THE UNIVERSITY OF WESTERN AUSTRALIA

Simple Group Factorisations and Applications in Combinatorics: Lecture 2

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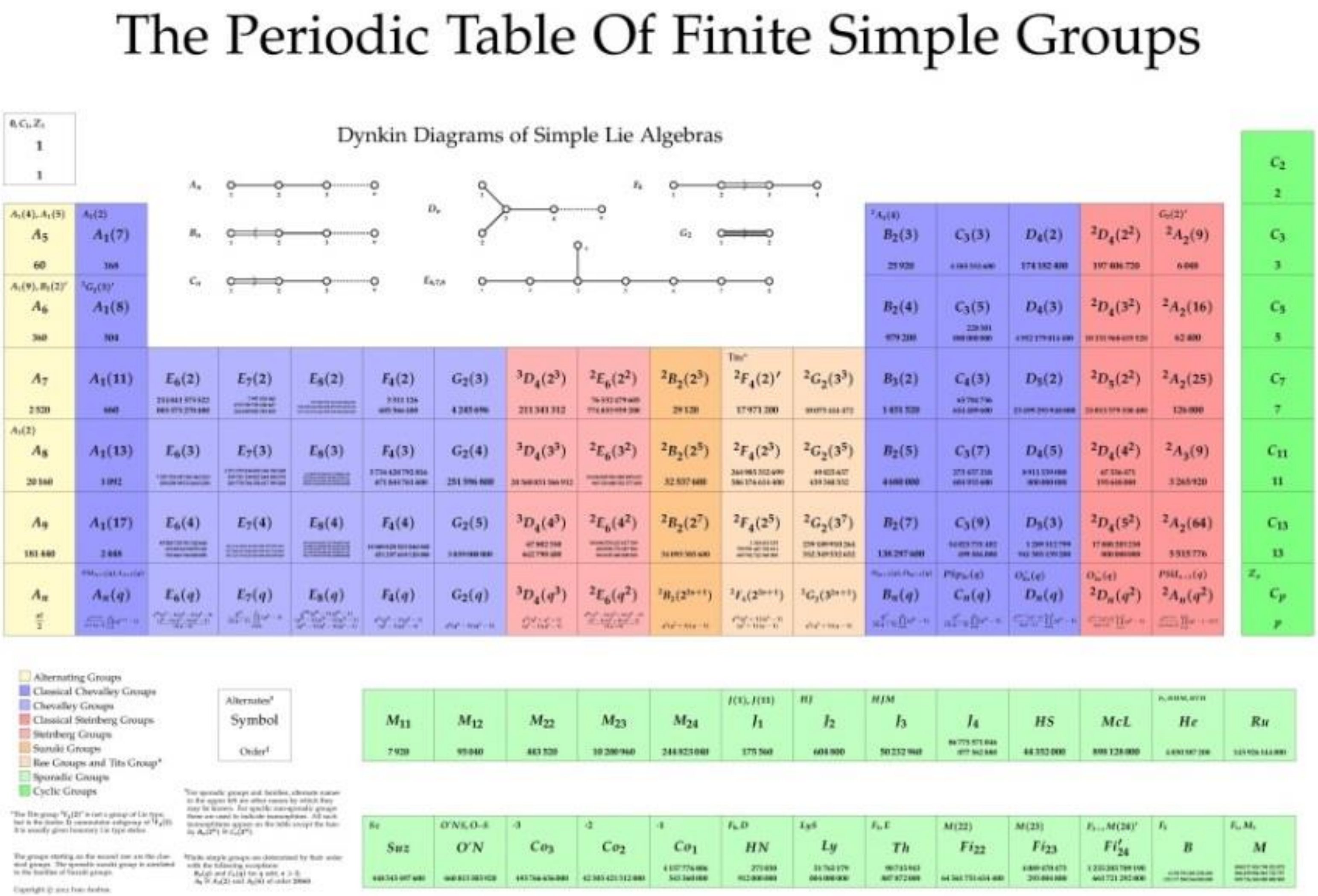
In lecture 1 we saw:

- Classifying “Maximal subgroups of Sym(X) and Alt(X)” (X finite) required
  - O’Nan—Scott Theorem for the primitive types
  - Maximal factorisations of all almost simple groups

- Studying symmetric (point-transitive) structures often requires knowledge of full automorphism group
  - Problem: finding overgroups of given transitive groups
  - Solving this: combination of “refined O’Nan—Scott” and almost simple group facrtoisations

- **This lecture:** a bit about the almost simple factorisations; a start on an using them.

