The fundamental theorems of invariant theory classical, quantum and super

Gus Lehrer

University of Sydney
NSW 2006
Australia

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If $\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = 1$ then $b^2 - ac = (b')^2 - a'c'$.

Thus the discriminant $\Delta = b^2 - ac$ is an invariant of quadratic forms under transformations by SL_2. (c.1000)

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• For a KG-module M (K a commutative ring), describe $K[M]^G$ (generators and relations). (Commutative algebra)

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Quantum group analogues of the above questions, with applications to representation theory, math. physics, unitarisability,...

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- sketch how to apply these ideas to super-algebras ($\mathbb{Z}_2$-graded algebras), using $\mathbb{Z}_2$-graded algebraic geometry, and to quantum groups.

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Then $A \subseteq Z_E(B)$ and $B \subseteq Z_E(A)$.

Now A is evidently semisimple, so (by double centraliser theory) $A = Z_E(B)$ if and only if $B = Z_E(A)$, and the former statement is the FFT.

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we see that $Z_E(A)$ is spanned by the elements $g \otimes g \otimes \ldots \otimes g$ ($g \in \mathrm{End}_{\mathbb{C}}(V)$), which proves the FFT. □

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We have a finite set of generators for $\mathrm{End}_{\mathrm{GL}(V)}(T^r(V))$, namely the permutations $\pi \in \mathrm{Sym}_r$. The SFT asks for all relations among these; thus the question is: what is $\ker(\omega_r : \mathbb{C}\mathrm{Sym}_r \longrightarrow \mathrm{End}_G(T^r(V))$?

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Theorem

SFT: If $r \leq m$, $\ker(\omega_r) = 0$. If $r \geq m+1$, $\ker(\omega_r)$ is the ideal of $\mathbb{C}\mathrm{Sym}_r$ generated by e_{m+1}^-.

Sketch of a proof of the SFT

Let $N = \ker(\omega_r)$. We have $\mathbb{C}\mathrm{Sym}_r = \oplus_\lambda I(\lambda)$, where λ runs over all partitions of r, and $I(\lambda)$ is a minimal 2-sided ideal.

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First inequality: done; second: show explicitly that if λ has at most m rows, then $\omega_r(I(\lambda)) \neq 0$. By Frobenius' theory, one knows an explicit idempotent $e(\lambda)$ which generates $I(\lambda)$; one constructs an element $\mathbf{v} \in T^r(V)$ such that $e(\lambda)\mathbf{v} \neq 0$.

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First inequality: done; second: show explicitly that if λ has at most m rows, then $\omega_r(I(\lambda)) \neq 0$. By Frobenius' theory, one knows an explicit idempotent $e(\lambda)$ which generates $I(\lambda)$; one constructs an element $\mathbf{v} \in T^r(V)$ such that $e(\lambda)\mathbf{v} \neq 0$.

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Sketch of a proof of the SFT

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This shows that $N = (e^-_{m+1})$, which is the SFT. □

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The SFT may be formulated in these contexts as well, but we leave that until we have the language of diagrams.

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$$\eta_r(\pi) = \begin{cases} \omega_r(\pi) \text{ if } G = O(V) (\textit{symmetric case}) \\ \varepsilon(\pi)\omega_r(\pi) \text{ if } G = \mathrm{Sp}(V) (\textit{symplectic case}) \end{cases}.$$

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If $e : v \otimes w \mapsto (v, w)c_0$, then $e \in \mathrm{End}_G(T^2(V))$, and since $e(c_0) = \sum_a (b_a, b'_a)c_0 = \pm m c_0$,

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we have $e^2 = \varepsilon m e$, where $\varepsilon = \begin{cases} 1 \text{ in the orthogonal case} \\ -1 \text{ in the symplectic case.} \end{cases}$

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Note that $-\mathrm{id}_V \in G$; since $-\mathrm{id}_V$ acts as $(-1)^r$ on $T^r(V)$, it follows that $\mathrm{Hom}_G(T^r(V), T^s(V)) \neq 0 \implies r \equiv s \bmod 2$.

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We've seen that $s_i^2 = 1$ and $e_i^2 = \varepsilon m e_i$ ($m = \dim V$). Also, the s_i satisfy the familiar braid relations (e.g. $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$), and we have $s_i e_i = e_i s_i = e_i$

Classical group invariants (cont). The Brauer category.

