THE UNIVERSITY OF WESTERN AUSTRALIA

Simple Group Factorisations and Applications in Combinatorics
Lecture 1

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ACHIEVE INTERNATIONAL EXCELLENCE

Group products and factorisations

- Group G and proper subgroups A, B such that G=AB
- Interesting **contrasts** between construction and decomposition
- Well known **constructions:**
- Direct product G = A x B where both A, B normal in G and $A \cap B = 1$
- Semidirect product G = A.B where A normal in G and $A \cap B = 1$
- "General product" G = A.B where $A \cap B = 1$ [B H Neumann, 1935]

A few words about "general products": $G = AB$, with $A \cap B = 1$

- Bernhard Neumann (1935) recognised interpretation
  - G acting on coset space [G:A] with B a regular subgroup
- Later rediscovered and called Zappa—Redei—Szep products
- But already occurred in de Seguier's book 1904
- 2014 Angore & Militaru "bicrossed product" construction for these general products

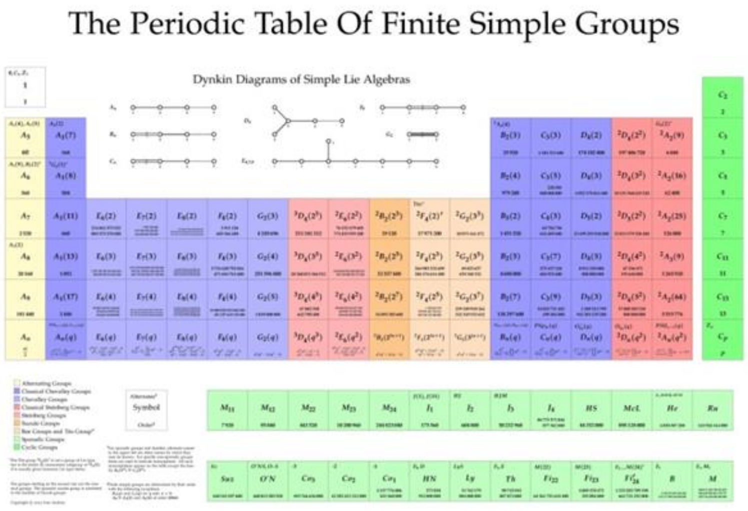
A few words about "general products": G= AB, with $A \cap B = 1$

- Bernhard Neumann (1935) recognised interpretation
  - G acting on coset space [G:A] with B a regular subgroup
- Coset space: [G:A] = { Ag | g in G }
- G-action: x in G maps Ag to Agx by "right multiplication"
- B regular: B is transitive (each coset of the form Ab for some b in B)
  & only the identity of B fixes any coset (Agb=Ag iff b=1)

Why factorisations? Why simple groups?

Simple group factorisations:

- What is known?
- How applied?



Courtesy: Ivan Andrus 2012

If extra time then

- Different kinds of factorisations

Why factorisations?

- Example Study “symmetric” “structures” X in which all “points” are “equivalent” under “structure-preserving maps”

Structure X		
Graph		
Linear space		
Group		

“Symmetric” will mean:
all “points” of X “equivalent” under “structure-preserving maps”

- “Points”?   “Structure-preserving maps”?

Structure X	Points	Maps
Graph	Vertices, or edges	Edge-preserving permutations of vertices
Linear space	Points or lines	Line-preserving permutations of points
Group	Involutions ($x^2=1$)	Group automorphisms

“Symmetric” will mean:
all “points” of X “equivalent” under “structure-preserving maps”

- Automorphism group: Aut(X) = { structure-preserving maps }

- “Equivalent”: Aut(X) transitive on points: for all points α, β there exists h in Aut(X) such that $\alpha^h = \beta$ (h maps α to β)

- Problem: Have found $G < \mathrm{Aut}(X)$, G transitive on points of X, how to decide if $G = \mathrm{Aut}(X)$?