Remember the kinds of finite simple groups:

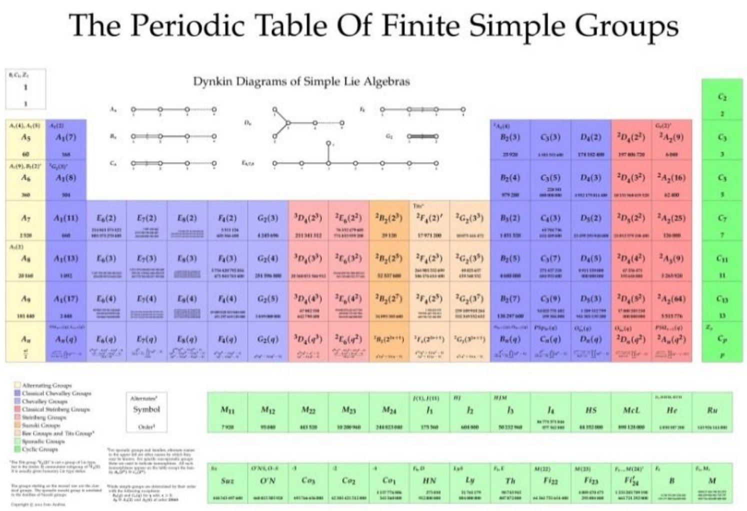


Courtesy: [Ivan Andrus](#) 2012

Alternating and symmetric groups G = Alt(n) and Sym(n)

- G = AB (and neither A not B contains Alt(n))
  - A, say, satisfies Alt(k) x Alt(n-k) ≤ A ≤ Sym(k) x Sym(n-k)
  - While B is k-homogeneous (transitive on k-subsets)
  - Where k = 1, 2, 3, 4, or 5
  - [or some extra cases when n = 6, 8, and 10]

- Comments
  - 1980 Wiegold & Williamson classified those with A ∩ B = 1
  - k-homogeneous groups known explicitly [using simple group classn.]



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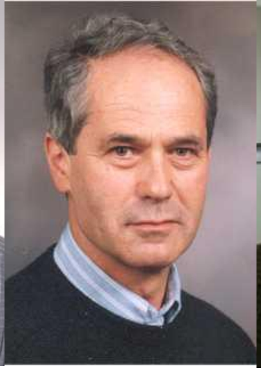
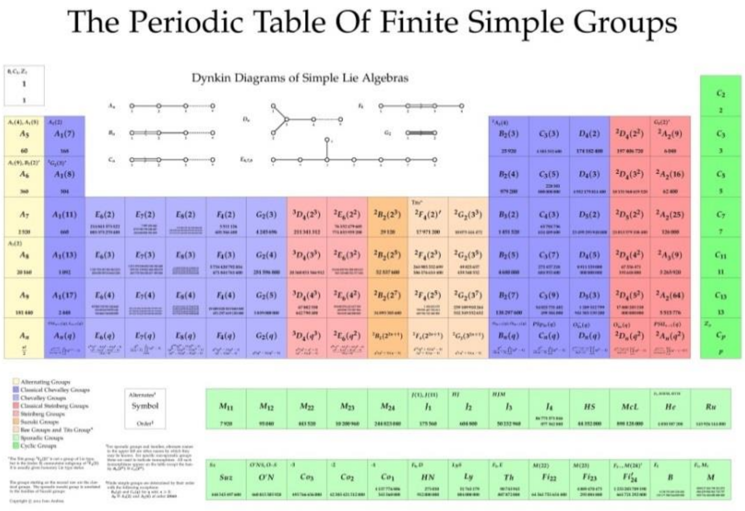
Sporadic almost simple groups