Less obvious relations: $e_i s_{i\pm1} e_i = e_i$, $e_i e_{i\pm1} e_i = e_i$, $s_i e_{i+1} s_i = s_{i+1} e_i s_{i+1}$.

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Brauer diagrams and the Brauer category

We next show how to interpret these endomorphisms as diagrams, which are morphisms in the Brauer category.

The Brauer category

Given a ring R and element $\delta \in R$, we define the Brauer category $\mathcal{B}(R, \delta) = \mathcal{B}(\delta)$.

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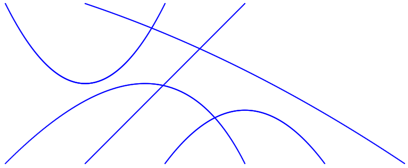

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and for diagrams D_1, D_2, $\boxed{D_1} \otimes \boxed{D_2} = \boxed{D_1 D_2}$ (juxtaposition)

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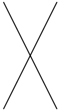
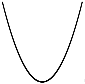
I= , X=  , A=  , U=  ,

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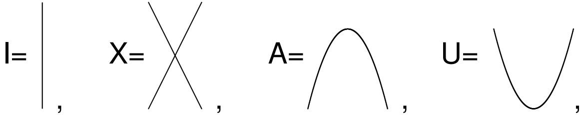

subject to a set of 7 relations, the most substantial of which can be depicted as follows.

Relations in the Brauer category

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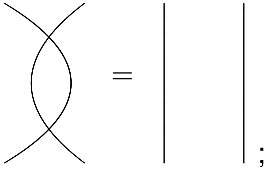

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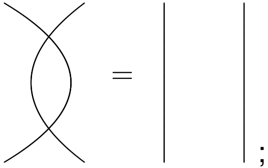

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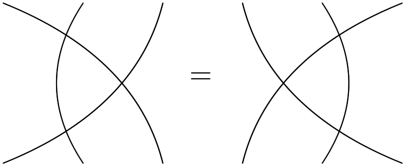

Braid relation

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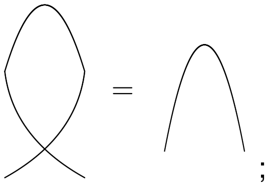
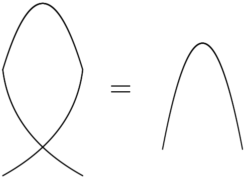


De-looping



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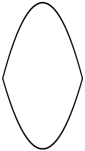
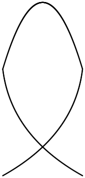
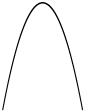
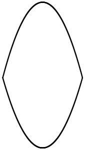


$= \delta : 0 \longrightarrow 0$

De-looping Loop Removal

 $=$  ;  $= \delta : 0 \longrightarrow 0$

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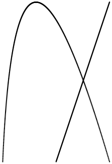
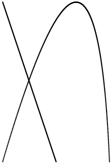
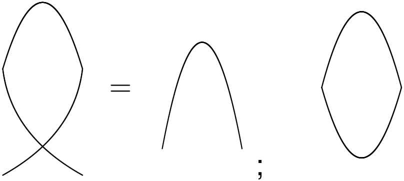
 $=$  ;

Sliding

$$(A \otimes I) \circ (I \otimes X) = (I \otimes A) \circ (X \otimes I);$$



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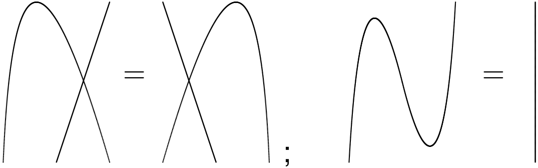

Sliding   Straightening

$(A \otimes I) \circ (I \otimes X) = (I \otimes A) \circ (X \otimes I);$ $(A \otimes I) \circ (I \otimes U) = I$

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Important fact (due to Rui & Si, using cellular structure): The Brauer algebra $\mathcal{B}_r(\mathbb{C}, z)$ is non-semisimple only if $z \in \mathbb{Z}$. If $z \in \mathbb{Z}_{\geq 0}$, the algebras $\mathcal{B}_r(z)$ and $\mathcal{B}_r(-2z)$ are semisimple $\iff r \leq z + 1$.