- Use fact from theory of group actions: all transitive group actions “permutationally isomorphic” to “coset actions”

Coset actions? We have $G < Aut(X)$ with G transitive

- Choose a point of X α
- Consider the stabiliser of α in $H = Aut(X)$ - call it H_α

Since G transitive does not matter which point chosen

- Identify "points of X" with "cosets of H_α" i.e. with $[H : H_\alpha]$
  - α corresponds to H_α
  - α^h corresponds to $H_\alpha h$ for each h in Aut(X)

Well defined?
Yes and bijective since
- $H_\alpha h = H_\alpha g$
- iff hg^{-1} in H_α
- iff $\alpha^h = \alpha^g$

- Right multiplication action on cosets: for g in H
  - g maps α^h to α^{hg} corresponds to
  - g maps $H_\alpha h$ to $H_\alpha hg$

Problem: Have $G < H$ = Aut(X) with G transitive how to decide if $G = H$?

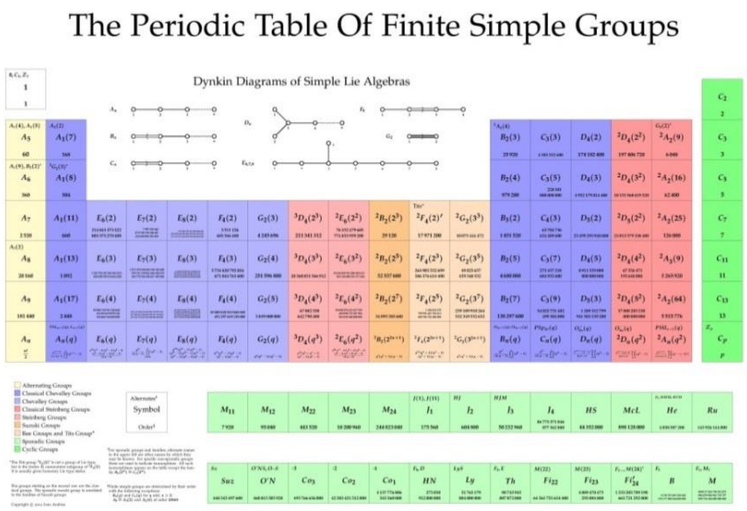
- G transitive (using coset action) means: $\{ H_\alpha g \mid g \text{ in } G \}$ = all the cosets
- Equivalently: factorisation $H_\alpha G = H$
- So if $G < H$ then G is transitive if and only if $H_\alpha G = H$
- Studying whether G = Aut(X) closely linked to "searching for factorisations"

Why simple group factorisations?   Since ….

- Early studies focused on questions like: Given G=AB and certain properties of A and B, does G inherit similar properties?

- 1911 W. Burnside's p^aq^b-Theorem - could be interpreted ….
- 1955 **Noboru Ito's famous theorem:** A, B abelian implies derived group G' is abelian and G is metabelian
- 1958, 1961 Wielandt & Kegel – A, B nilpotent implies G soluble



Courtesy: Ivan Andrus 2012

Why simple groups? [when studying factorisations]

- Classifying the finite simple groups

"one of the greatest achievements of twentieth century mathematics"