- Sporadic almost simple group G = AB

- 1986 Gentchev — if both A, B simple
- 1990 Liebeck, CEP, Saxl — if both A and B maximal
  - Generous help from Rob Wilson
- 2006 Giudici — all of them — [J. Algebra]

- Comments: Mathieu groups have many; some (e.g. Monster) have none

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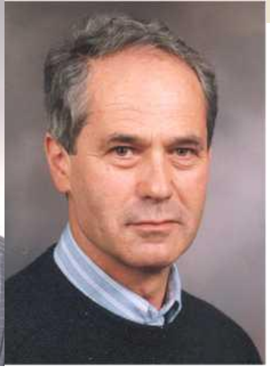
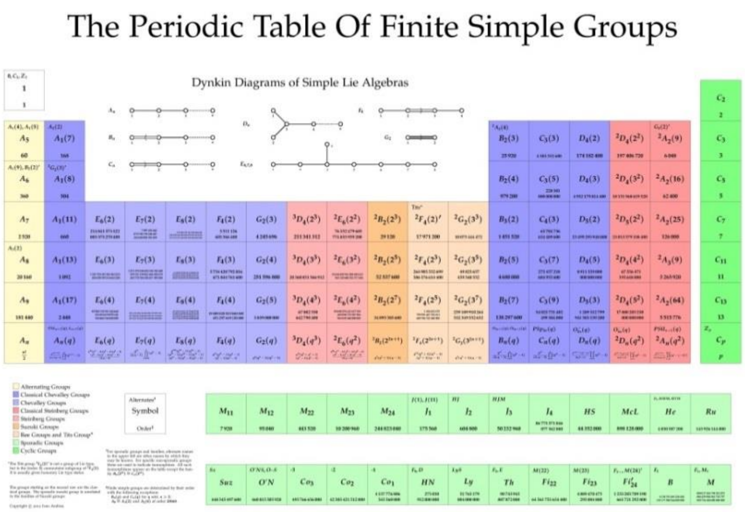
Exceptional Lie type groups G

- G = AB
- 1987 Herring Liebeck Saxl found all of them
  - Only groups factorisable G are $G_2(3^c)$, $G_2(4)$ and $F_4(2^c)$
- This left the classical groups to be dealt with [most difficult case!]

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Classical Lie type groups G

- The simple groups PSL, PSp, PSU, $P\Omega^{+}$, $P\Omega^{-}$, $P\Omega^{o}$
- G = AB

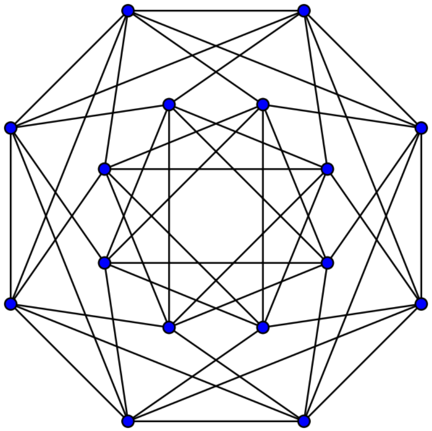
- 1990 Liebeck CEP Saxl found all maximal factorisations
  - All families of groups factorise except odd dimensional PSU
  - Five pages of tables -- published in AMS Memoir

- Why so hard? What more known?
- 2010 Liebeck CEP Saxl A maximal and $A \cap B = 1$ [exact factorisations]
  - Just really hard – complete classification not in sight

Applications of factorisations

- Recall: if $G < H < Sym(X)$ then G is transitive if and only if $H_\alpha G = H$
- Often use factorisations to explore existence of larger groups preserving a point-transitive structure.
- "Algebraic example": Maximal subgroup problem. Deciding if an almost simple primitive group is maximal
- We consider two applications: to graphs and cartesian dcompositions

