Buildings

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Introduction

All non-abelian finite simple groups are either

- alternating OR
- sporadic OR
- automorphism groups of buildings.

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The results in chapters one, two and four are due to Jacques Tits, the results in the third chapter, to Bernhard Mühlherr.

Moufang polygons

Generalized n-gons

Definition

A generalized n-gon is a bipartite graph of diameter n such that the length of a shortest circuit is $2n$.

Generalized n-gons

Definition

A generalized n-gon is thick if each vertex has at least three neighbors.

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Examples

- generalized 2-gons = complete bipartite graphs
- generalized 3-gons = projective planes

Generalized n-gons

We always assume that

- Γ is thick.
- $n \geq 3$.

Definitions

A root is a path of length n.
An apartment is a circuit of length $2n$.

- Every path of length $n+1$ lies on a unique apartment.

The Moufang property

Definition

Let
$$\alpha = (x_0, x_1, x_2, \ldots, x_{n-1}, x_n)$$
be a root. The root group U_α is the pointwise stabilizer of
$$\Gamma_{x_1} \cup \Gamma_{x_2} \cup \cdots \cup \Gamma_{x_{n-1}}.$$

Definition

Γ is Moufang if for every root α, the root group U_α acts transitively on the set of apartments containing α.

Root group sequences

Let Σ be an apartment. We number its vertices consecutively

$$x_0, x_1, x_2, \ldots$$

(with indices modulo $2n$) and let U_i denote the root group

$$U_{(x_i, x_{i+1}, \ldots, x_{i+n})}.$$

$U_1, U_2, \ldots, U_n$ are the root groups fixing the vertices x_{n-1} and x_n.

Let

$$U_+ = \langle U_1, U_2, \ldots, U_n \rangle.$$

Uniqueness

Definition

The sequence

$$(U_+, U_1, U_2, \ldots, U_n)$$

is called the root group sequence of Γ.

Theorem (Uniqueness)

Γ is uniquely determined by its root group sequence.

Commutator relations

Let

$$U_{[k,s]} = U_k U_{k+1} \cdots U_s$$

for all k, s with $1 \leq k \leq s \leq n$ and $U_{[k,s]} = 1$ if $s < k$.

- $[U_i, U_j] \subset U_{[i+1,j-1]}$ for all i, j with $1 \leq i < j \leq n$.
- $[U_i, U_{i+1}] = 1$.

Thus $U_+ = U_1 U_2 \cdots U_n$.

Key observation

The group $U_+ = \langle U_1, U_2, \ldots, U_n \rangle$ is uniquely determined by the individual U_i and the commutator relations of the form

$$[u_i, u_j] = u_{i+1} \cdots u_{j-1},$$

where $u_k \in U_k$ for all k.

$n = 3$

- K is a field.
- $x_i \colon K \to U_i$ is an isomorphism for $i = 1, 2, 3$:

$$x_i(s)x_i(t) = x_i(s + t) \text{ for all } s, t \in K.$$

- $[x_1(s), x_3(t)] = x_2(st)$.

This construction works also if K is a skew field or an octonion division algebra. The Moufang triangles we obtain are

- *algebraic* if K is finite dimensional over its center
- *classical* if K is a skew field.
- *exceptional* if K is octonion.

Quaternions

Let E/K be a separable quadratic extension with norm N, so $N(a) = a \cdot a^\sigma$. Let α be in $K \backslash N(E)$ and let

$$Q = \{a + eb \mid a, b \in E\},$$

where

$$a \cdot eb = e(a^\sigma b), \quad eb \cdot a = e(ab), \quad ea \cdot eb = \alpha a^\sigma b.$$

Then Q is a division algebra with center K. Its norm N is given by

$$N(a + eb) = N(a) - \alpha N(b)$$

and its standard involution σ is given by

$$(a + eb)^\sigma = a^\sigma - eb.$$

Octonions

Let Q be a quaternion division algebra with center K and standard involution σ.

Let β be in $K \backslash N(Q)$ and let

$$A = \{a + fb \mid a, b \in Q\},$$

where

$$a \cdot fb = f(a^\sigma b), \quad fb \cdot a = f(ab), \quad fa \cdot fb = \beta a^\sigma b.$$

Then A is a (non-associative) division algebra with center K.