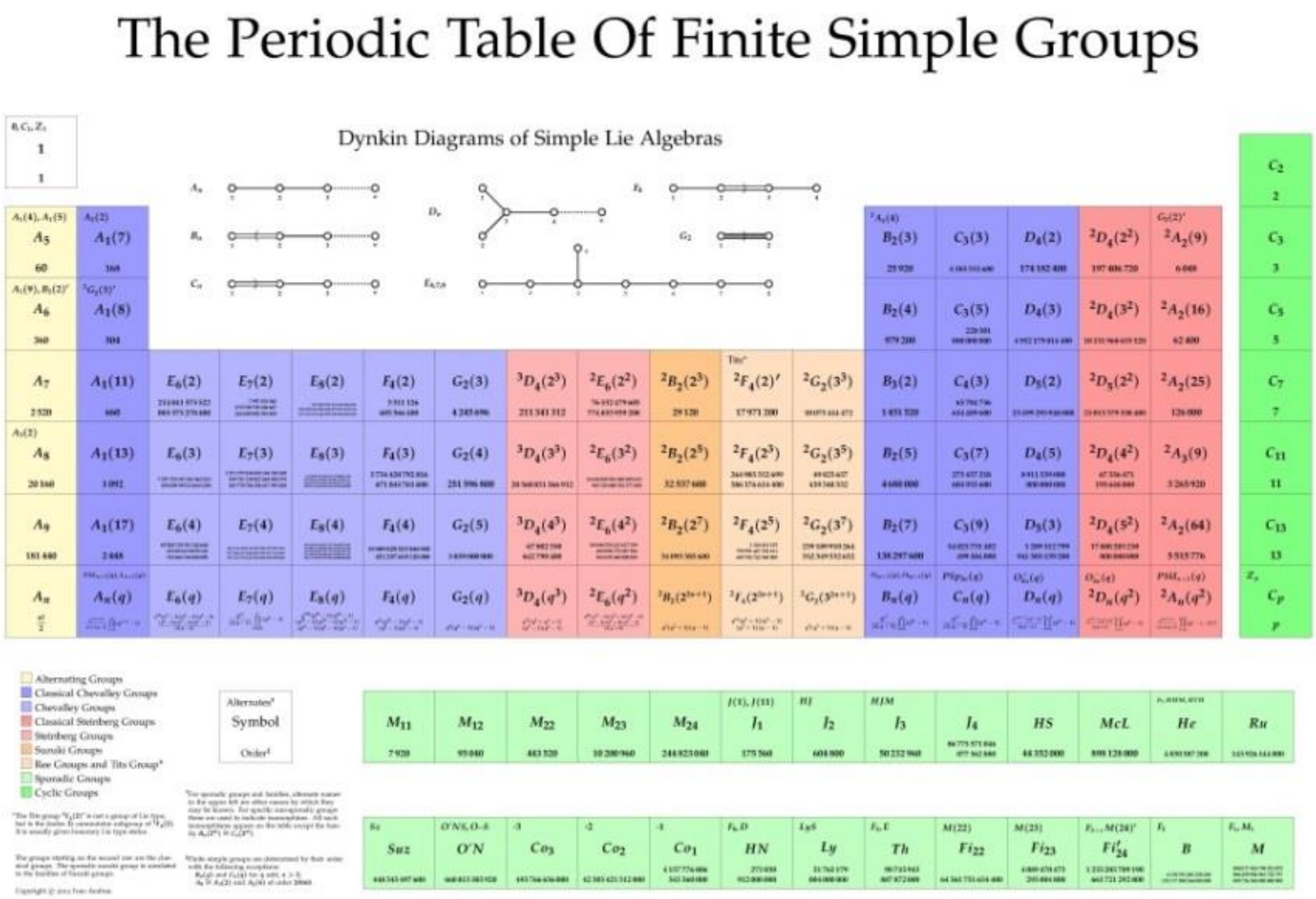
[From 2008 Abel Prize citation for J. G. Thompson and Jacques Tits]

Although known "by name" even simply stated problems remain open:

What do all the largest (maximal) subgroups of the simple groups look like?

"Periodic table" depicts simple groups: columns are infinite families, bottom green rows are sporadic simple groups

A closer look “for fun”



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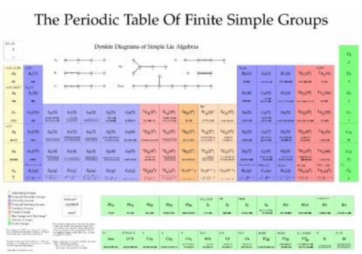
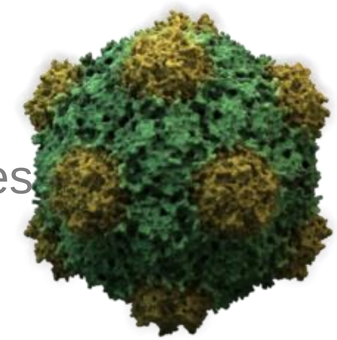


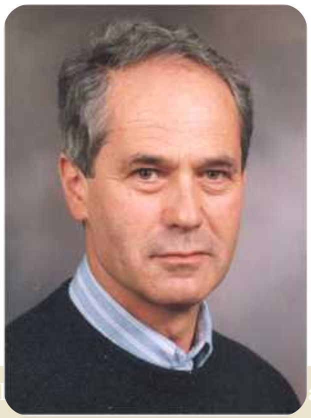
Left hand column of periodic table

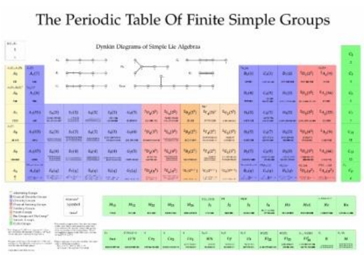
Smallest A_5 symmetry group of several small viruses