2002 Liebeck, CEP, Saxl



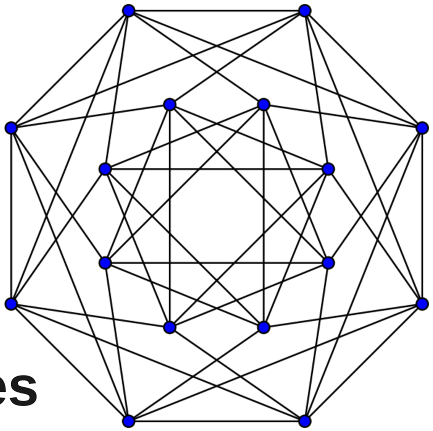
Let Γ be a graph and G < Aut(Γ) be transitive on arcs and primitive on vertices [arcs: "directed edges"]

- Is it possible for Aut(Γ) to be "very much bigger" than G?
- Could we have G < H ≤ Aut(Γ) and G, H have different socles?

Socle is the subgroup generated by all the minimal normal subgroups

- Surely yes, sometimes.
- Example: Γ the "triangle graph" with vertices pairs from {1,2,…,n} and edges {A, B} if the pairs A, B meet. Aut(Γ) = Sym(n)
  - Take G any 3-transitive subgroup of Sym(n); G is arc-transitive and usually vertex-primitive
  - E.g. If n=q+1 then G = PGL(2,q) < Sym(q+1)
  - E.g. $M_{11} < M_{12} <$ Sym(12)

2002 Liebeck, CEP, Saxl



$G < H \leq Aut(\Gamma)$ with G transitive on arcs and primitive on vertices, and G, H with different socles

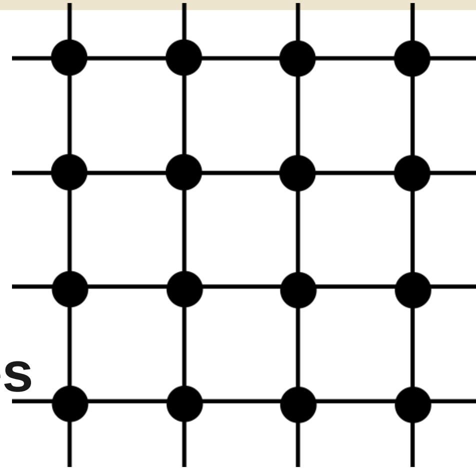
- How could we classify them all?

- Understand what happens in the groups: let X = set of vertices.
- Then the set of arcs (directed edges) is an orbit for both G and H in X x X
- Also the vertex stabilisers: G_α maximal in G, and H_α maximal in H

- And we have factorisations: $H = G H_\alpha$ and for an arc (α, β), $H_\alpha = G_\alpha H_{\alpha\beta}$

- Tools/Methods: O'Nan—Scott Theorem and factorisations

- Lead first to source of generic examples: …..



2002 Liebeck, CEP, Saxl

$G < H \leq \mathrm{Aut}(\Gamma)$ with G transitive on arcs and primitive on vertices, and G, H with different socles

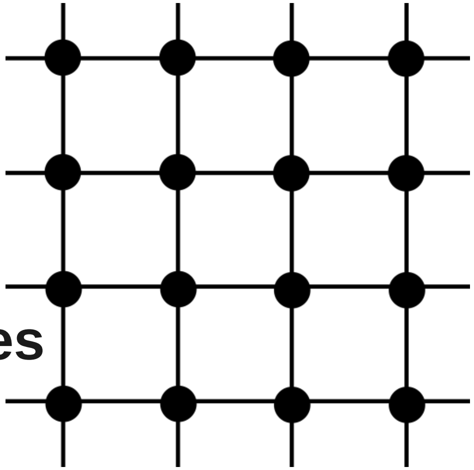
- ONS-product-type: H preserves cartesian decomposition $X = Y^k$ with $k > 1$
  - Then $H < \mathrm{Sym}(Y) \text{ wr } \mathrm{Sym}(k)$ in "product action"
  - Each of G, H "induces" a primitive $G_0 < H_0 < \mathrm{Sym}(Y)$
  - Gives $H < H_0 \text{ wr } \mathrm{Sym}(k)$ and $G < G_0 \text{ wr } \mathrm{Sym}(k)$

