



THE UNIVERSITY OF  
WESTERN AUSTRALIA

# Simple Group Factorisations and Applications in Combinatorics: Lecture 2

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ACHIEVE INTERNATIONAL EXCELLENCE



## In lecture 1 we saw:

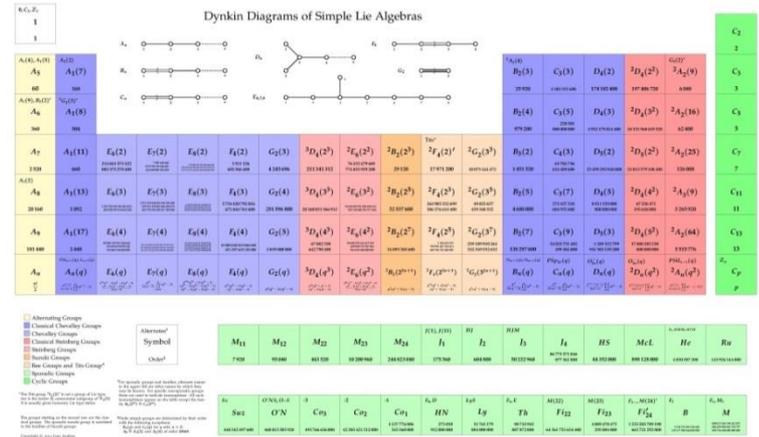
- ↘ Classifying “Maximal subgroups of  $\text{Sym}(X)$  and  $\text{Alt}(X)$ ” ( $X$  finite) required
  - O’Nan—Scott Theorem for the primitive types
  - Maximal factorisations of all almost simple groups
  
- ↘ Studying symmetric (point-transitive) structures often requires knowledge of full automorphism group
  - Problem: finding overgroups of given transitive groups
  - Solving this: combination of “refined O’Nan—Scott” and almost simple group factorisations
  
- ↘ **This lecture:** a bit about the almost simple factorisations; a start on an using them.



## Alternating and symmetric groups $G = \text{Alt}(n)$ and $\text{Sym}(n)$

- ↘  $G = AB$  (and neither  $A$  nor  $B$  contains  $\text{Alt}(n)$ )
  - $A$ , say, satisfies  $\text{Alt}(k) \times \text{Alt}(n-k) \leq A \leq \text{Sym}(k) \times \text{Sym}(n-k)$
  - While  $B$  is  $k$ -homogeneous (transitive on  $k$ -subsets)
  - Where  $k = 1, 2, 3, 4$ , or  $5$
  - [or some extra cases when  $n = 6, 8$ , and  $10$ ]
  
- ↘ Comments
  - 1980 Wiegold & Williamson classified those with  $A \cap B = 1$
  - $k$ -homogeneous groups known explicitly [using simple group classn.]

# The Periodic Table Of Finite Simple Groups



## Sporadic almost simple groups

➤ Sporadic almost simple group  $G = AB$

- 1986 Gantchev if both A, B simple
- 1990 Liebeck, CEP, Saxl if both A and B maximal
  - Generous help from Rob Wilson
- 2006 Giudici all of them

Courtesy: [Ivan Andrus](#) 2012

[J. Algebra]

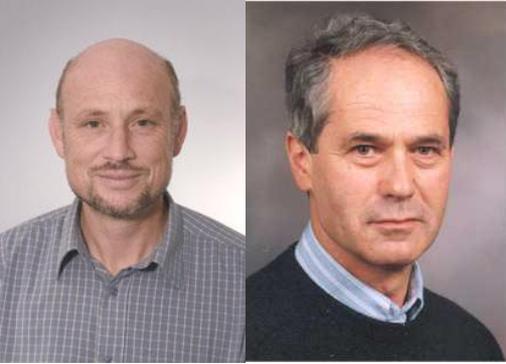
➤ Comments: Mathieu groups have many; some (e.g. Monster) have none



