The University of Western Australia

Simple Group Factorisations and Applications in Combinatorics: Lecture 3

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ACHIEVE INTERNATIONAL EXCELLENCE

In lecture 1 and 2 we met:

- O'Nan—Scott Theorem for primitive permutation groups
- Maximal factorisations of all almost simple groups
- Primitive Inclusions: G < H < Sym(X)

- Discussed using these tools to solve problems:
  - Classifying maximal subgroups of Sym(X) and Alt(X)
  - Deciding when a graph could have "very different" vertex-primitive, arc-transitive G < H < Aut(Γ)
  - Detecting whether a permutation group preserves a cartesian decomposition

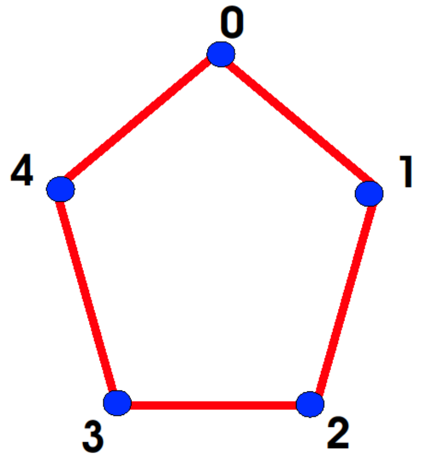
- **This last lecture:** using these tools to study Cayley graphs

Comparison of the applications

Previous applications: Have transitive group $G < Sym(X)$ and searched for overgroups H using factorisation $H = G H_\alpha$

What is different in new application: we search for transitive subgroups B of the given G. Again we have a factorisation $G = B G_\alpha$

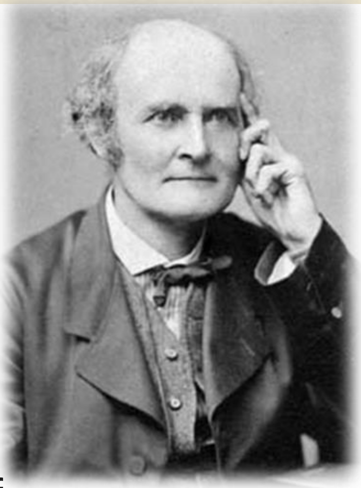
Long history (discuss later): first look at Cayley graphs – why factorisations might be involved



Is a given graph a Cayley graph?

- Cayley graphs: visualisations of groups with given generating set
- Input:
  - Group H = < S > where s in S iff s^{-1} in S [inverse-closed]
- Construction:
  - Cay(H,S) has vertex set H. Edges { h sh } for h in H, s in S
- Example:
  - H = Z_5 under addition and S = { 1, 4 }

Some facts about Cayley graphs



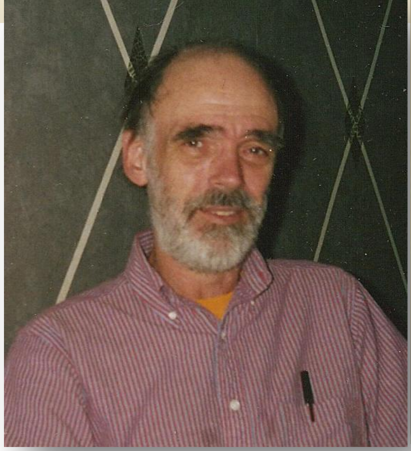
- Cay(H,S) admits the right multiplication action of H as a group of automorphisms
  - Right multiplication by u in H maps the edges { h, sh } to edge { hu, shu}
  - The only u in H which fixes ANY vertex is $u=1_H$

