ON THE SYMMETRIC

CROSSCAP NUMBER

OF GROUPS

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Let S be a Klein surface, that is a compact surface endowed with a dianalytic atlas.

Our maximum programme would be find the automorphism group of ALL surfaces of a given type... IMPOSSIBLE!

Reduce our problem in two different ways: MAXIMUM ORDER AND MINIMUM GENUS.

Given a finite group G, it is always the group of automorphism of a non-orientable, unbordered Klein surface (E. BUJALANCE)

The lowest algebraic genus of the non-orientable, unbordered Klein surface where the group G acts as an automorphism group it calls the SYMMETRIC CROSSCAP NUMBER of the group G, and it is denoted by $\tilde{\sigma}(G)$

For the study of this symmetric crosscap number, the NEC groups are essential.

Let $X = \mathcal{H}/\Gamma$ be a non-orientable unbordered Klein surface on which G acts as an automorphism group. Then there exists another NEC group Λ such that $G = \Lambda/\Gamma$.

From the Riemann-Hurwitz relation we have $g - 2 = o(G)|\Lambda|$, *where* $o(G)$ *denotes the order of* G. *Then*

$$\tilde{\sigma}(G) \leq g = 2 + o(G)|\Lambda|,$$

and so to obtain the symmetric crosscap number is equivalent to find a group Λ *and an epimorphism* $\theta : \Lambda \to G$, *such that* $\Gamma = ker\,\theta$ *is a surface NEC group (and so, without elements with finite order) and* $G = \theta(\Lambda^+)$, *where* Λ^+ *is the subgroup consisting of the orientation-preserving elements of* Λ, *and minimal* $|\Lambda|$.

We know about symmetric crosscap number:

Symmetric crosscap number of some infinite families: $C_n \times D_m$, $D_n \times D_m$, $DC_3 \times C_n$, $C_n \times A_4$

Symmetric crosscap number of groups with order less than 32. (J.J. Etayo, E. Martínez)

Groups with symmetric crosscap number less than 6. (T. Tucker, C.L. May, E. Bujalance, J.J. Etayo, E. Martínez)

M. Conder published in his website two list related with the symmetric crosscap number: one up to order 127 and one up to symmetric crosscap number 65.

These two list include the GAP notation, the associated NEC group (s), and its order or its symmetric crosscap number, depende on the list.

Results: Algebraic structure, presentation and a epimorphism for each NEC group up to order 63 and up to symmetric crosscap number 17.

SPECTRUM:

Real genus: 2 (E. Bujalance, J. M. Gamboa), 12 (Mockiewicz), 24 (May), ¿72?

Strong Symmetric genus: All of them (May, Zimmerman)

Symmetric genus: All of them (Conder, Tucker)

Symmetric Crosscap number: Problems for n such that $n = 12k + 3$

Going to study low k to analyze what happens, using the list of Conder. Taking $k = 1, 2, 3, 4, 5$ we have:

GAP	G	Relations
[24,1]	<-2,2,3>	$a^8, b^8, (a^3b)^3, a^2b^6, a^2(b^{-1}a^{-1})^3, b^2(b^{-1}a^{-1})^3$
[39,1]	$C_{13} \rtimes C_3$	$a^3, b^{13}, bab^{10}a^{-1}$
[48,15]	$(C_3 \times D_4) \rtimes C_2$	$a^2, b^8, c^3, (ab)^2, (ac)^2, b^{-1}cbc$
[78,1]	$(C_{13} \rtimes C_3) \rtimes C_2$	$a^2, b^3, c^{13}, (ca)^2, cbc^{10}b^{-1}$
[1092,25]	$PSL(2,13)$	$a^3, b^7, c^2, (ab)^2, (cb)^2, (ac)^2, b^{-1}(ab^{-2})^6a^{-1}c$

Symmetric crosscap number 15

GAP	G	Relations
[40,3]	$C_5 \rtimes C_8$	a^8, b^5, bab^3a^{-1}
[75,2]	$(3,3\|3,5)$	$a^3, b^3, (ab)^3, (a^{-1}b)^5$
[150,5]	$(3,3\|3,5) \rtimes C_2$	$a^3, b^3, c^2, (ab)^3, (a^{-1}b)^5, (ca)^2, (cb)^2$
[150,6]	$(3,3\|3,5) \rtimes C_2$	$a^3, b^3, c^2, (ab)^3, (a^{-1}b)^5, (cb)^2$
[300,25]	$((3,3\|3,5) \rtimes C_2) \rtimes C_2$	$a^3, b^3, c^2, d^2, (ab)^3, (a^{-1}b)^5, (ca)^2, (cb)^2, (ad)^2$