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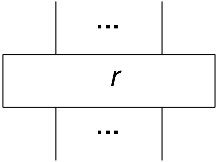

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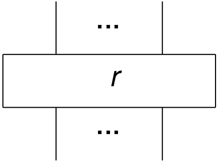

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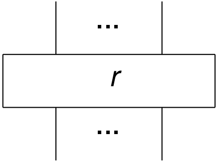

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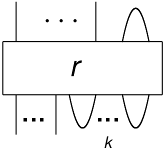

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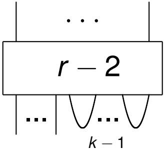

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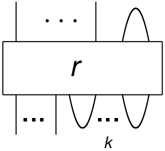

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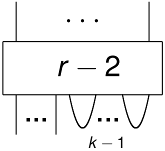

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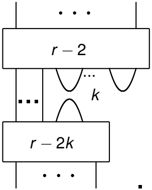
$(r - 2 - 2k)!^{-1}$



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Return to G, the isometry group of $V = \mathbb{C}^m, (-,-)$. Define $\mathcal{T}_G(V)$ to be the full subcategory of G-modules with objects $T^r(V), r = 0, 1, 2, \ldots$.

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Statements of the fundamental theorems.

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Theorem (SFT L-Zhang, 2012-13) Let $d = m$ if $G = O(V)$, and $d = \frac{m}{2}$ if $G = \mathrm{Sp}(V)$. The kernel of the map η_r is zero if $r \leq d$. If $r \geq d + 1$ the kernel is generated by a single idempotent in $\mathcal{B}_{d+1}(\varepsilon m)$ which is explicitly described in terms of diagrams as follows.

Generating idempotent: the symplectic case.

Here $m = 2d$, and we first take $r = d + 1$. For $k \leq \frac{d}{2}$, let Ψ_k be the element of $\mathcal{B}_{d+1}(-2d)$ described below

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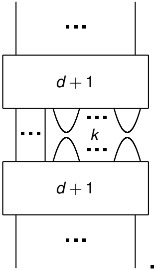

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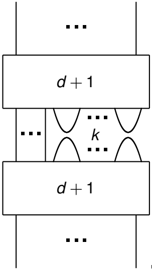

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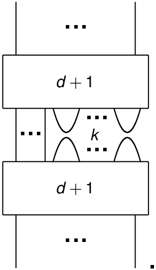
$\Psi_k =$ 

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$$\Phi = \sum_{k=0}^{\left[\frac{n+1}{2}\right]} a_k \Psi_k \quad \text{where} \quad a_k = \frac{1}{(2^k k!)^2 (n+1-2k)!}. \tag{1}$$

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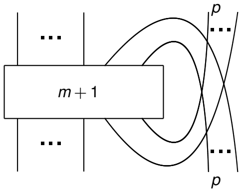
$E_{m+1-p} =$ 

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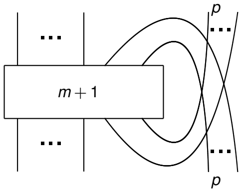
$E_{m+1-p} =$ 

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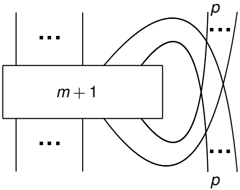

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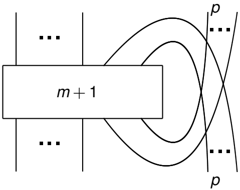
$E_{m+1-p} =$ 

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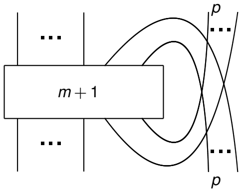
$E_{m+1-p} =$ 

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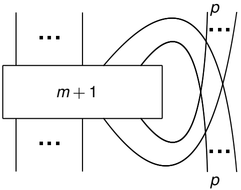

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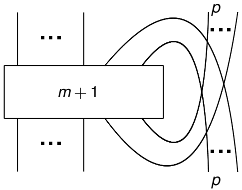
$E_{m+1-p} =$ 

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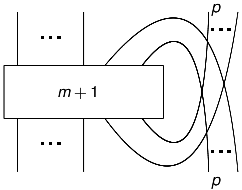
$E_{m+1-p} =$ 

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We will next sketch an independent proof, which essentially reduces the problem to the case of $\mathrm{GL}(V)$.

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Write $\mathbb{C}[Y]$ for the coordinate ring (ring of polynomial functions) of an affine variety Y over $\mathbb{C}$. Note that $\mathbb{C}[\mathbb{E}]$ and $\mathbb{C}[\mathbb{E}^+]$ are graded algebras.