Such H called **regular**

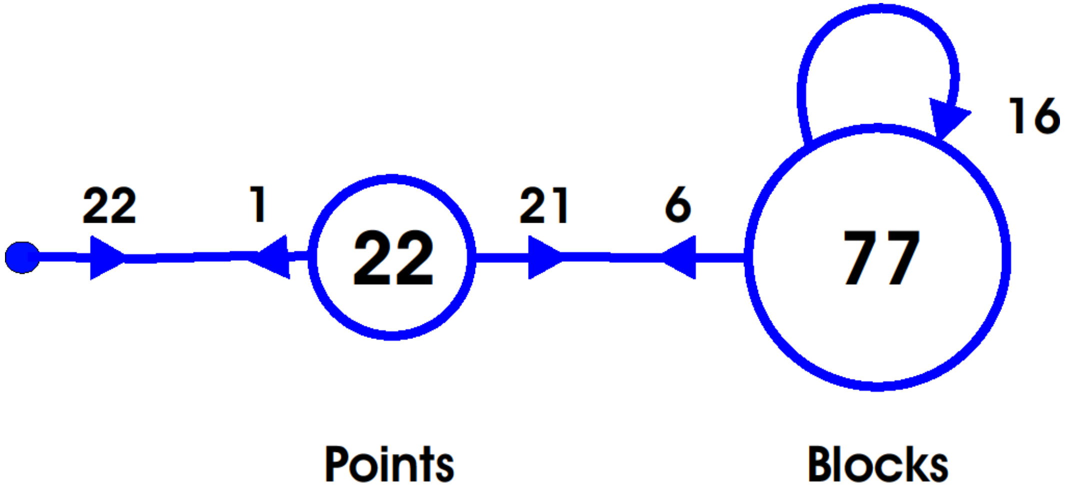
- So Cayley graphs are vertex-transitive

- Arise in many areas
  - Circulant graphs [Cayley graphs for cyclic groups]
    - Experimental layouts for statistical expermients, and many constructions in combinatorics
  - Expander graphs
    - Difficut to find explicit constructions – Ramanujan graphs of Lubotzky/Phillips/Sarnak are Cayley
  - Random selection for group computation
    - Modelled and analysed as random walk on a Cayley graph

Is a given graph a Cayley graph?



- It had better be vertex transitive!
- Sometime not obvious whether a famous graph is a Cayley graph
- Higman Sims graph Γ(HS) 100 vertices, valency 22 Aut Γ(HS) = HS.2
- Related to the Steiner system S(3,6,22) vertex stab. Is $M_{22}.2$
- Not obvious that
- Γ(HS) = Cay(H,S) for
  H = (Z_5 x Z_5) : [4]



Criterion:

A given (vertex-transitive) graph Γ is a Cayley graph

- If and only if Aut(Γ) contains a **regular** subgroup R

Regular means transitive and only the identity fixes a vertex

- In this case $\Gamma \approx$ Cay(R, S) for some S

- If G = Aut(Γ) and $R < G$ then R is regular if and only if
  1. R is transitive — $G = R\,G_\alpha$
  2. Vertex stabiliser $R_\alpha = 1$ — $R \cap G_\alpha = 1$

- G is a "general product" –
- these days we say $G = R\,G_\alpha$ is an **exact factorisation**

Now we go back in history

The story starts with

Primitive groups G < Sym(n) containing an n-cycle

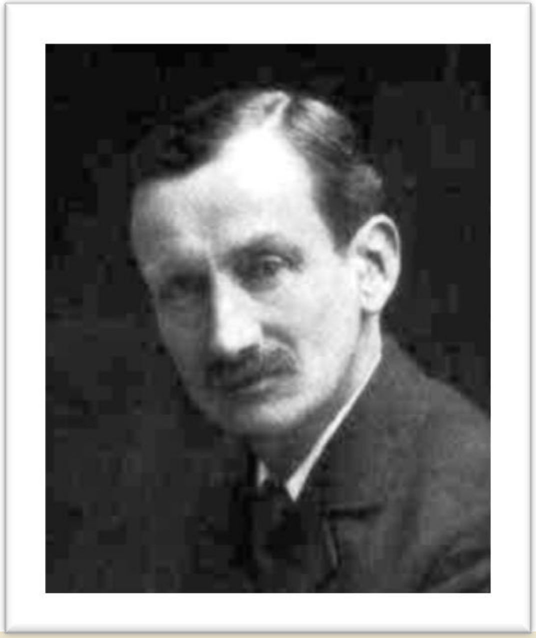
Old problem: goes back more than 100 years to work of William Burnside