Symmetric crosscap number 27

GAP	G	Relations
[60,1]	$DC_3 \times C_5$	$a^4, b^3, c^5, baba^{-1}$
[111,1]	$C_{37} \rtimes C_3$	$a^3, b^{37}, bab^{27}a^{-1}$
[120,12]	$(DC_3 \times C_5) \rtimes C_2$	$a^4, b^3, c^5, d^2, baba^{-1}, (ad)^2, (cd)^2, (db)^2$
[222,1]	$(C_{37} \rtimes C_3) \rtimes C_2$	$a^3, b^{37}, c^2, bab^{27}a^{-1}, (bc)^2$

Symmetric crosscap number 39

GAP	G	Relations
[84,2]	$C_4 \times (C_7 \rtimes C_3)$	$a^3, b^7, c^4, bab^5a^{-1}$
[147,1]	$C_{49} \rtimes C_3$	$a^3, b^{49}, bab^{31}a^{-1}$
[147,5]	$(3,3\|3,7)$	$a^3, b^3, (ab)^3, (a^{-1}b)^7$
[294,1]	$(C_{49} \rtimes C_3) \rtimes C_2$	$a^3, b^{49}, c^2, bab^{31}a^{-1}, (ac)^2$
[294,7]	$(3,3\|3,7) \rtimes C_2$	$a^3, b^3, c^2, (ab)^3, (a^{-1}b)^7, (ca)^2, (cb)^2$
[294,14]	$(3,3\|3,7) \rtimes C_2$	$a^3, b^3, c^2, (ab)^3, (a^{-1}b)^7, (cb)^2$
[588,35]	$((3,3\|3,7) \rtimes C_2) \rtimes C_2$	$a^3, b^3, c^2, d^2, (ab)^3, (a^{-1}b)^7, (ca)^2, (cb)^2, (ad)^2$

Symmetric crosscap number 51

GAP	G	Relations
[96,1]	$C_3 \rtimes C_{32}$	$a^3, b^{32}, abab^{-1}$
[183,1]	$C_{61} \rtimes C_3$	$a^3, b^{61}, bab^{48}a^{-1}$
[192,78]	$(C_3 \rtimes C_{32}) \rtimes C_2$	$a^3, b^{32}, c^2, abab^{-1}, (ca)^2, (cb)^2$
[366,1]	$(C_{61} \rtimes C_3) \rtimes C_2$	$a^3, b^{61}, c^2, bab^{48}a^{-1}, (bc)^2$

Symmetric crosscap number 63

Theorem 1: Let $n = 12k + 3$ be such that $n - 2$ has all its prime factors congruent to $1(mod\ 3)$. Then $(C_{12k+1} \rtimes C_3) \rtimes C_2$ has symmetric crosscap number n.

Theorem 2: Let $n = 12k + 3$ be such that $n - 2 = m^2$ is a square. Then we have that there are two groups with algebraic structure $(3, 3|3, m) \rtimes C_2$, namely $(2, 3, 2m; 3)$ and $(2, 3, 6; m)$, that have symmetric crosscap number n.

Theorem 3: Let $n = 12k + 3$ be such that $n - 2 = m^2$ is a square, with m odd. The symmetric crosscap number of the group $((3, 3|3, m) \rtimes C_2) \rtimes C_2 \approx G^{3,6,2m}$ is n.

Theorem 4: Let n be such that $n = 48k + 39$. The symmetric crosscap number of $(DC_3 \times C_{6k+5}) \rtimes C_2$ is n.

Theorem 5: Let $n = 84k+51$ be such that $12k+7$ has all its prime factors congruent to $1(mod\ 3)$. Then the symmetric crosscap number of $C_4 \times (C_{12k+7} \rtimes C_3)$ is n.

Theorem 6: Let n be such that $n = 60k + 27$. Then the symmetric crosscap number of $C_5 \rtimes C_{16k+8}$ is n.

Theorem 7: Let n be such that $n = 24k + 15$. Then the symmetric crosscap number of $C_3 \rtimes C_{12k+8}$ is n.

699 next gap. Only 699 and 913 to 1000. 8 gaps to 2000.

Open problems: Only 4 classes mod 120 $\rightarrow$ 3,51,75,99 with

exceptions.

THANK YOU!