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Main Lemma (d'après Atiyah) Let $f : \mathbb{E} \longrightarrow \mathbb{C}$ be an element of $\mathbb{C}[\mathbb{E}]$ such that $f(gA) = f(A)$ for all $g \in G$.

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We now show how to prove the FFT, given the main lemma.

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Theorem: $(T^r(V)^*)^G = \begin{cases} 0 \text{ if } r \text{ is odd} \\ \text{Span}\{\kappa_\pi \mid \pi \in \mathrm{Sym}_{2d} \text{ if } r = 2d, \end{cases}$

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Now take $r = 2d$, and let $L \in (T^r(V)^*)^G$. Define $\Psi_L : \mathbb{E} \times T^r(V) \longrightarrow \mathbb{C}$ by:

$$\Psi_L(A, v_1 \otimes \ldots \otimes v_{2d}) = L(Av_1 \otimes Av_2 \otimes \ldots \otimes Av_{2d}).$$

Then $\Psi_L \in \mathbb{C}^{2d}[\mathbb{E}] \otimes T^{2d}(V)^*$.

Proof of FFT for classical (later super) G

Statement of

Theorem: $(T^r(V)^*)^G = \begin{cases} 0 \text{ if } r \text{ is odd} \\ \mathrm{Span}\{\kappa_\pi \mid \pi \in \mathrm{Sym}_{2d} \text{ if } r = 2d, \end{cases}$

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For $g \in G$, by invariance of L, $\Psi_L(gA, \mathbf{v}) = \Psi_L(A, \mathbf{v})$, so we may apply the Main Lemma:

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(*) This is because: (i) $\omega : \mathbb{E} \to \mathbb{E}^+$ has generic fibre G; (ii) The Cayley transform $C : A \mapsto (1 - A)(1 + A)^{-1}$ defines a birational equivalence between $\mathbb{E}^-$ and G, and (iii) so $\dim(\mathbb{E}^+) = \dim(\mathbb{E}) - \dim(\mathbb{E}^-) = \dim(\mathbb{E}) - \dim(G) = \dim(\omega(\mathbb{E}))$.

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INJECTIVE MAP $h : ((T^{2d}(V))^*)^G \to \left((T^d(\mathbb{E}) \otimes T^{2d}(V))^*\right)^{\mathrm{GL}(V)}$.

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Next, we show how the injective map $h : ((T^{2d}(V))^*)^G \to \left((T^{2d}(V^*)\otimes T^{2d}(V))^*\right)^{\mathrm{GL}(V)}$ may be used to prove the SFT for classical (and orthosymplectic super-)groups.

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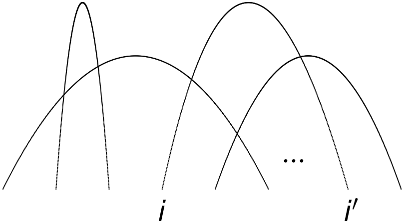

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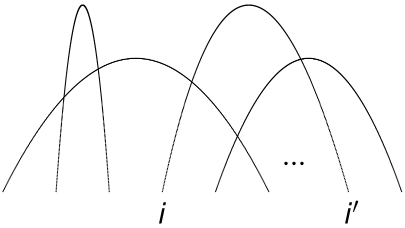

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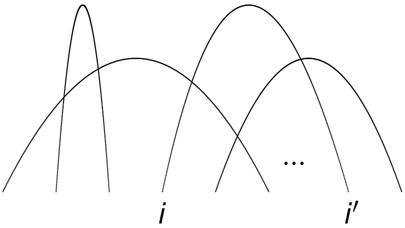

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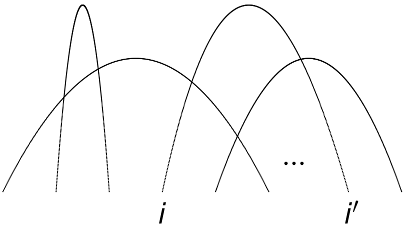

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The SFT identifies $\ker(\kappa)$.

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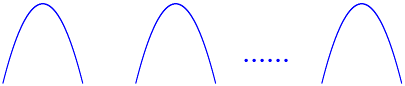

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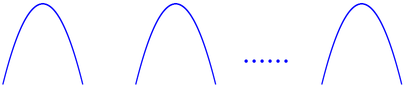

Its stabiliser C in Sym_{2d} is the centraliser of the involution $(1,2)(3,4)\ldots(2d-1,2d)$, a group of order $2^d d!$.