# The Periodic Table Of Finite Simple Groups

Dynkin Diagrams of Simple Lie Algebras

Order	Symbol														
2	$C_2$	3	$C_3$	4	$C_4$	5	$C_5$	6	$C_6$	7	$C_7$	8	$C_8$	9	$C_9$
10	$C_{10}$	11	$C_{11}$	12	$C_{12}$	13	$C_{13}$	14	$C_{14}$	15	$C_{15}$	16	$C_{16}$	17	$C_{17}$
18	$C_{18}$	19	$C_{19}$	20	$C_{20}$	21	$C_{21}$	22	$C_{22}$	23	$C_{23}$	24	$C_{24}$	25	$C_{25}$
26	$C_{26}$	27	$C_{27}$	28	$C_{28}$	29	$C_{29}$	30	$C_{30}$	31	$C_{31}$	32	$C_{32}$	33	$C_{33}$
34	$C_{34}$	35	$C_{35}$	36	$C_{36}$	37	$C_{37}$	38	$C_{38}$	39	$C_{39}$	40	$C_{40}$	41	$C_{41}$
42	$C_{42}$	43	$C_{43}$	44	$C_{44}$	45	$C_{45}$	46	$C_{46}$	47	$C_{47}$	48	$C_{48}$	49	$C_{49}$
50	$C_{50}$	51	$C_{51}$	52	$C_{52}$	53	$C_{53}$	54	$C_{54}$	55	$C_{55}$	56	$C_{56}$	57	$C_{57}$
58	$C_{58}$	59	$C_{59}$	60	$C_{60}$	61	$C_{61}$	62	$C_{62}$	63	$C_{63}$	64	$C_{64}$	65	$C_{65}$
66	$C_{66}$	67	$C_{67}$	68	$C_{68}$	69	$C_{69}$	70	$C_{70}$	71	$C_{71}$	72	$C_{72}$	73	$C_{73}$
74	$C_{74}$	75	$C_{75}$	76	$C_{76}$	77	$C_{77}$	78	$C_{78}$	79	$C_{79}$	80	$C_{80}$	81	$C_{81}$
82	$C_{82}$	83	$C_{83}$	84	$C_{84}$	85	$C_{85}$	86	$C_{86}$	87	$C_{87}$	88	$C_{88}$	89	$C_{89}$
90	$C_{90}$	91	$C_{91}$	92	$C_{92}$	93	$C_{93}$	94	$C_{94}$	95	$C_{95}$	96	$C_{96}$	97	$C_{97}$
98	$C_{98}$	99	$C_{99}$	100	$C_{100}$	101	$C_{101}$	102	$C_{102}$	103	$C_{103}$	104	$C_{104}$	105	$C_{105}$
106	$C_{106}$	107	$C_{107}$	108	$C_{108}$	109	$C_{109}$	110	$C_{110}$	111	$C_{111}$	112	$C_{112}$	113	$C_{113}$
114	$C_{114}$	115	$C_{115}$	116	$C_{116}$	117	$C_{117}$	118	$C_{118}$	119	$C_{119}$	120	$C_{120}$	121	$C_{121}$
122	$C_{122}$	123	$C_{123}$	124	$C_{124}$	125	$C_{125}$	126	$C_{126}$	127	$C_{127}$	128	$C_{128}$	129	$C_{129}$
130	$C_{130}$	131	$C_{131}$	132	$C_{132}$	133	$C_{133}$	134	$C_{134}$	135	$C_{135}$	136	$C_{136}$	137	$C_{137}$
138	$C_{138}$	139	$C_{139}$	140	$C_{140}$	141	$C_{141}$	142	$C_{142}$	143	$C_{143}$	144	$C_{144}$	145	$C_{145}$
146	$C_{146}$	147	$C_{147}$	148	$C_{148}$	149	$C_{149}$	150	$C_{150}$	151	$C_{151}$	152	$C_{152}$	153	$C_{153}$
154	$C_{154}$	155	$C_{155}$	156	$C_{156}$	157	$C_{157}$	158	$C_{158}$	159	$C_{159}$	160	$C_{160}$	161	$C_{161}$
162	$C_{162}$	163	$C_{163}$	164	$C_{164}$	165	$C_{165}$	166	$C_{166}$	167	$C_{167}$	168	$C_{168}$	169	$C_{169}$
170	$C_{170}$	171	$C_{171}$	172	$C_{172}$	173	$C_{173}$	174	$C_{174}$	175	$C_{175}$	176	$C_{176}$	177	$C_{177}$
178	$C_{178}$	179	$C_{179}$	180	$C_{180}$	181	$C_{181}$	182	$C_{182}$	183	$C_{183}$	184	$C_{184}$	185	$C_{185}$
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194	$C_{194}$	195	$C_{195}$	196	$C_{196}$	197	$C_{197}$	198	$C_{198}$	199	$C_{199}$	200	$C_{200}$	201	$C_{201}$