We give a construction:
A cartesian product of graphs

  - Each example Γ_0 with $G_0 < H_0 < \mathrm{Aut}(\Gamma_0)$ lifts to an example Γ for $G < H$
  - With $\mathrm{soc}(G) = \mathrm{soc}(G_0)^k$ and $\mathrm{soc}(H) = \mathrm{soc}(H_0)^k$

How typical are these examples?

2002 Liebeck, CEP, Saxl



$G < H \leq Aut(\Gamma)$ with G transitive on arcs and primitive on vertices, and G, H with different socles

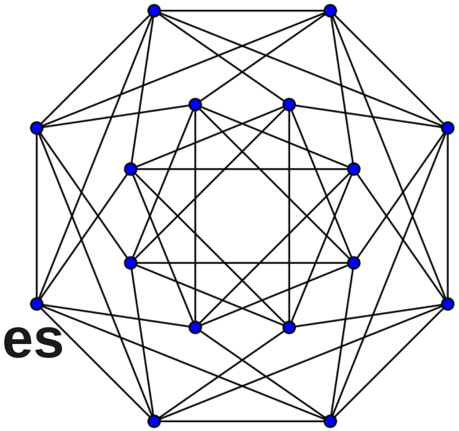
↘ Analysis tricky: if $H < H_0$ wr Sym(k) and $X = Y^k$ with $k > 1$ then

- Either $soc(G) = soc(G_0)^k$ and $soc(H) = soc(H_0)^k$ and we find all possibilities for G_0 and H_0

- or ~3 exceptional cases: e.g. G = Sym(6).2 < H = Sym(6) wr Sym(2) [other two have $G_0 = M_{12}$ and $G_0 = Sp(4,4)$]

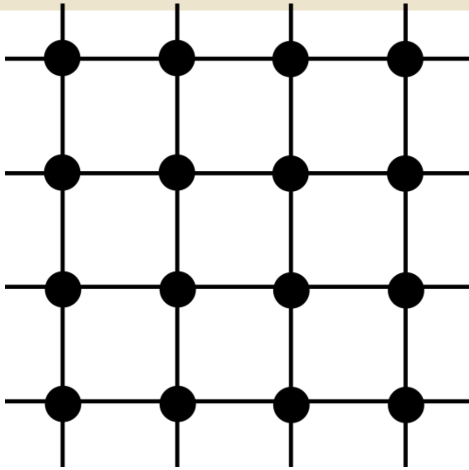
Unexpected cartesian decompositions preserved by simple groups – more on this later

2002 Liebeck, CEP, Saxl



$G < H \leq Aut(\Gamma)$ with G transitive on arcs and primitive on vertices, and G, H with different socles

- Lot of hard work dealing with all other ONS-types for H:
- Tool: “Primitive Inclusions”: classification of possible ONS-types for (G, H) 1990 CEP
- Suppose H does not preserve a cartesian decomposition. We show
  - If one of G or H is affine then
    - Γ is complete graph K_n and G = [affine] < H = Alt(n) or Sym(n)
      [or one exception G = PSL(2,7) < H = AGL(3,2)]
  - The only other possibility is that G, H are both almost simple.
    - Then $H = G H_\alpha$ is a maximal factorisation and also $H_\alpha = G_\alpha H_{\alpha\beta}$
    - Two pages of examples – giving values for G, H, vertex action, valency



Second application: decide if a permutation group preserves a cartesian decomposition

- Given $G < Sym(X)$. Can we identify $X = Y^k$ such that $G \leq Sym(Y)$ wr $Sym(k)$ with $k > 1$?