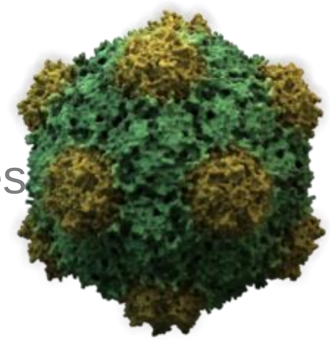
Alternating groups A_n: an infinite family of simple groups

- 1980 Start with H = Alt(n) or Sym(n) – find all maximal subgroups G
- By 1986 some cases solved
  - 1983 Guralnick n prime power
  - 1985 Liebeck & Saxl n=kp with p prime and k < p
  - 1985, 87 Kantor, Liebeck & Saxl n odd





Left hand column of periodic table
Smallest A_5 symmetry group of several small viruses



Alternating groups A_n: an infinite family of simple groups

- 1990 Liebeck, CEP, Saxl: Reduced the problem of classifying maximal subgroups of simple groups A_n
- To a problem involving all simple groups:

Classify all factorisations S=AB of all simple groups S
with A, B maximal subgroups of S.

- Solution occupies a research monograph: useful for many applications
- Will try to explain where this reduction comes from and how it can be used

Really need to know maximal subgroups of "almost simple" groups H where H between simple group S and Aut(S)

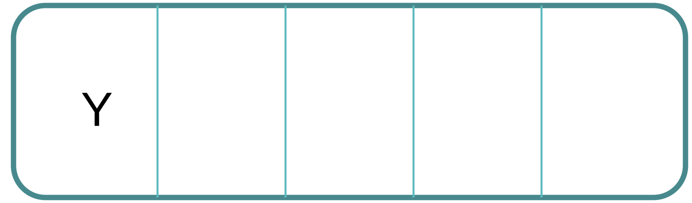
Maximal subgroups G of H = Sym(X) or Alt(X) where X = { 1, 2, … , n }

- For simplicity take H = Sym(X), and G < H [maximal]
- G is a permutation group on X - analyse properties of the G-action

- G intransitive on X means G preserves a proper subset Y of X
- So G contained in the largest such group Sym(Y) x Sym(X\Y)
- Question then is: when is Sym(Y) x Sym(X\Y) maximal in H?

- Answer is: almost always maximal - exception when 2.|Y| = n
  [when swapping Y and X\Y gives a larger subgroup]

- This gives one "type" of maximal subgroup
- And in all other cases G is transitive on X

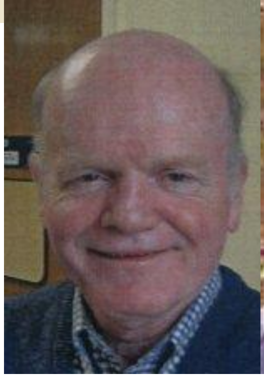


r equal-sized parts, permuted by G

Maximal subgroups G of H = Sym(X) where X = { 1, 2, … , n }

- Take analysis further so G transitive on X Next case:
- G imprimitive on X means G preserves a nontrivial partition P of X
- So G contained in the largest such group Stab(P) = Sym(Y) wr Sym(r)
- Question: when is Stab(P) maximal in H?
- Answer: Stab(P) is always maximal
- [a tiny exception when H = Alt(8) with $\text{Stab}_H(P)$ contained in an affine group AGL(3,2)]
- This gives second "type" of maximal subgroup
- And in all other cases G is primitive on X – G preserves no nontrivial partitions
- Equivalently stabiliser G_α is maximal in G

O'Nan—Scott theory is the
"post-classification standard"
for analysing finite group actions



Maximal subgroups G of H = Sym(X) where X = { 1, 2, … , n }

- Analysing the primitive groups G in a similar manner

- One of the first initiatives involving group actions following the simple group classification. First done independently by Michael O'Nan and Leonard Scott

- Original formulation of "O'Nan—Scott Theorem"
- Description of (primitive hopefully maximal) subgroups G of Sym(X) of Alt(X)

- Affine type: X = finite vector space and G = AGL(X)
- Diagonal type: maximal of this type $S^k.(Out(S) \times Sym(k))$ where S simple, k > 1
- Product type: maximal of this type stabilisers of cartesian decompositions of $X=Y^k$
- Almost simple type: $S \leq G \leq Aut(S)$