202	$C_{202}$	203	$C_{203}$	204	$C_{204}$	205	$C_{205}$	206	$C_{206}$	207	$C_{207}$	208	$C_{208}$	209	$C_{209}$
210	$C_{210}$	211	$C_{211}$	212	$C_{212}$	213	$C_{213}$	214	$C_{214}$	215	$C_{215}$	216	$C_{216}$	217	$C_{217}$
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234	$C_{234}$	235	$C_{235}$	236	$C_{236}$	237	$C_{237}$	238	$C_{238}$	239	$C_{239}$	240	$C_{240}$	241	$C_{241}$
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258	$C_{258}$	259	$C_{259}$	260	$C_{260}$	261	$C_{261}$	262	$C_{262}$	263	$C_{263}$	264	$C_{264}$	265	$C_{265}$
266	$C_{266}$	267	$C_{267}$	268	$C_{268}$	269	$C_{269}$	270	$C_{270}$	271	$C_{271}$	272	$C_{272}$	273	$C_{273}$
274	$C_{274}$	275	$C_{275}$	276	$C_{276}$	277	$C_{277}$	278	$C_{278}$	279	$C_{279}$	280	$C_{280}$	281	$C_{281}$
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370	$C_{370}$	371	$C_{371}$	372	$C_{372}$	373	$C_{373}$	374	$C_{374}$	375	$C_{375}$	376	$C_{376}$	377	$C_{377}$
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386	$C_{386}$	387	$C_{387}$	388	$C_{388}$	389	$C_{389}$	390	$C_{390}$	391	$C_{391}$	392	$C_{392}$	393	$C_{393}$
394	$C_{394}$	395	$C_{395}$	396	$C_{396}$	397	$C_{397}$	398	$C_{398}$	399	$C_{399}$	400	$C_{400}$	401	$C_{401}$
402	$C_{402}$	403	$C_{403}$	404	$C_{404}$	405	$C_{405}$	406	$C_{406}$	407	$C_{407}$	408	$C_{408}$	409	$C_{409}$
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434	$C_{434}$	435	$C_{435}$	436	$C_{436}$	437	$C_{437}$	438	$C_{438}$	439	$C_{439}$	440	$C_{440}$	441	$C_{441}$
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450	$C_{450}$	451	$C_{451}$	452	$C_{452}$	453	$C_{453}$	454	$C_{454}$	455	$C_{455}$	456	$C_{456}$	457	$C_{457}$
458	$C_{458}$	459	$C_{459}$	460	$C_{460}$	461	$C_{461}$	462	$C_{462}$	463	$C_{463}$	464	$C_{464}$	465	$C_{465}$
466	$C_{466}$	467	$C_{467}$	468	$C_{468}$	469	$C_{469}$	470	$C_{470}$	471	$C_{471}$	472	$C_{472}$	473	$C_{473}$
474	$C_{474}$	475	$C_{475}$	476	$C_{476}$	477	$C_{477}$	478	$C_{478}$	479	$C_{479}$	480	$C_{480}$	481	$C_{481}$
482	$C_{482}$	483	$C_{483}$	484	$C_{484}$	485	$C_{485}$	486	$C_{486}$	487	$C_{487}$	488	$C_{488}$	489	$C_{489}$
490	$C_{490}$	491	$C_{491}$	492	$C_{492}$	493	$C_{493}$	494	$C_{494}$	495	$C_{495}$	496	$C_{496}$	497	$C_{497}$
498	$C_{498}$	499	$C_{499}$	500	$C_{500}$	501	$C_{501}$	502	$C_{502}$	503	$C_{503}$	504	$C_{504}$	505	$C_{505}$
506	$C_{506}$	507	$C_{507}$	508	$C_{508}$	509	$C_{509}$	510	$C_{510}$	511	$C_{511}$	512	$C_{512}$	513	$C_{513}$
514	$C_{514}$	515	$C_{515}$												