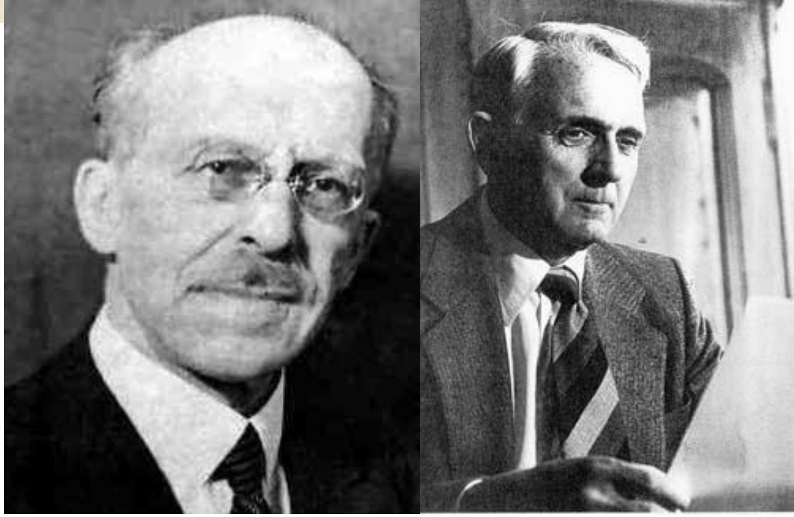
- **Burnside (1911):** if $n = p^m$ with $m > 1$ and G primitive contains an n-cycle then G is 2-transitive

[all ordered pairs equivalent under G-action]

- **Burnside's Question (1911):** Is the same true for ANY non-prime n ?   [known false if n prime]

According to PM Neumann: generalisation of Burside's Theorem: a transitive group of prime degree is either 2-transitive or soluble

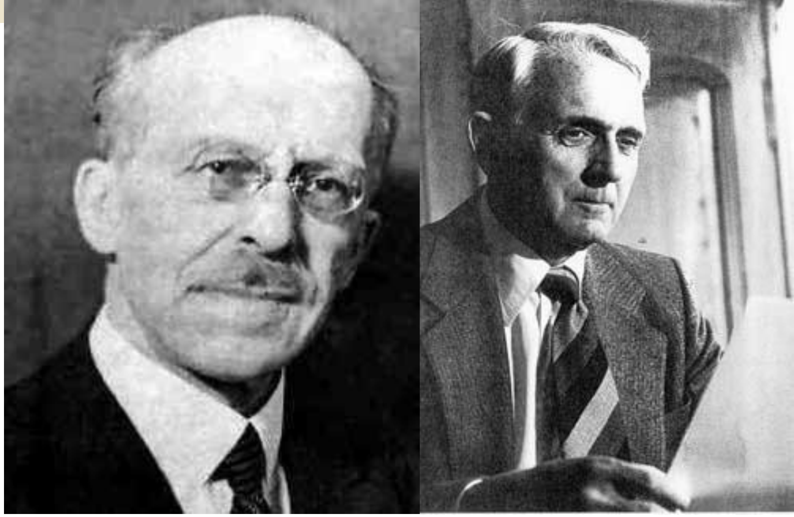




A lot of work inspired by Burnside's work

- 1921 Burnside had tried to prove that every primitive group containing a regular subgroup B that is abelian but not elementary abelian must be 2-transitive -- but his proof wrong -- his error was pointed out by Dorothy Manning in 1936
- 1933 Schur: G primitive contains an n-cycle and n is not prime, then G is 2-transitive
  - Schur's methods led to Schur's theory of **S-rings** (Wielandt school), **coherant configurations** (D. G. Higman), and **centraliser algebras** and **Hecke algebras**
- 1935, 1950, 1955 Wielandt: various kinds of regular subgroups B force G to be 2-transitive