Its norm N is given by

$$N(a + fb) = N(a) - \beta N(b)$$

and its standard involution σ is given by

$$(a + fb)^\sigma = a^\sigma - fb.$$

$n = 4$: Quadratic form type

Let (K, V, q) be an anisotropic quadratic space:

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- V is a vector space over K.
- $q \colon V \to K$

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such that

- $f(a, b) = q(a + b) - q(a) - q(b)$ is bilinear.
- $q(ta) = t^2 q(a)$.
- $q(a) = 0$ if and only if $a = 0$.

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Let $x_i\colon K \to U_i$ for $i = 1$ and 3 and $x_i\colon L \to U_i$ for $i = 2$ and 4.

$$[x_1(t), x_4(a)] = x_2(ta)x_3(tq(a)) \text{ and } [x_2(a), x_4(b)] = x_3(f(a, b)).$$

Anisotropic quadratic forms

Examples

- $V = K$ and $q(t) = t^2$.
- The norm of a quadratic extension.
- The norm of a quaternion or octonion division algebra.
- If K is finite, then $\dim_K L \leq 2$.

If $\mathrm{char}(K) \neq 2$, then $q(a) = f(a, a)/2$.

$n = 4$: Involutory type

Let K be a field or skew field and let σ be an involution of K:

- σ is an additive automorphism of K.
- $(ab)^\sigma = b^\sigma a^\sigma$.
- σ is of order 2.

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An involutory set is a triple (K, K_0, σ), where K_0 be an additive subgroup of K containing 1 such that

- $K_\sigma = \{a + a^\sigma \mid a \in K\} \subset K_0 \subset K^\sigma = \{a \in K \mid a^\sigma = a\}$.
- $a^\sigma K_0 a \subset K_0$.

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Let $x_i \colon K_0 \to U_i$ for $i = 1$ and 3 and $x_i \colon K \to U_i$ for $i = 2$ and 4.

$[x_1(t), x_4(u)] = x_2(tu)x_3(u^\sigma tu)$ and $[x_2(u), x_4(v)] = x_3(u^\sigma v + v^\sigma u)$.

Involutory sets

Let (K, K_0, σ) be an involutory set.

- If $\mathrm{char}(K) \neq 2$, then $a = (a/2) + (a/2)^\sigma$ for $a \in K^\sigma$, so $K_\sigma = K^\sigma$.
- If $\mathrm{char}(K) = 2$, let $(u + K_\sigma)t = t^\sigma ut + K_\sigma$. This makes K^σ/K_σ into a right vector space over K!!
- If K is commutative, then $F := K_\sigma = K_0 = K^\sigma$ is a subfield and K/F is a separable quadratic extension.

- Either $K = \langle K_0 \rangle$ (as a subring) or
  - K is commutative.
  - K is a quaternion division algebra algebra and σ is the standard involution of K.

Pseudo-quadratic forms

Let (K, K_0, σ) be an involutory set, let L be a right vector space over K and let f be a skew-hermitian form on L:

- $f(u + v, w) = f(u, w) + f(v, w)$
- $f(u, wt) = f(u, w)t$ and $f(ut, w) = t^{\sigma} f(u, w)t$
- $f(u, w)^{\sigma} = -f(u, w)$

Pseudo-quadratic forms

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- $f(u, w)^\sigma = -f(u, w)$

A map $q\colon L \to K$ is a pseudo-quadratic form if for some skew-hermitian form f:

- $q(u + w) \equiv q(u) + q(w) + f(u, w) \pmod{K_0}$
- $q(ut) \equiv t^\sigma q(u)t \pmod{K_0}$

q is anisotropic if

- $q(u) \equiv 0 \pmod{K_0}$ iff $a = 0$.

Anisotropic pseudo-quadratic forms

Example

- Let (K, K_0, σ) be an involutory set.
- Let $\gamma \in K \backslash K_0$.
- Let $q \colon K \to K$ be given by $q(t) = t^\sigma \gamma t$.
- Let $f(s,t) = s^\sigma(\gamma - \gamma^\sigma)t$ for all s, t.
- Let $L = K$.