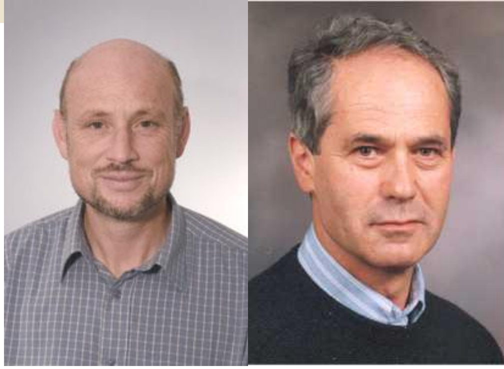
- Big Question: if G is maximal of its ONS-type when is G maximal in Sym(X) or Alt(X)?

Solving the Question: if G is maximal of its ONS-type when is G maximal in Sym(X) or Alt(X)?

- Affine type: X = finite vector space and G = AGL(X)
- Diagonal type: maximal of this type $S^k.(Out(S) \times Sym(k))$ where S simple, k > 1
- Product type: maximal of this type stabilisers of cartesian decompositions of $X=Y^k$
- Almost simple type: $S \leq G \leq Aut(S)$

- Affine type: always [four exceptions if H=Alt(X) and n = |X| = 7, 11, 17, 23]
- Diagonal type: always
- Product type: always
- Almost simple type: $S \leq G \leq Aut(S)$ The Difficult Case!!

- [Liebeck, CEP, Saxl 1987]



Solving the Question: if G is maximal with simple socle S, when is G maximal in Sym(X) or Alt(X)?

- Almost simple type: $S \leq G \leq Aut(S)$ S is the socle of G

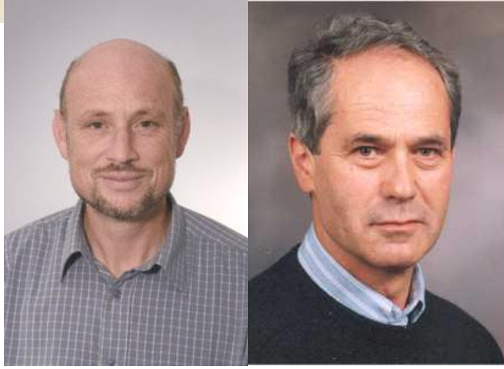
Socle is the subgroup generated by all the minimal normal subgroups

- Take $G = N_{Sym(X)}(S)$ [largest of its ONS type with socle S]
- If $G < H \leq Sym(X)$ with H maximal of its ONS type, then H is almost simple or

If such an H exists

S	H	Type of H
PSL(2,7)	AGL(3,2)	Affine
A_6	S_6 wr S_2	Product
M_{12}	S_{12} wr S_2	Product
Sp(4,q) q even q > 2	S_m wr S_2 $m=q^2(q^2-1)/2$	Product

- [Liebeck, CEP, Saxl]



Major question: when can a primitive G with simple socle S be properly contained in another group H with simple socle T in Sym(X) or Alt(X)?

- Almost simple type: $S \leq G \leq Aut(S)$ and $T \leq H \leq Aut(T)$ and $G < H < Sym(X)$

When does such an H exist?

If no such H exists then G is maximal

- Already know the answer: we need a factorisation:
  - $H = G H_\alpha$ with H_α maximal in H and
  - G does not contain T and (if we wish) G maximal in H
  - This is called a maximal factorisation of H
  - We need to know all maximal factorisations of all almost simple groups H with one factor G also almost simple and intersection $G \cap H_\alpha$ maximal in G
- [Liebeck, CEP, Saxl]




Summarising where we are:

- Classifying “Maximal subgroups of Sym(X) and Alt(X)” (X finite) required
  - O’Nan—Scott Theorem for the primitive types
  - Maximal factorisations of all almost simple groups

- Studying symmetric (point-transitive) structures often requires knowledge of full automorphism group
  - Problem: finding overgroups of given transitive groups
  - Solving this: combination of “refined O’Nan—Scott” and almost simple group facrtoisations

- Next steps: wee bit about how to find factorisations; lots more about problems where we want to use them.

Thank you

Photo. Courtesy: Joan Costa [joancostaphoto.com]