A lot of work inspired by Burnside's result

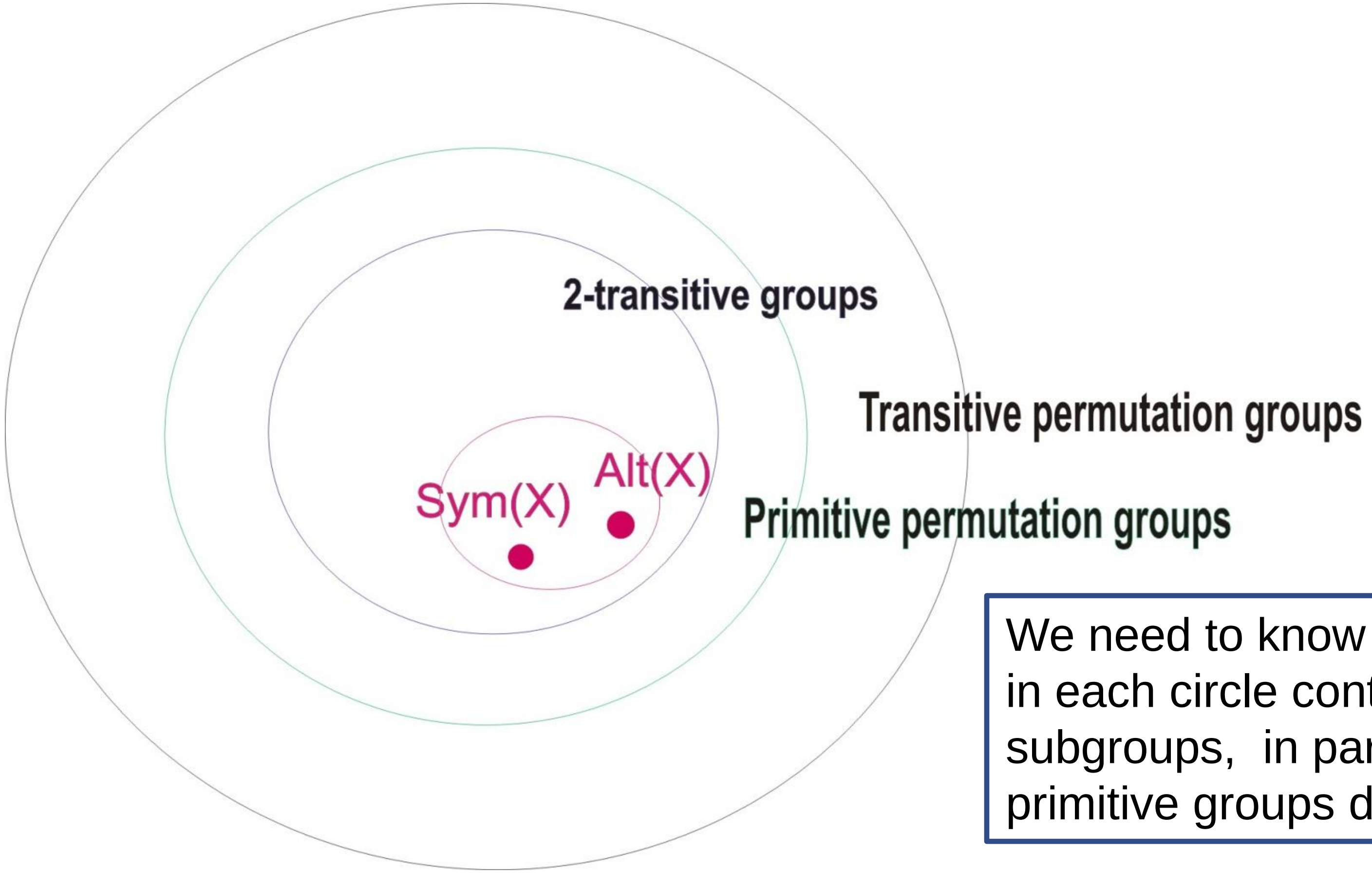


- 1955 Wielandt: named such groups B-groups in honour of Burnside
- A group B is a B-group if every primitive permutation group G containing B as a regular subgroup is 2-transitive
- So Wielandt knew that most cyclic groups, many abelian groups, all dihedral groups are B-groups
- Nowadays not so interested in 2-transitivity: general study led to **"Regular subgroup problem"**
  - Find all pairs (G, B) with **G primitive and B regular**