Then f is a skew-hermitian form on L and

$$\begin{aligned} q(s+t) &= s^\sigma \gamma s + t^\sigma \gamma t + s^\sigma \gamma t + t^\sigma \gamma s \\ &= q(s) + q(t) + f(s,t) + s^\sigma \gamma^\sigma t + t^\sigma \gamma s \\ &= q(s) + q(t) + f(s,t) + (t^\sigma \gamma s)^\sigma + (t^\sigma \gamma s) \end{aligned}$$

and $(t^\sigma \gamma s)^\sigma + (t^\sigma \gamma s) \in \{a + a^\sigma \mid a \in K\} \subset K_0$.

Anisotropic pseudo-quadratic forms

- $q(u) = f(u,u)/2$ if $\mathrm{char}(K) \neq 2$.
- If K is finite, then $\dim_K L \leq 1$.

Moufang sets

Let X be a set. For each $x \in X$, let U_x be a subgroup of the symmetric group $\mathrm{Sym}(X)$ and let G be a subgroup of $\mathrm{Sym}(X)$ containing

$$\langle U_x \mid x \in X \rangle.$$

The pair $(G, \{U_x \mid x \in X\})$ is a Moufang set if

- For each $x \in X$, U_x fixes x and acts *sharply transitively* on $X \backslash \{x\}$; and
- $\{U_x \mid x \in X\}$ is a conjugacy class of subgroups in G.

Moufang sets

Examples

- The group of special fractional linear maps

$$x \mapsto \frac{ax + b}{cx + d}$$

acting on the projective line $K \cup \{\infty\}$.

- The set of neighbors of a fixed vertex of a Moufang polygon.

Spherical Buildings

Coxeter groups

A square symmetric matrix $(m_{st})_{s,t\in S}$ is a Coxeter matrix if

$$m_{ss} = 1 \text{ and } m_{st} \in \{2, 3, 4, 5, \dots, \infty\}.$$

Let $M = (m_{st})_{s,t\in S}$ be a Coxeter matrix. Then

$$W = \langle s_i \mid (st)^{m_{st}} = 1 \text{ for all } s, t \in S \text{ such that } m_{st} < \infty \rangle$$

is the corresponding Coxeter group and the pair (W, S) is the corresponding Coxeter system.

The graph with vertex set S and edges all pairs $\{s, t\}$ such that $m_{st} \geq 3$ labeled by the quantity m_{st} is called the corresponding Coxeter diagram.

Coxeter groups

Example

The Coxeter group corresponding to the Coxeter diagram having just two vertices and one edge with label $n \in \{3, 4, 5, \ldots, \infty\}$ is the dihedral group D_{2n}.

Irreducible and spherical Coxeter matrices

Definition

A Coxeter matrix is irreducible if the Coxeter diagram is connected.

Definition

A Coxeter matrix is spherical if the Coxeter group W is finite.

The spherical Coxeter matrices were classified by Coxeter in the 1930's.

Chamber systems

Let S be a set of "colors." An S-colored chamber system is a connected graph whose *edges* each have a color from the set S such that for each vertex x, the following hold:

- For each $s \in S$, there exists a vertex y such that $\{x, y\}$ is an edge of color s.
- If y, z are two vertices such that $\{x, y\}$ and $\{x, z\}$ are both edges of color s, then $\{y, z\}$ is also an edge of color s.

Chamber systems

Definitions

A chamber system is *thick* if for each vertex x and each color $s \in S$, there exists at least two s-colored edges containing x.

A chamber system is *thin* if for each vertex x and each color $s \in S$, there exists exactly one s-colored edges containing x.

Examples of chamber systems

Let (W, S) be a Coxeter system.

Let $\Sigma = \Sigma_M$ be the S-colored graph with vertex set W whose s-colored edges (for each $s \in S$) are all pairs of the form

$$\{x, y\}$$

for some $x, y \in W$ such that $x^{-1}y = s$.

Σ is a *thin* chamber system.

Examples of chamber systems

Let $\Gamma = (V, E)$ be a connected bipartite graph in which every vertex has at least two neighbors.

Thus V is a disjoint union $B \cup W$ such that every edge contains one vertex in B and one in W.

Let S be the 2-element set $\{B, W\}$.

Let Δ_Γ be the graph whose vertices are the edges of Γ, where two edges of Γ are joined by an edge of color $s \in S$ in Δ_Γ precisely when the two edges of Γ intersect in a vertex of Γ contained in s.

Δ_Γ is a chamber system with two colors.

Δ_Γ is thick if and only if every vertex of Γ has at least three neighbors.