# The Periodic Table Of Finite Simple Groups

## Classical Lie type groups G

- The simple groups  $PSL$ ,  $PSp$ ,  $PSU$ ,  $P\Omega^+$ ,  $P\Omega^-$ ,  $P\Omega^\circ$
- $G = AB$

Courtesy: [Ivan Andrus](#) 2012

- 1990 Liebeck CEP Saxl found all **maximal** factorisations
  - All families of groups factorise except odd dimensional  $PSU$
  - Five pages of tables -- published in AMS Memoir

➤ Why so hard? What more known?

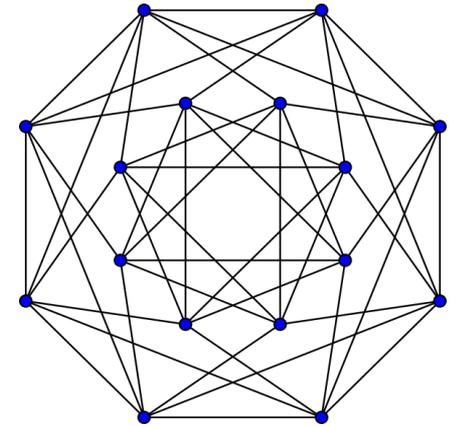
➤ 2010 Liebeck CEP Saxl

$A$  maximal and  $A \cap B = 1$   
[exact factorisations]

- Just really hard – complete classification not in sight

## Applications of factorisations

- ↘ **Recall: if  $G < H < \text{Sym}(X)$  then**  $G$  is transitive if and only if  $H_\alpha G = H$
- ↘ Often use factorisations to explore existence of larger groups preserving a point-transitive structure.
- ↘ **“Algebraic example”**: Maximal subgroup problem. Deciding if an almost simple primitive group is maximal
- ↘ We consider two applications: to graphs and cartesian decompositions



Let  $\Gamma$  be a graph and  $G < \text{Aut}(\Gamma)$  be transitive on arcs and primitive on vertices [arcs: “directed edges”]

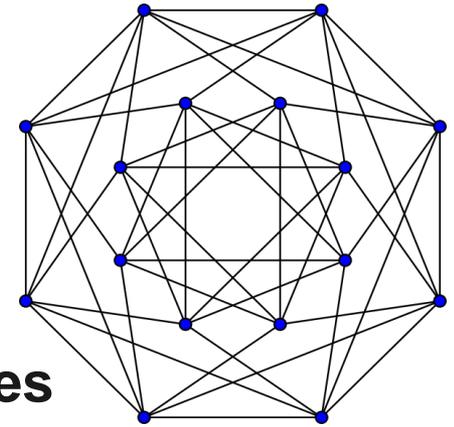
- ↘ Is it possible for  $\text{Aut}(\Gamma)$  to be “very much bigger” than  $G$ ?
- ↘ Could we have  $G < H \leq \text{Aut}(\Gamma)$  and  $G, H$  have different socles?

↘ Surely yes, sometimes.