G=AB
A maximal
$A \cap B = 1$

Permutation group hierarchy

- For studying Cayley graphs we want to decide existence of regular subgroups maybe in primitive groups – but a graph has a 2-transitive automorphism group only if it is empty (no edges) or complete



We need to know which groups in each circle contain regular subgroups, in particular which primitive groups do

Regular subgroup problem: Find all pairs (G, B) with G primitive and B regular

- Equivalently: find all exact factorisations of finite primitive groups G

G=AB
A maximal
$A \cap B = 1$

- Even this was an old problem:

- 1935 G. A. Miller: gave examples of integers n such that the ONLY exact factorisations of G isomorphic to Alt(n) have A = Alt(n-1)

- 1980 Wiegold & Williamson: classified all exact factorisations with G isomorphic to Alt(n) or Sym(n)

A cute reality check: a "density result"

- 1982 Cameron, Neumann, Teague: for "almost all n" the only primitive subgroups of Sym(n) are Alt(n) and Sym(n)
- More precisely: If N(x) := Number of n ≤ x such that there exists primitive G < Sym(n) with G ≠ Sym(n) or Alt(n) then $N(x)/x \to 1$ as $x \to \infty$

Proof uses simple group classification and gives more refined information

- Consequence for us: for "almost all n" every group of order n (that is, every possibility for B) is a B-group!

But we still want answers

- Finding all exact factorisations of finite primitive groups G implies
- Finding all vertex-primitive Cayley graphs
- First determines all [G=AB, A maximal, $A \cap B = 1$] then G-action yields all S for Cay(B,S)
- Generic example (to avoid): For any group B of order n take S = B \ { 1 } Then Cay(B,S) is the complete graph K_n with primitive automorphism group Sym(n)
- So every group B has this generic primitive Cayley graph!

G=AB
A maximal
$A \cap B = 1$

Chronologically

- ~ 2000 know all exact factorisations with B cyclic
  - Hence know all vertex-primitive circulants [details next slide]
- By 2007 know all exact factorisations for certain other B …
- By 2010 know all exact factorisations for all ONS-types of G EXCEPT product action
- Identified explicitly lots more B-groups
- Details following

Finite primitive groups containing an n-cycle known explicitly

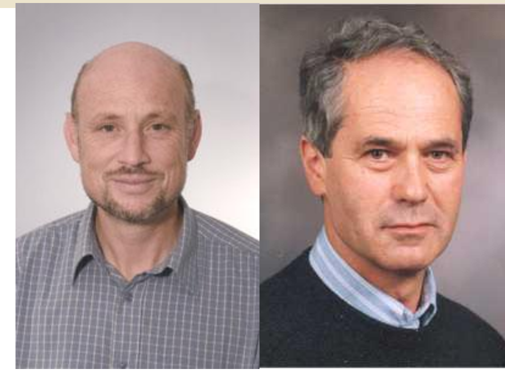
- Early important work of (Galois, Schur, Ritt)
- Application of finite simple group classification (Feit)
- Final details (McSorley 1997, Jones 2002)

By 2007 know all exact factorisations G=AB for certain other G, B …

Who	When	What
Liebeck, CEP, Saxl	2000	All possible ONS-types
Cai Heng Li	2003, 2007	All G with B abelian or dihedral
Cai Heng Li and Akos Seress	2005	All G if n square-free and $B \leq$ Soc(G)
Michael Giudici	2007	All G, B if G sporadic almost simple
Barbara Baumeister	2006, 2007	All G, B with G sporadic, exceptional Lie type, PSU, or $\Omega^{+}(8,q)$

Open cases left after these results: G classical simple (heart of the problem)
And G of product action type – still unresolved

2010 AMS Memoir: Liebeck CEP Saxl



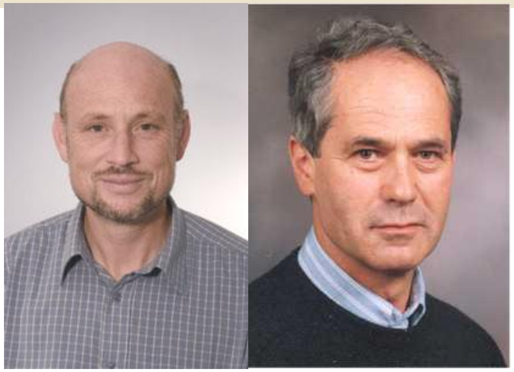
Exact factorisations G = AB of almost simple classical groups G

- Principal tool: Maximal factorisations yield all possibilities for G = A M with M maximal subgroup of G and M containing B Then the hard work begins!