Γ is a circuit of length $2n$ if and only if Δ_Γ is a circuit of length $2n$.

Subgraphs

Let $\Gamma = (V, E)$ be a graph with vertex set V and edge set E.

Definition

A subgraph is a pair (X, E'), where

- $X \subset V$ and
- E' is a subset of E consisting of 2-element subsets of X.

Definition

Let $X \subset V$. The subgraph spanned by X is the subgraph (X, E_X), where E_X denotes the set of *all* edges of E consisting of 2-element subsets of X.

Residues and panels in chamber systems

- Let $\Delta = (V, E)$ be an S-colored chamber system.
- Let J be a subset of S.
- Let E_J be the set of edges whose color is contained in J.

Definition

A J-residue of Δ is a connected component of the subgraph (V, E_J).

Residues and panels in chamber systems

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A J-residue of Δ is a connected component of the subgraph (V, E_J).

- Each vertex of Δ lies in a unique J-residue.

Residues and panels in chamber systems

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A J-residue of Δ is a connected component of the subgraph (V, E_J).

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- The set J is the type of a J-residue and the cardinality of J is the rank of the a J-residue.

Residues and panels in chamber systems

- Let $\Delta = (V, E)$ be an S-colored chamber system.
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Definition

A *J-residue* of Δ is a connected component of the subgraph (V, E_J).

- Each vertex of Δ lies in a unique J-residue.
- The set J is the type of a J-residue and the cardinality of J is the rank of the a J-residue.
- The cardinality of S is the rank of Δ.

Residues and panels in chamber systems

- A residue of rank one is called a panel.
- Panels are complete graphs having at least two vertices.

Convexity

Let $\Gamma = (V, E)$ be a graph.

Definition

A subgraph (X, E') of Γ is convex if for all $x, y \in X$ and for all paths $(x_0, x_1, \ldots, x_k)$ in Γ from $x_0 = x$ to $x_k = y$ of minimal length:

- $x_i \in X$ for all $i \in [0, k]$ and
- $\{x_{i-1}, x_i\} \in E'$ for all $i \in [1, k]$.

Buildings

- Let M be a Coxeter diagram with vertex set S.
- Let $\Sigma = \Sigma_M$ be the corresponding S-colored thin chamber system.
- Let Δ be an arbitrary S-colored thick chamber system.

Definition

An apartment in Δ is a subgraph isomorphic to Σ.

Buildings

Let M be our Coxeter diagram with vertex set S.

Definition

A building of type M is an S-colored chamber system Δ such that the following hold:

- For each vertex x and each panel P, there exists a unique vertex in P nearest to x.
- Every two vertices are contained in an apartment.
- Apartments are convex.

Irreducible buildings

Let Δ be a building of type M.

Definition

- Δ is called irreducible if the Coxeter diagram M is connected.

Every building is the direct product of irreducible buildings in a suitable sense.

Spherical buildings

Definition

- A building Δ is called spherical if its apartments are finite.

Examples of buildings

Example

A building of rank one is just a complete graph whose apartments are the subgraphs spanned by its 2-element subsets.

Example

- Let M be a Coxeter diagram with vertex set S.
- Let Σ be the corresponding thin S-colored chamber system.

Then Σ itself is the unique thin building of type M.

The chamber system associated with a bipartite graph

Let Γ be a connected bipartite graph in which every vertex has at least two neighbors. We have observed that the edge graph Δ_Γ is a chamber system of rank 2.

In fact, every chamber system of rank 2 arises in this way.

Thus:

Connected bipartite graphs every vertex of which has at least two neighbors and chamber systems of rank 2 are essentially the same thing!

Buildings and generalized polygons

Let M be an irreducible Coxeter diagram with two vertices and let n be the label on the unique edge of M.

Let Δ be a building of type M.

Let Γ be the corresponding bipartite graph.

- If $n < \infty$, then Γ is a generalized n-gon.
- If $n = \infty$, then Γ is a tree, every vertex of which has at least two neighbors.

A basic property of buildings

Let M be a Coxeter diagram with vertex set S.

Let Δ be a building of type M.

Let $J \subset S$, let M_J be the subdiagram spanned by the set J and let R be a J-residue of Δ.