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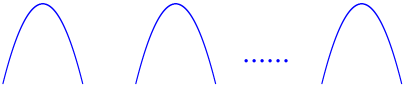

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Every diagram $D \in \mathcal{B}^0_{2d}$ may be expressed $D = D_0\pi_D$ with $\pi_D \in \mathrm{Sym}_{2d}$ unique up to premultiplication by an element $c \in C$, as shown below.

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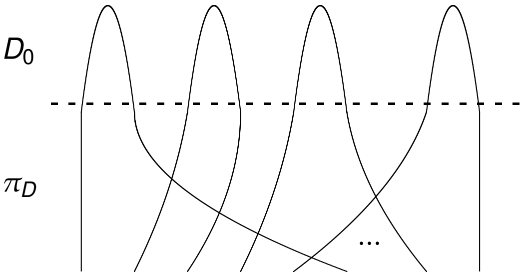

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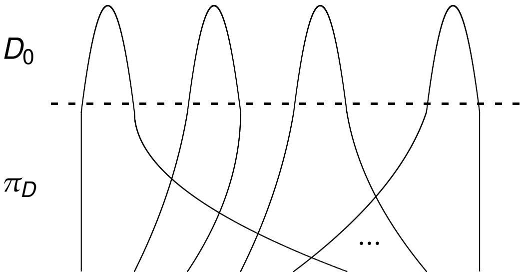

Now in addition to $\kappa : \mathcal{B}_{2d}^{0} \longrightarrow ((T^{2d}(V))^*)^G$, we also have $\delta : \mathcal{B}_{2d}^{2d} \longrightarrow \left((T^{2d}(V^*) \otimes T^{2d}(V))^*\right)^{\mathrm{GL}(V)}$,

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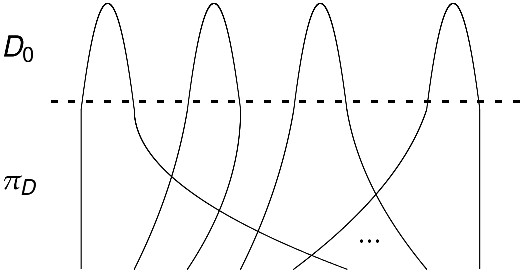

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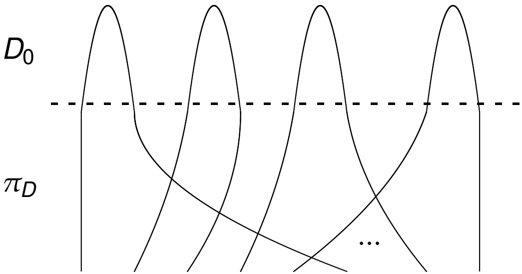

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where $\delta_\pi(\phi_1 \otimes \ldots \otimes \phi_{2d} \otimes v_1 \otimes \ldots \otimes v_{2d}) = \prod_{i=1}^{2d} \phi_i(v_{\pi(i)})$.

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- The maps κ and δ are as just described.
- h is the injective map we have constructed geometrically using the Main Lemma.
- The map $\chi : \mathcal{B}_{2d}^{0} \to \mathcal{B}_{2d}^{2d}$ is given by $D \mapsto e(C)\pi_D$, where π_D is such that $D = D_0\pi_D$ and $e(C) = |C|^{-1} \sum_{c \in C} c$.

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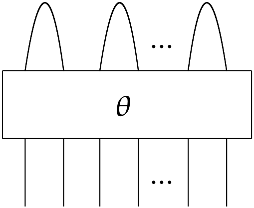

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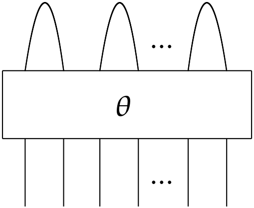

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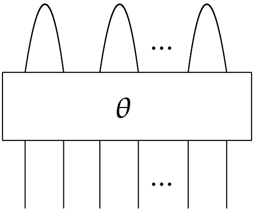

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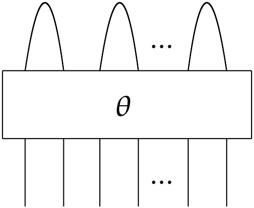