- An “easy example”: G = PGL(d,q), A = stabiliser of k-subspace of V(d,q)

  - Maxl Factns gives all maximal M that are transitive on k-subspaces
  - Need to search in each M for a minimal transitive B – hoping B regular

  - Special case k=1: apply Hering’s classification of transitive linear groups – find metacyclic examples B < $\Gamma L(1, q^d)$



2010 AMS Memoir: Liebeck CEP Saxl

Exact factorisations G = AB of almost simple classical groups G

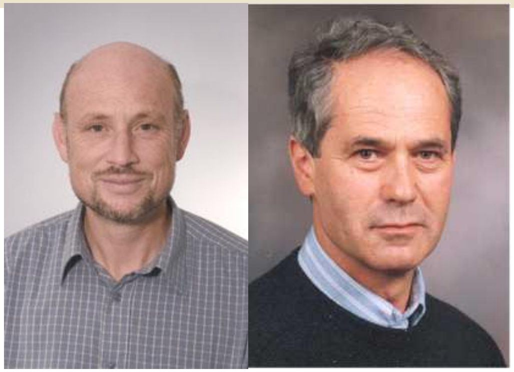
- Strategy: Proved sequence of lemmas – for each kind of classical group (PSL, PSU, PSp, $P\Omega^{\varepsilon}$) classifying subgroups which are transitive on various kinds of subspaces

- Factorisations "propagate": If G = A M and B < M then
  - also $M = (A \cap M) B$ and sometimes this helps.
  - also if K normal in M then $M/K = ((A \cap M)K/K)\ (BK/K)$ – good if M/N al't simple

- Main Theorem: Complete lists of all possibilities for G, A, B

- Many small cases but: if degree $n > 3 \times 29!$ and $G \neq \mathrm{Alt}(n)$ or $\mathrm{Sym}(n)$ then

  - B metacyclic of order $(q^d-1)/(q-1)$ or
  - B of odd order $q(q-1)/2$ in $A\Gamma L(1, q)$ with $q \equiv 3 \pmod 4$
  - $B = \mathrm{Alt}(p)$, $\mathrm{Sym}(p)$ (p prime) or $B=\mathrm{Alt}(p-2) \times Z_2$ (p prime, $p \equiv 1 \pmod 4$), or $B=\mathrm{Alt}(p^2-2)$ (p prime, $p \equiv 3 \pmod 4$)

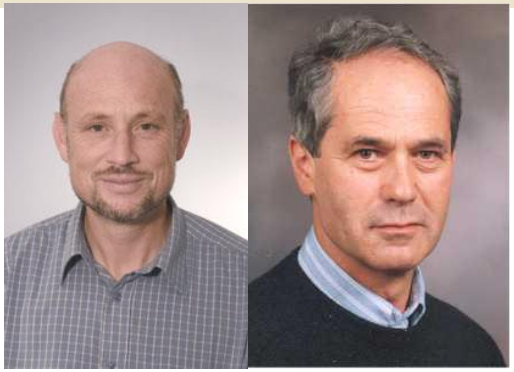


What did we learn?