Socle is the subgroup generated by all the minimal normal subgroups

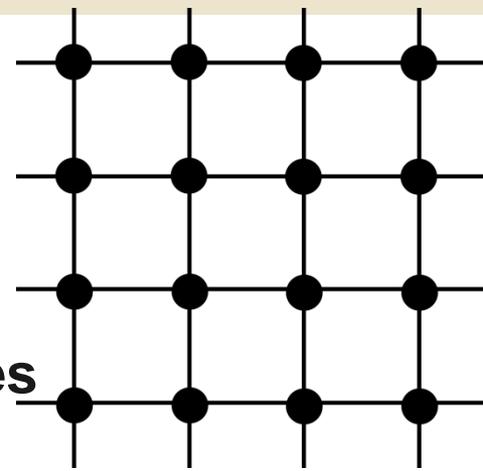
↘ **Example:**  $\Gamma$  the “triangle graph” with vertices pairs from  $\{1, 2, \dots, n\}$  and edges  $\{A, B\}$  if the pairs  $A, B$  meet.  $\text{Aut}(\Gamma) = \text{Sym}(n)$

- Take  $G$  any 3-transitive subgroup of  $\text{Sym}(n)$ ;  $G$  is arc-transitive and usually vertex-primitive
- E.g. If  $n=q+1$  then  $G = \text{PGL}(2, q) < \text{Sym}(q+1)$
- E.g.  $M_{11} < M_{12} < \text{Sym}(12)$



**$G < H \leq \text{Aut}(\Gamma)$  with  $G$  transitive on arcs and primitive on vertices, and  $G, H$  with different socles**

- ↘ How could we classify them all?
- ↘ Understand what happens in the groups: let  $X =$  set of vertices.
- ↘ Then the set of arcs (directed edges) is an orbit for both  $G$  and  $H$  in  $X \times X$
- ↘ Also the vertex stabilisers:  $G_\alpha$  maximal in  $G$ , and  $H_\alpha$  maximal in  $H$
- ↘ And we have factorisations:  $H = G H_\alpha$  and for an arc  $(\alpha, \beta)$ ,  $H_\alpha = G_\alpha H_{\alpha\beta}$
- ↘ Tools/Methods: O’Nan—Scott Theorem and factorisations
- ↘ Lead first to source of generic examples: .....



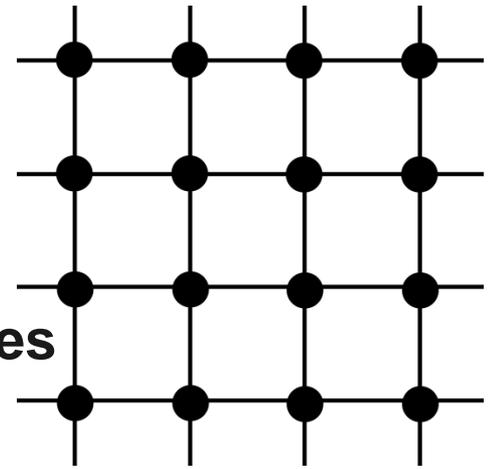
**$G < H \leq \text{Aut}(\Gamma)$  with  $G$  transitive on arcs and primitive on vertices, and  $G, H$  with different socles**

- ↘ ONS-product-type:  **$H$  preserves cartesian decomposition  $X = Y^k$  with  $k > 1$**
- Then  $H < \text{Sym}(Y)$  wr  $\text{Sym}(k)$  in “product action”
  - Each of  $G, H$  “induces” a primitive  $G_0 < H_0 < \text{Sym}(Y)$
  - Gives  $H < H_0$  wr  $\text{Sym}(k)$  and  $G < G_0$  wr  $\text{Sym}(k)$

We give a construction:  
A cartesian product of graphs

- Each example  $\Gamma_0$  with  $G_0 < H_0 < \text{Aut}(\Gamma_0)$  lifts to an example  $\Gamma$  for  $G < H$
- With  $\text{soc}(G) = \text{soc}(G_0)^k$  and  $\text{soc}(H) = \text{soc}(H_0)^k$

How typical are these examples?