Then R is a convex subgraph. It is also a building of type M_J whose apartments are the intersections

$$R \cap \Sigma$$

for all apartments Σ of Δ containing chambers of R.

Roots in buildings

Suppose: Δ is a building and Σ is an apartment of Δ.

If e is an edge and x a vertex of Σ, then x is nearer to one vertex in e then it is to the other. The nearer vertex in e is called $\mathrm{proj}_e(x)$.

Two edges e and e' of Σ are *parallel* if the map proj_e is a bijection from e' to e. This is an equivalence relation.

A root of Σ is a connected component of the graph obtained from Σ by removing all the edges in a parallel class.

A root of Δ is a root of one of its apartments. A root can be the a root in many apartments simultaneously.

Moufang buildings

Let Δ be a *thick irreducible spherical building of rank at least two*.

Let α be a root of Δ.

The root group U_α is the pointwise stabilizer in $\mathrm{Aut}(\Delta)$ of the set of all vertices adjacent to at least two chambers in α.

The root group U_α acts trivially on α.

Δ is Moufang if for every root α, the root group U_α acts transitively on the set of apartments containing α.

A local-to-global principle

Definition

For each vertex x of a building Δ, let $E_2(x)$ be the subgraph spanned by all the irreducible rank 2 residues of Δ containing x.

Theorem

Let Δ and Δ' be two thick irreducible spherical buildings of the same type M and let $x \in \Delta$ and $x' \in \Delta'$ be vertices. Suppose that φ is an isomorphism from $E_2(x)$ to $E_2(x')$. Then φ extends to an isomorphism from Δ to Δ'.

Thus a spherical building is uniquely determined by the irreducible rank 2 residues containing a fixed vertex.

A local-to-global principle

Corollary

Every thick irreducible spherical building of rank at least three is Moufang, as is every irreducible residue of rank at least two of such a building.

The classification of thick buildings of type H_3 and H_4

The classification of thick buildings of type H_3 and H_4

There aren't any.

The classification of simply laced spherical buildings

Let M be one of the Coxeter diagrams A_ℓ for $\ell \geq 3$, D_ℓ for $\ell \geq 4$, E_6, E_7 or E_8.

Let Δ be a thick building of type M.

The classification of simply laced spherical buildings

Let M be one of the Coxeter diagrams A_ℓ for $\ell \geq 3$, D_ℓ for $\ell \geq 4$, E_6, E_7 or E_8.

Let Δ be a thick building of type M.

Then all irreducible rank 2 residues of Δ are Moufang triangles defined by the same field or skew field K.

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Then all irreducible rank 2 residues of Δ are Moufang triangles defined by the same field or skew field K.

Δ is uniquely determined by M and K.

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Let M be one of the Coxeter diagrams A_ℓ for $\ell \geq 3$, D_ℓ for $\ell \geq 4$, E_6, E_7 or E_8.

Let Δ be a thick building of type M.

Then all irreducible rank 2 residues of Δ are Moufang triangles defined by the same field or skew field K.

Δ is uniquely determined by M and K.

If the Coxeter diagram M has a vertex of degree 3, then K must be commutative.

The classification of spherical buildings

Suppose that M is the Coxeter diagram B_ℓ for $\ell \geq 3$.

Let K be the field or skew field or octonion division algebra defining the residue of type $A_{\ell-1}$ containing a fixed chamber x.

Then Δ is uniquely determined by

- An anisotropic quadratic space (K, L, q) OR
- An involutory set (K, K_0, σ) OR
- An anisotropic pseudo-quadratic space (K, K_0, σ, L, q) OR
- An honorary involutory set (K, K_0, σ).

This last case can only occur if $\ell = 3$.

The classification of spherical buildings

An honorary involutory set is a triple (K, K_0, σ), where

- K is an octonion division algebra
- K_0 is its center
- σ is its standard involution.

Buildings of type F_4

Buildings of type F_4 are classified by the following families of anisotropic quadratic spaces (F, K, q):

- $\mathrm{char}(F) = 2$, K is a purely inseparable extension of F of exponent 1 and $q(x) = x^2$.
- $F = K$ and $q(x) = x^2$.
- K/F is a separable quadratic extension and q is its norm.
- K is a quaternion division algebra, F is its center and q is its norm.
- K is an octonion division algebra, F is its center and q is its norm.