↘ Complete information about almost simple groups B: when they are B-groups and if not which primitive groups arise

- B is a B-group $\Leftrightarrow$ B not simple and not one of Sym(p-2) (p prime), PSL(2,16).4, PSL(3,4).2

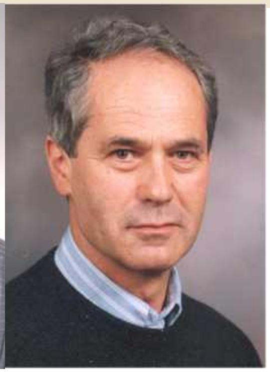
- If B is simple or one of Sym(p-2) (p prime), PSL(2,16).4, PSL(3,4).2 and if B is a regular subgroup of a primitive $G < \mathrm{Sym}(n)$ (and $G \neq \mathrm{Alt}(n)$) then
  - (generic case) $B \times B \leq G \leq$ Holomorph of B
  - or G, A, B in short explicit list of possibilities



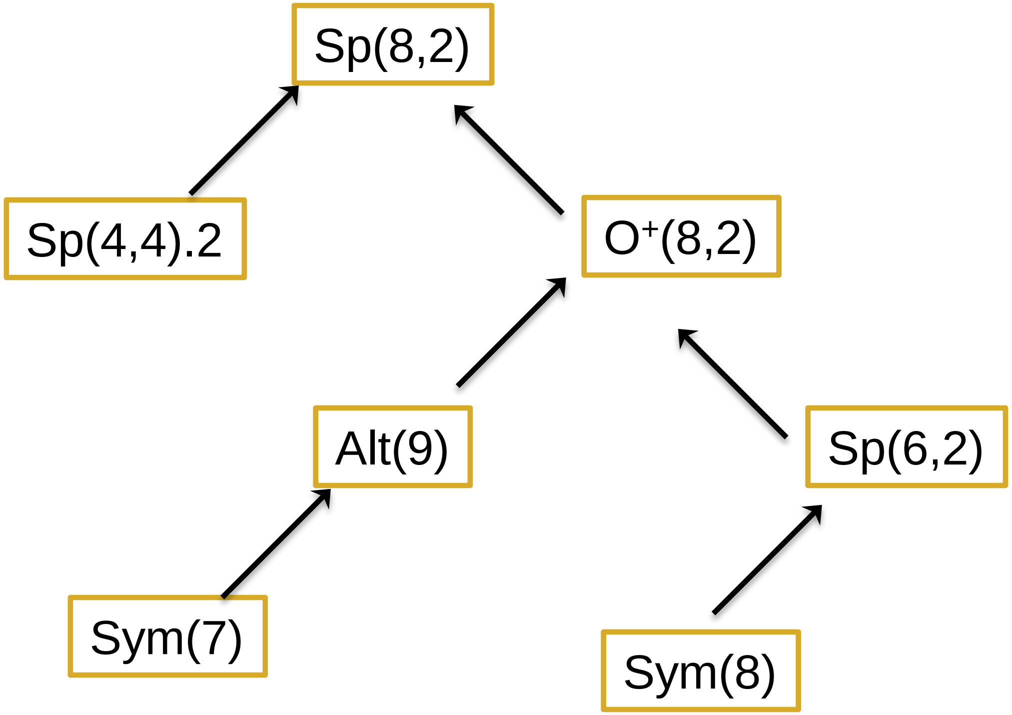
What did we learn about primitive Cayley graphs?

- B simple: Cay(B,S) is vertex-primitive but not a complete graph then
  - Either S is a union of B-conjugacy classes
  - Or B = Alt(p^2-2) (p prime, p $\equiv$ 3 (mod 4))
- In both cases examples exist (for each S, and each p respectively)

What did we notice: interesting coincidences



- Among the examples:
  - sometimes several primitive groups share the same regular subgroup
  - notably SEVEN primitive groups on 120 points contain a regular subgroup Sym(5)   [lattice of containments below]



Some open problems

1. Regular subgroups of primitive product action groups
   - Does there exist an almost simple primitive H < Sym(Y) with NO regular subgroup such that H wr Sym(k) acting on Y^k has a regular subgroup?

2. Determine the kinds of regular subgroups of affine primitive groups apart from the translation subgroup (Some exist: Hegedus 2000)

3. Find groups with a regular subgroup among the quasiprimitive and innately transitive permutation groups – hence find Cayley graphs admitting these actions

4. Extend the classification of almost simple group factorisations (not just maximal ones)

Thank you

- I tried to
- Describe simple groups factorisations
- Sample of applications in group theory and combinatorics

Photo. Courtesy: Joan Costa joancostaphoto.com