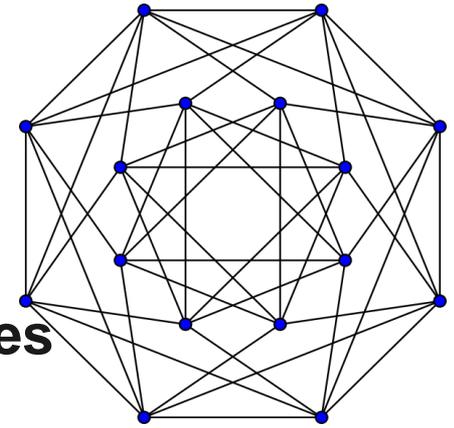


**$G < H \leq \text{Aut}(\Gamma)$  with  $G$  transitive on arcs and primitive on vertices, and  $G, H$  with different socles**

↘ **Analysis tricky:** if  $H < H_0$  wr  $\text{Sym}(k)$  and  $X = Y^k$  with  $k > 1$  then

- Either  $\text{soc}(G) = \text{soc}(G_0)^k$  and  $\text{soc}(H) = \text{soc}(H_0)^k$  **and we find all possibilities for  $G_0$  and  $H_0$**
- or ~3 exceptional cases: e.g.  $G = \text{Sym}(6).2 < H = \text{Sym}(6)$  wr  $\text{Sym}(2)$  [other two have  $G_0 = M_{12}$  and  $G_0 = \text{Sp}(4,4)$  ]

Unexpected cartesian decompositions  
preserved by simple groups – more on this later



**$G < H \leq \text{Aut}(\Gamma)$  with  $G$  transitive on arcs and primitive on vertices, and  $G, H$  with different socles**

↘ Lot of hard work dealing with all other ONS-types for  $H$ :

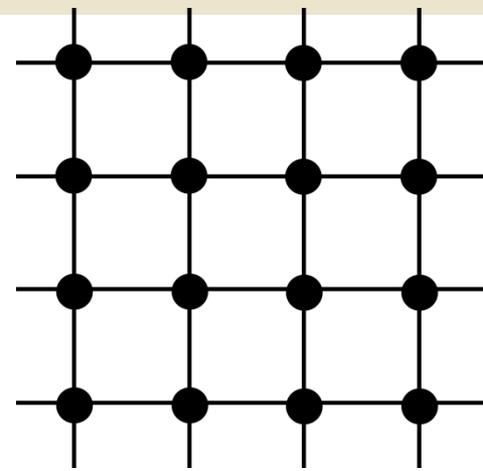
↘ Tool: “Primitive Inclusions”: classification of possible ONS-types for  $(G, H)$

1990 CEP

↘ Suppose  $H$  does not preserve a cartesian decomposition. We show

- If one of  $G$  or  $H$  is affine then
  - $\Gamma$  is complete graph  $K_n$  and  $G = [\text{affine}] < H = \text{Alt}(n)$  or  $\text{Sym}(n)$   
[or one exception  $G = \text{PSL}(2,7) < H = \text{AGL}(3,2)$ ]
- The only other possibility is that  $G, H$  are both almost simple.
  - Then  $H = G H_\alpha$  is a maximal factorisation and also  $H_\alpha = G_\alpha H_{\alpha\beta}$
  - Two pages of examples – giving values for  $G, H$ , vertex action, valency

## Second application: decide if a permutation group preserves a cartesian decomposition



- ↘ Given  $G < \text{Sym}(X)$ . Can we identify  $X = Y^k$  such that  $G \leq \text{Sym}(Y)$  wr  $\text{Sym}(k)$  with  $k > 1$  ?
- Question underlies O’Nan—Scott Theorem for primitive groups
  - Solution needed to decide maximality/inclusions of “quasiprimitive” groups
  - More general question. Can  $X = Y_1 \times \dots \times Y_k$  with  $Y_i$  different sizes
- ↘ **Easy “normal” example:** If say  $G = \text{Sym}(Y)$  wr  $\text{Sym}(k)$  then the cartesian decomposition corresponds to a direct decomposition of  $\text{soc}(G) = \text{Alt}(Y)^k$

## Not so obvious example.

- ↘  $G = M_{12}$  has two classes of subgroups of index 12 [isomorphic to  $M_{11}$ ]
- ↘ If  $A, B$  are representatives then  $G = AB$  so
- ↘ the  $G$ -coset action on  $X := [G : A \cap B]$  of size 144 preserves a cartesian decomposition  $X = Y \times Y$  with  $|Y|=12$
- ↘ So  $G < M_{12}$  wr  $\text{Sym}(2)$
  