Buildings of type F_4

Buildings of type F_4 are classified by the following families of involutory sets (K, F, σ):

- $\mathrm{char}(K) = 2$, K is a purely inseparable extension of the field F of exponent 1 and $\sigma = \mathrm{id}$.
- $F = K$ and $\sigma = \mathrm{id}$.
- K/F is a separable quadratic extension and σ is the non-trivial element in $\mathrm{Gal}(K/F)$.
- K is a quaternion division algebra, F is its center and σ is its standard involution.
- K is an octonion division algebra, F is its center and σ is its standard involution.

The field of definition

In almost every case the relevant algebraic structure is defined over a field or a skew field or an octonion division algebra K. We call K the field of definition of the spherical building Δ. It is an invariant of Δ.

The algebraic structure itself is also an invariant, more or less. For example, two anisotropic quadratic spaces if and only if they are similar.

In the remaining cases, the relevant algebraic structure is defined over a purely inseparable field extension K/F in characteristic $p = 2$ or 3 such that $K^p \subset F$. Tits calls these the mixed cases.

Conclusion

There is a Moufang spherical building corresponding to every absolutely simple algebraic group of F-rank at least 2. Here F is the center $Z(K)$ of the defining field K or, in some cases, $F = Z(K) \cap K^\sigma$ for some involution σ of K.

The only Moufang spherical buildings which do not arise in this way are those that involve:

- an infinite dimensional vector space,
- a skew field of infinite dimension over its center,
- a bilinear (or skew-hermitian form) that is degenerate or
- a purely inseparable field extensions in characteristic 2 or 3.

The classification of Moufang polygons

There are triangles, hexagons, octagons and six families of quadrangles.

Descent in buildings

The opposite map

Let M be a Coxeter diagram, let J be a subset of the vertex set S of M and let Σ_J be the chamber system associated with the subdiagram M_J.

Suppose that the subdiagram M_J is spherical.

The opposite map

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Then there is an automorphism of Σ_J which maps each vertex to the unique opposite vertex. This automorphism induces an automorphism of the Coxeter diagram M_J which we denote by op_J.

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The map op_J stabilizes each connected component of M_J.

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The map op_J stabilizes each connected component of M_J.

The map op_J acts non-trivially on a connected component X of M_J

iff

X is A_n for arbitrary $n \geq 2$, E_6, D_n for $n \geq 4$ odd or $I_2(n)$ for $n \geq 3$ odd.

Tits indices

Definition

A Tits index is a triple (M, Θ, A), where

- M is a Coxeter diagram with vertex set S.
- Θ is a subgroup of $\mathrm{Aut}(M)$.
- A is a Θ-invariant subset S

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- A is a Θ-invariant subset S such that for each $s \in S \backslash A$,
  - the subdiagram $M_{\Theta(s) \cup A}$ is spherical and
  - A is $\mathrm{op}_{\Theta(s) \cup A}$-invariant.

The longest element

Let M be a Coxeter diagram, let J be a subset of the vertex set S of M and let Σ_J be the chamber system associated with the subdiagram M_J.

Let $W_J = \langle J \rangle$. Thus W_J is both a finite subgroup of W and the vertex set of Σ_J.

The unique vertex of Σ_J opposite the vertex 1 is called the longest element of W_J. We denote this element by w_J.

The relative Coxeter group

Theorem

Let (M, Θ, A) be a Tits index. For each $s \in S \backslash A$, let $\tilde{s}$ be the product of the longest element in the Coxeter group W_A and the longest element in the Coxeter group $W_{\Theta(s) \cup A}$.

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Let $\tilde{S}$ denote the set consisting of all the elements $\tilde{s}$ and let $\tilde{W} = \langle \tilde{S} \rangle$.

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Then

$$(\tilde{W}, \tilde{S})$$

is a Coxeter system called the relative Coxeter system of (M, Θ, A).

Γ-chambers

Let Δ be a building of type M and let Γ be a subgroup of $\mathrm{Aut}(\Delta)$.

A Γ-residue is a residue stabilized by Γ.

A Γ-chamber is a minimal Γ-residue.

A Γ-panel is a Γ-residue P such that for some Γ-chamber C, P is minimal among all the Γ-residues containing C.

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Definition

Let Δ^{Γ} be the graph whose vertex set is the set of all Γ-chambers, where two Γ-chambers are adjacent whenever they are contained in a Γ-panel.