  - Question underlies O'Nan—Scott Theorem for primitive groups
  - Solution needed to decide maximality/inclusions of "quasiprimitive" groups

  - More general question. Can $X = Y_1 \times \ldots \times Y_k$ with Y_i different sizes

- Easy "normal" example: If say $G = Sym(Y)$ wr $Sym(k)$ then the cartesian decomposition corresponds to a direct decomposition of $soc(G) = Alt(Y)^k$

Not so obvious example.

- $G = M_{12}$ has two classes of subgroups of index 12 [isomorphic to M_{11}]
- If A, B are representatives then G = AB so
- the G-coset action on $X := [G : A \cap B]$ of size 144 preserves a cartesian decomposition X = Y x Y with |Y|=12
- So $G < M_{12}$ wr Sym(2)

- This behaviour is unusual but not unique

- 2004 Baddeley, CEP, Schneider determined all transitive actions of simple groups which preserve a cartesian decomposition.
  - All on Y^2 – 2 individual examples and two families [involving $P\Omega^+(8,q)$ and Sp(4,q)]

Links with group factorisations

- Suppose $G < Sym(X)$ and G has a transitive minimal normal subgroup M
  - True for primitive, quasiprimitive, innately transitive groups

- Choose point α in X
- Each cartesian decomposition $Y_1 \times \ldots \times Y_k$ of X preserved by G determines Cartesian Factorisation of M   a set of k subgroups $K_1,\ldots,K_k$ of M such that
  - $K_1 \cap \ldots \cap K_k = M_\alpha$
  - For all i=1,…,k, $M = K_i (\cap_{j \neq i} K_j)$ [k factorisations of M]

- 2004 Baddeley, CEP Schneider   One-to-one correspondence between the G-invariant cartesian decompositions of X and the cartesian factorisations of M (relative to α)

Examples: G preserves $X = Y_1 \times \dots \times Y_k$; minimal normal subgroup M

- "Normal" Case:
  - $M = T_1 \times \dots \times T_k$
  - let $\alpha = (y_1, \dots, y_k)$ and $L_i = (T_i)_{y_i}$
  - Define cartesian factorisation by
  - $K_1 = L_1 \times T_2 \times \dots \times T_k, \dots, K_k = T_1 \times \dots \times T_{k-1} \times L_k$

- Conditions:
  - $K_1 \cap \dots \cap K_k = L_1 \times \dots \times L_k = M_\alpha$
  - and each $M = K_i \left(\cap_{j \neq i} K_j \right)$ holds

Role of simple group factorisations: one simple example

- T nonabelian simple group with factorisation T = AB
  - Diagonal D = { (t , t) | t in T } copy of T in T x T a "strip"
  - Define E = { (t , t) | t in $A \cap B$ }
- Critical property: T x T = D (A x B)
  - To write arbitrary (u,v) as (t,t)(a,b)
  - Express $u^{-1}v = a^{-1}b$ with a in A, b in B and note that $t := ua^{-1} = vb^{-1}$
  - Then (t, t)(a, b) = (ua^{-1} , vb^{-1})(a,b) = (u,v)
- The Example:
  - M = T x T x T x T
  - K_1 = A x B x D
  - K_2 = D x A x B
- Conditions:
  - $K_1 K_2 = M$ and $K_1 \cap K_2 = E \times E = M_\alpha$

Set acted on: X = Y x Y
Where Y = [TxT : E]
And α = (E, E) in X

Rich theory of cartesian decompositions preserved by groups with a transitive minimal normal subgroup

- Involves
  - Cartesian factorisations of characteristically simple groups T^k
  - Factorisations of characteristically simple groups
- Leads to
  - Understanding of subgroup lattice above a (quasi)primitive group
  - Tools for studying overgroups of such groups arising as automorphism groups

Summary

- What is known about maximal factorisations of almost simple groups
- Using ONS Theory & factorisations to
  - study graph automorphisms
  - Detect if cartesian decompositions preserved
- **Third lecture:** different kind of application – Cayley graphs

Thank you

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