- ↘ This behaviour is unusual but not unique
  
- ↘ **2004 Baddeley, CEP, Schneider** determined all transitive actions of **simple** groups which preserve a cartesian decomposition.
  - All on  $Y^2 - 2$  individual examples and two families [involving  $P\Omega^+(8,q)$  and  $\text{Sp}(4,q)$ ]

## Links with group factorisations

- ↘ Suppose  $G < \text{Sym}(X)$  and  $G$  has a transitive minimal normal subgroup  $M$ 
  - True for primitive, quasiprimitive, innately transitive groups
- ↘ Choose point  $\alpha$  in  $X$
- ↘ Each cartesian decomposition  $Y_1 \times \dots \times Y_k$  of  $X$  preserved by  $G$  determines **Cartesian Factorisation of  $M$**  a set of  $k$  subgroups  $K_1, \dots, K_k$  of  $M$  such that
  - $K_1 \cap \dots \cap K_k = M_\alpha$
  - For all  $i=1, \dots, k$ ,  $M = K_i \left( \bigcap_{j \neq i} K_j \right)$  [k factorisations of  $M$ ]
- ↘ **2004 Baddeley, CEP Schneider** One-to-one correspondence between the  $G$ -invariant cartesian decompositions of  $X$  and the cartesian factorisations of  $M$  (relative to  $\alpha$ )

**Examples:  $G$  preserves  $X = Y_1 \times \dots \times Y_k$ ; minimal normal subgroup  $M$**

↘ **“Normal” Case:**

- $M = T_1 \times \dots \times T_k$
- let  $\alpha = (y_1, \dots, y_k)$  and  $L_i = (T_i)_{y_i}$
- Define cartesian factorisation by
- $K_1 = L_1 \times T_2 \times \dots \times T_k, \dots, K_k = T_1 \times \dots \times T_{k-1} \times L_k$

↘ **Conditions:**

- $K_1 \cap \dots \cap K_k = L_1 \times \dots \times L_k = M_\alpha$
- and each  $M = K_i (\cap_{j \neq i} K_j)$  holds

## Role of simple group factorisations: one simple example

- ↘  $T$  nonabelian simple group with factorisation  $T = AB$ 
  - Diagonal  $D = \{ (t, t) \mid t \in T \}$  copy of  $T$  in  $T \times T$  a “strip”
  - Define  $E = \{ (t, t) \mid t \in A \cap B \}$
- ↘ **Critical property:**  $T \times T = D(A \times B)$ 
  - To write arbitrary  $(u, v)$  as  $(t, t)(a, b)$
  - Express  $u^{-1}v = a^{-1}b$  with  $a$  in  $A$ ,  $b$  in  $B$  and note that  $t := ua^{-1} = vb^{-1}$
  - Then  $(t, t)(a, b) = (ua^{-1}, vb^{-1})(a, b) = (u, v)$
- ↘ **The Example:**
  - $M = T \times T \times T \times T$
  - $K_1 = A \times B \times D$
  - $K_2 = D \times A \times B$
- ↘ **Conditions:**
  - $K_1 K_2 = M$  and  $K_1 \cap K_2 = E \times E = M_\alpha$

Set acted on:  $X = Y \times Y$   
 Where  $Y = [T \times T : E]$   
 And  $\alpha = (E, E)$  in  $X$

## **Rich theory of cartesian decompositions preserved by groups with a transitive minimal normal subgroup**

### ↘ Involves

- Cartesian factorisations of characteristically simple groups  $T^k$
- Factorisations of characteristically simple groups

### ↘ Leads to

- Understanding of subgroup lattice above a (quasi)primitive group
- Tools for studying overgroups of such groups arising as automorphism groups

## Summary

- ↘ What is known about maximal factorisations of almost simple groups
- ↘ Using ONS Theory & factorisations to
  - study graph automorphisms
  - Detect if cartesian decompositions preserved
- ↘ **Third lecture:** different kind of application – Cayley graphs



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**Thank you**

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