Main theorem of descent

Theorem

Let Δ be a building of type M, let Γ be a subgroup of $\mathrm{Aut}(\Delta)$ and let Θ be the subgroup of $\mathrm{Aut}(M)$ induced by Γ. Suppose that there is a Γ-chamber C of type A and

- *The subdiagram M_A is spherical.*

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- *(M, Θ, A) is a Tits index.*

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- *The subdiagram M_A is spherical.*
- *Every Γ-panel containing C contains at least two other Γ-chambers.*

Then the following hold:

- *Every Γ-chamber has type A.*
- *(M, Θ, A) is a Tits index.*
- *The graph Δ^Γ is a building of type $(\tilde{W}, \tilde{S})$, where $(\tilde{W}, \tilde{S})$ is the relative Coxeter diagram of (M, Θ, A).*

Affine Buildings

Affine Coxeter matrices

The affine Coxeter diagrams are the Coxeter diagrams underlying the extended Dynkin diagrams.

Every affine Coxeter diagram if of the form $\tilde{M}$, where M is one of the spherical Coxeter diagrams $A_\ell, B_\ell, \ldots, G_\ell$.

The number of vertices of $\tilde{M}$ is one more than the number of vertices of the spherical diagram M.

Affine buildings

An (irreducible) affine building is a building of type $\tilde{M}$ for some affine Coxeter diagram $\tilde{M}$.

The apartments of an affine building of type $\tilde{M}$ have a canonical representation as a tessellation of Euclidean space of dimension ℓ.

Example

An apartment A of a building X of type $\tilde{A}_2$ looks like a Euclidean space of dimension 2 tessellated by regular hexagons, each subdivided into 6 equilateral triangles. These triangles are the chambers of A.

The building at infinity

Let X be a building of type $\tilde{M}$.

Apartments contain sectors. A sector of X is a sector in one of its apartments.

Two sectors are equivalent if their intersection is a sector.

The set of sector classes is the vertex set of a building X^∞ of type M. The building X^∞ is called the building at infinity of X. It is spherical, its rank is one less than the rank of X and

$$A \mapsto A^\infty$$

is a bijection from the set of apartments of X to the set of apartments of X^∞.

Bruhat-Tits buildings

Definition

A Bruhat-Tits building is an irreducible affine building whose building at infinity is Moufang.

The root groups of X^∞

Let X is a Bruhat-Tits building, let A be an apartment of X and let a be a “half-space” of A. Its parallel class consists of all half-spaces contained in or containing a. There exists a unique root α of the apartment A^∞ of $\Delta = X^\infty$ such that the following hold:

- Every element g in the root group U_α of X^∞ is induced by a unique element $\hat{g} \in \mathrm{Aut}(X)$.
- Let g be a non-trivial element of U_α. The fixed point set in A of $\hat{g}$ is a half-space of A parallel to a. This observation gives rise to a function $\varphi_\alpha \colon U_\alpha^* \to \mathbb{Z}$ such that

  $$\varphi_\alpha(g) = \varphi_\alpha(-g) \quad \text{and} \quad \varphi_\alpha(g_1 + g_2) \geq \min\{\varphi_\alpha(g_1), \varphi_\alpha(g_2)\}.$$

- The map

  $$d_\alpha(g_1, g_2) = 2^{-\varphi_\alpha(g_1 - g_2)}$$

  is a metric on U_α.
- U_α is complete with respect to the metric d_α.

The classification of Bruhat-Tits buildings

Theorem

A Bruhat-Tits building is uniquely determined by its building at infinity.

Theorem

Let X be a Bruhat-Tits building and let $\Delta = X^\infty$. Then there is a canonical isomorphism from $\mathrm{Aut}(X)$ to $\mathrm{Aut}(\Delta)$.

(A Bruhat-Tits building is not, however, uniquely determined by its residues.)

The classification of Bruhat-Tits buildings

Theorem

Let Δ be a spherical building satisfying the Moufang condition and let K be its field of definition. Then Δ is the building at infinity of a Bruhat-Tits building iff

- *K is complete with respect to a discrete valuation and*
- *for each root α, the root group U_α is complete with respect to the metric d_α.*

The second condition follows from the first if Δ is the spherical building associated with an absolutely simple algebraic group or if Δ is simply laced.

The End