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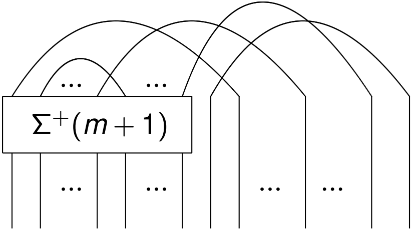

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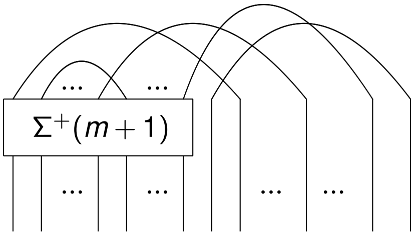

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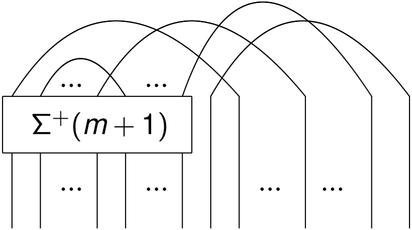

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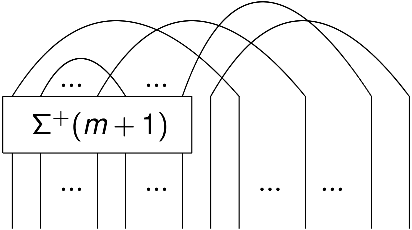

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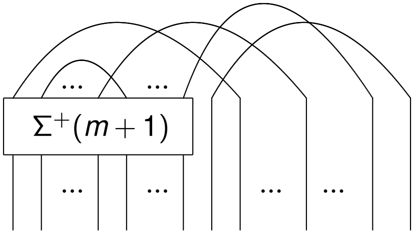

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Step 3: If $A = (x_{ij})$, impose the (linear) relations on the x_{ij} implied by the equation $A^\dagger = A$. These depend on whether one is in the orthogonal, symplectic or orthosymplectic case.

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Step 7: By the irreducibility of $\mathbb{E}$, $f - F \circ \omega$ is zero on all of $\mathbb{E}$.$\square$

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If V, W are $\mathbb{Z}_2$-graded , so are $V^*, V \otimes_{\mathbb{C}} W$ and $\mathrm{Hom}_{\mathbb{C}}(V, W) \simeq W \otimes_{\mathbb{C}} V^*$. In particular, so is $\mathrm{End}_{\mathbb{C}}(V)$.

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• The **orthosymplectic Lie algebra** $\mathrm{osp}(m|2n)$ is the $\mathbb{Z}_2$-graded subalgebra of $\mathfrak{gl}(m|2n)$ defined by $\{X \in \mathfrak{gl}(m|2n) \mid (Xv, w) + (-1)^{[X][v]}(v; Xw) = 0\}$.

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The Lie superalgebra $\mathfrak{gl}(V)$ acts on $V^{\otimes r}$ via

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The subalgebra $\mathrm{osp}(m|2n)$ acts correspondingly on $V^{\otimes r}$.

Further the group $G := O(V_{\bar{0}}) \times \mathrm{Sp}(V_{\bar{1}})$ also acts on $V^{\otimes r}$, compatibly with osp.

We have the endomorphisms $\tau, e \in \mathrm{End}_{\mathrm{osp},G}(V \otimes V)$:

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This leads to a functor F from $\mathcal{B}(m-2n)$ to $Rep(\mathrm{osp}(m|2n), G)$; so the Brauer algebra $B_r(m-2n)$ acts on $V^{\otimes r}$. This action commutes with that of osp and of G.

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NOTE: the image of η_r is almost always non-semisimple,

as are the modules $T^r(V)$.

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Def: $\mathrm{GL}(V) := \{g \in \mathrm{End}_\Lambda(V)_{\bar{0}} \mid g \text{ is invertible}\}$.

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- One develops a theory of affine varieties over Λ, and proves the (geometric) Main Lemma in this context. $\square$

Quantum invariant theory.

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There is a universal R-matrix $R \in (\widehat{\mathrm{U}_q \otimes \mathrm{U}_q})$ such that

(i) For $u \in \mathrm{U}_q(\mathfrak{g})$, $R\Delta(u)R^{-1} = \Delta'(u)$, and

(ii) We have $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \in \mathrm{U}_q(\mathfrak{g})^{\otimes 3}$ (Y-B equation).

