S_n. Suppose that g contains a cycle of prime length p such that:

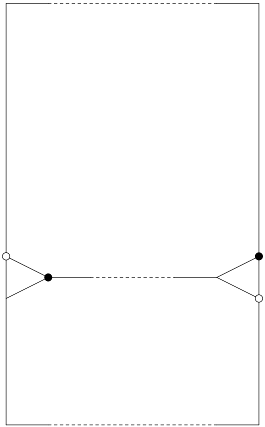
a) $p < n - 2$.

b) p divides the length of no other cycle of g.

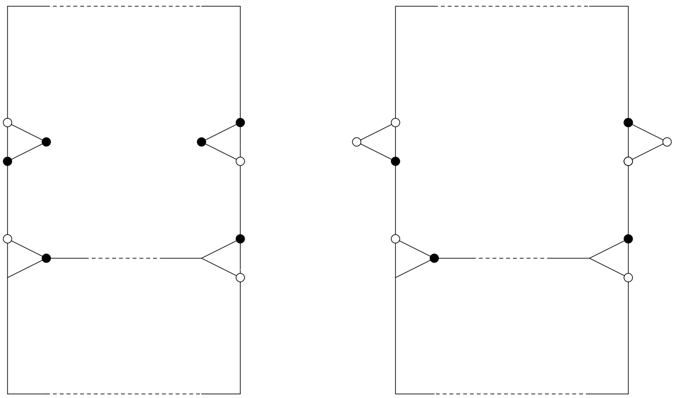
c) The p-cycle contains either a fixed point of h, or the points from a cycle of h.

Then, the subgroup generated by g and h is either A_n or S_n.

Initial diagram



Insider and Outsider blocks



The existence of diagrams for each t

Proposition: Let m be a prime number. For each prime s such that

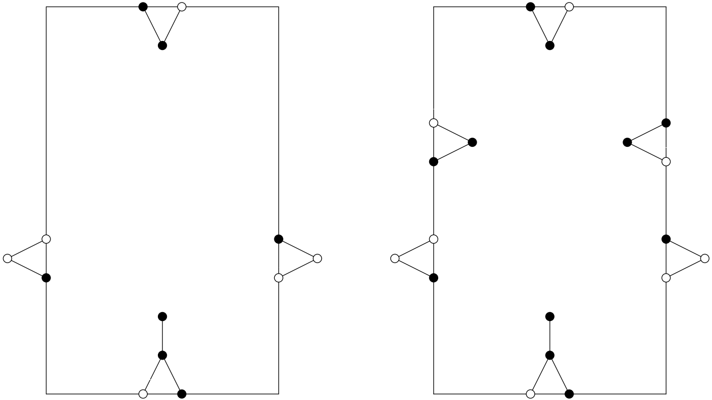
$$2(t - t_0) + m \leq s \leq 4(t - t_0) + m,$$

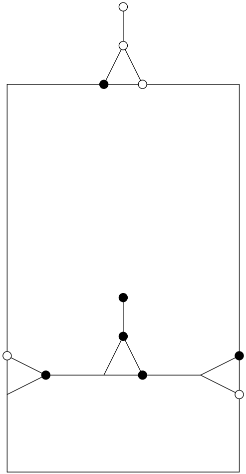
the system

$$\begin{aligned} s &= m + 4x + 2y \\ t - t_0 &= x + y \end{aligned}$$

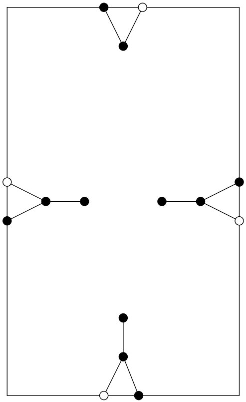
has solution in $\mathbb{Z}^+$.

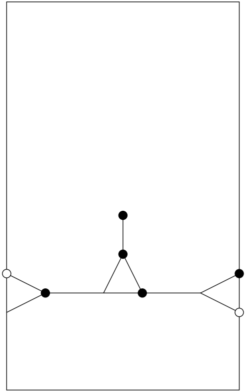
S_{13} and S_{19}



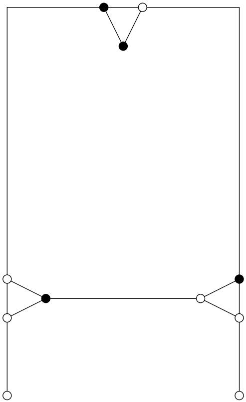


S_{15}

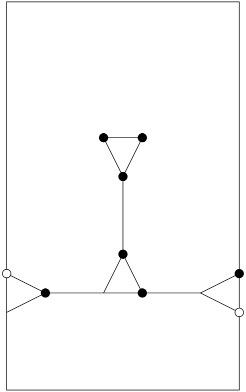




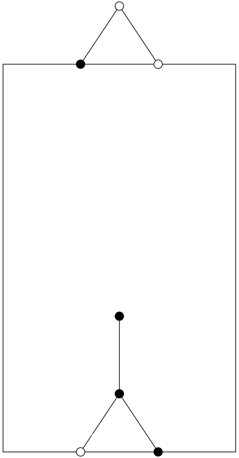
S_{11}

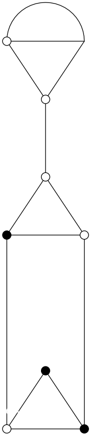


S_{12}



S_7 and S_9





S_5

$$\begin{aligned} a &= (2,4) \\ b &= (1,5) \\ c &= (1,4)(2,3) \end{aligned}$$

$$a^2 = b^2 = c^2 = (ab)^2 = (ac)^4 = 1$$

$$\mathrm{o}(cab) = 5$$

$$\langle a, b, c \rangle = S_5$$

$$\begin{aligned} a &= (1,4)(2,3) \\ b &= (5,6) \\ c &= (1,6)(2,4) \end{aligned}$$

$$a^2 = b^2 = c^2 = (ab)^2 = (ac)^5 = 1$$

$$acb = (1,2,3,4,5,6)$$

$$\langle a, b, c \rangle = S_6$$

S_8

$$\begin{aligned} a &= (1,2)(3,4)(5,6)(7,8) \\ b &= (1,2)(3,5)(4,6) \\ c &= (1,3)(5,7)(6,8) \end{aligned}$$

$$a^2 = b^2 = c^2 = (ab)^2 = (ac)^4 = 1$$

$$\langle a, b, c \rangle = S_8$$

The real genus of S_n

Theorem: Let $n \geq 4$. The symmetric group S_n has real genus

$$\rho(S_n) = \frac{n!}{12} + 1$$

for all $n \notin \{5, 6, 8\}$, and

$$\begin{aligned} \rho(S_5) &= 16 \\ \rho(S_6) &= 109 \\ \rho(S_8) &= 5041 \end{aligned}$$