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for any $V_q \in \mathcal{C}_q$, where $\mathbb{B}_r$ is the r-string braid group.

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In certain cases we can understand the action of $\check{R}$ on $V \otimes V$ 'generically'.

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Therefore (by Schur's Lemma) C acts on the simple module L_λ as a scalar, and that scalar is given by $\chi_\lambda(C) = (\lambda + 2\rho, \lambda)$, where $\rho =$ half-sum of the positive roots, and $(-,-)$ is a W-invariant form on $\mathfrak{h}^*$, normalised so that $(\alpha, \alpha) = 2$ for short roots.

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Therefore (by Schur's Lemma) C acts on the simple module L_λ as a scalar, and that scalar is given by $\chi_\lambda(C) = (\lambda + 2\rho, \lambda)$, where $\rho =$ half-sum of the positive roots, and $(-,-)$ is a W-invariant form on $\mathfrak{h}^*$, normalised so that $(\alpha, \alpha) = 2$ for short roots.

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Assume $L_{\lambda,q}^{\otimes 2} = L_{\mu_1,q} \oplus \cdots \oplus L_{\mu_s,q}$, where the μ_i are distinct weights. Then $\check{R}$ acts on $L_{\mu_i,q}$ as the scalar

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where $\varepsilon(i)$ is the sign by which the simple interchange acts on L_{μ_i} in the classical limit.

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For any r, let $e_q^-(r) = \sum_{w \in \mathrm{Sym}_r}(-q)^{-\ell(w)}T_w \in H_r(q)$ (The $H_r(q)$-alternator). This is the 'q-analogue' of the alternator e_r^-.

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Now let's look at the braid group action on the RHS:

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Let y, z be indeterminates over $\mathbb{C}$ and write $\mathcal{A} = \mathbb{C}[y^{\pm 1}, z]$. The BMW algebra $BMW_r(y, z)$ over $\mathcal{A}$ is the associative $\mathcal{A}$-algebra with generators $g_1^{\pm 1}, \ldots, g_{r-1}^{\pm 1}$ and $e_1, \ldots, e_{r-1}$, subject to the following relations:

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$$g_i e_i = e_i g_i = y e_i;$$
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Then: $(g_i - y)(g_i^2 - z g_i - 1) = 0$ (cubic!!)

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One uses an integrality argument, as well as the FFT for the classical limit ($\lim_{q \to 1}$), to prove the result.

Quantum SFT and further analysis-cellular algebras

To compare the quantum and classical cases of $O(V)$ and $\mathrm{Sp}(V)$, it is convenient to use the notion of a cellular algebra. These are algebras which are generically semisimple, but which have non-semisimple specialisations.

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- In general $L(\lambda)$ is simple or is zero. The set of non-zero $L(\lambda)$ forms a complete set of non-isomorphic A-modules.

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We work over the ring $\mathbb{C}[q]_1$, the localisation of $\mathbb{C}[q]$ at $(q-1)$, and define $\psi_1 : \mathbb{C}[q]_1 \to \mathbb{C}$ by $\psi_1(q) = 1$.

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In the symplectic case, there is a canonical choice. In the orthogonal case an explicit formula is known only for the case $m = 3$. (It is complicated!)

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where

$$\begin{aligned} a &= 1 + (1 - q^{-2})^2, \\ b &= 1 + (1 - q^2)^2 + (1 - q^{-2})^2, \\ c &= \frac{1 + (2 + q^{-2})(1 - q^{-2})^2 + (1 + q^2)(1 - q^{-2})^4}{([3]_q - 1)^2}, \\ d &= (q - q^{-1})^2 = q^2(a - 1). \end{aligned}$$

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Tilting modules for quantum $\mathfrak{gl}_n, \mathfrak{o}_m, \mathfrak{sp}_{2d}$ (i.e. indecomposable summands of $T^r(V)$) at roots of unity may be analysed using the above results/methods (e.g. Andersen-L-Zhang, to appear, Pacific J M.)

Unsolved problems–directions for future research.

Determine, in the orthosymplectic case, the smallest r for which the homomorphism $B_r(m-2n) \to T^r(V)$ has a non-trivial kernel. (We know $2r \geq (m+1)(2n+1)$ is necessary).

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When does the above map factor through a cellular algebra? (cf. ALZ)

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