A more detailed classification of symmetric cubic graphs

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Abstract

A graph Γ is *symmetric* if its automorphism group acts transitively on the arcs of Γ, and *s-regular* if its automorphism group acts regularly on the set of s-arcs of Γ. Tutte (1947, 1959) showed that every cubic finite symmetric cubic graph is s-regular for some $s \leq 5$. Djoković and Miller (1980) proved that there are seven types of arc-transitive group action on finite cubic graphs, characterised by the stabilisers of a vertex and an edge. A given finite symmetric cubic graph, however, may admit more than one type of arc-transitive group action. In this paper we determine exactly which combinations of types are possible. Some combinations are easily eliminated by existing theory, and others can be eliminated by elementary extensions of that theory. The remaining combinations give 17 classes of finite symmetric cubic graph, and for each of these, we prove the class is infinite, and determine at least one representative. For at least 14 of these 17 classes the representative we give has the minimum possible number of vertices (and we show that in two of these 14 cases every graph in the class is a cover of the smallest representative), while for the other three classes, we give the smallest examples known to us. In an Appendix, we give a table showing the class of every symmetric cubic graph on up to 768 vertices.

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1 Introduction

By a graph we mean an undirected finite graph, without loops or multiple edges. For a graph Γ, we denote by $V(\Gamma)$, $E(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ its vertex set, its edge set and its automorphism group, respectively.

An *s-arc* in a graph Γ is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of Γ such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and also $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$; in other words,

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a directed walk of length s which never includes the reverse of an arc just crossed. A graph Γ is said to be *s-arc-transitive* if its automorphism group $\mathrm{Aut}(\Gamma)$ is transitive on the set of all s-arcs in Γ. In particular, 0-arc-transitive means *vertex-transitive*, and 1-arc-transitive means *arc-transitive*, or *symmetric*. An arc-transitive graph Γ is said to be *s-regular* if for any two s-arcs in Γ, there is a unique automorphism of Γ mapping one to the other. For $s \geq 1$, an s-regular graph is a union of isomorphic s-regular connected graphs and isolated vertices. Hence in what follows, we consider only non-trivial connected graphs. Every connected vertex-transitive graph is regular in the sense of all vertices having the same valency (degree), and when this valency is 3 the graph is called *cubic*.

Tutte [20, 21] proved that every finite symmetric cubic graph is s-regular for some $s \leq 5$. The stabiliser of a vertex in any group acting regularly on the s-arcs of a (connected) cubic graph is isomorphic to either the cyclic group $\mathbb{Z}_3$, the symmetric group S_3, the direct product $S_3 \times \mathbb{Z}_2$ (which is dihedral of order 12), the symmetric group S_4 or the direct product $S_4 \times \mathbb{Z}_2$, depending on whether $s = 1, 2, 3, 4$ or 5 respectively. In the cases $s = 2$ and $s = 4$ there are two different possibilities for the edge-stabilisers, while for $s = 1, 3$ and 5 there are just one each. Taking into account the isomorphism type of the pair consisting of a vertex-stabiliser and edge-stabiliser, this gives seven classes of arc-transitive actions of a group on a finite cubic graph. These classes correspond also to seven classes of 'universal' groups acting arc-transitively on the infinite cubic tree with finite vertex-stabiliser (see [15, 17]). It follows that the automorphism group of any finite symmetric cubic graph is an epimorphic image of one of these seven groups, called G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2 and G_5 by Conder and Lorimer in [11].

We will use the following presentations for these seven groups, as given by Conder and Lorimer in [11] based on the analysis undertaken in [15, 17]:

G_1 is generated by two elements h and a, subject to the relations $h^3 = a^2 = 1$;

G_2^1 is generated by h, a and p, subject to $h^3 = a^2 = p^2 = 1$, $apa = p$, $php = h^{-1}$;

G_2^2 is generated by h, a and p, subject to $h^3 = p^2 = 1$, $a^2 = p$, $php = h^{-1}$;

G_3 is generated by h, a, p, q, subject to $h^3 = a^2 = p^2 = q^2 = 1$, $apa = q$, $qp = pq$, $ph = hp$, $qhq = h^{-1}$;

G_4^1 is generated by h, a, p, q and r, subject to $h^3 = a^2 = p^2 = q^2 = r^2 = 1$, $apa = p$, $aqa = r$, $h^{-1}ph = q$, $h^{-1}qh = pq$, $rhr = h^{-1}$, $pq = qp$, $pr = rp$, $rq = pqr$;

G_4^2 is generated by h, a, p, q and r, subject to $h^3 = p^2 = q^2 = r^2 = 1$, $a^2 = p$, $a^{-1}qa = r$, $h^{-1}ph = q$, $h^{-1}qh = pq$, $rhr = h^{-1}$, $pq = qp$, $pr = rp$, $rq = pqr$;

G_5 is generated by h, a, p, q, r and s, subject to $h^3 = a^2 = p^2 = q^2 = r^2 = s^2 = 1$, $apa = q$, $ara = s$, $h^{-1}ph = p$, $h^{-1}qh = r$, $h^{-1}rh = pqr$, $shs = h^{-1}$, $pq = qp$, $pr = rp$, $ps = sp$, $qr = rq$, $qs = sq$, $sr = pqrs$.

Given a quotient G of one of the seven groups above by some normal torsion-free subgroup, the corresponding arc-transitive graph $\Gamma = (V, E)$ can be constructed in the the way described in [11]. Let X be the generating set for G consisting of images of the above generators $h, a, \ldots,$ and let H be the subgroup generated by $X \setminus \{a\}$. For convenience, we will use the same symbol to denote a generator and its image. Now take as vertex-set the coset space $V = \{Hg \,|\, g \in G\}$, and join two vertices Hx and Hy an edge whenever $xy^{-1} \in HaH$. This adjacency relation is symmetric since $HaH = Ha^{-1}H$ (indeed $a^2 \in H$) in each of the seven cases. The group G acts on the right cosets by multiplication, preserving the adjacency relation. Since $HaH = Ha \cup Hah \cup Hah^{-1}$ in each of the seven cases, the graph Γ is cubic and symmetric. This 'double-coset graph' will be denoted by $\Gamma = \Gamma(G, H, a)$.

In some cases, the full automorphism group $\mathrm{Aut}(\Gamma)$ may contain more than one subgroup acting transitively on the arcs of Γ. When G' is any such subgroup, G' will be the image of one of the seven groups G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2 and G_5, and Γ will be obtainable as the double-coset graph $\Gamma(G', H', a')$ for the appropriate subgroup H' and element a' of G'. Such a subgroup G' of $\mathrm{Aut}(\Gamma)$ will said to be of type 1, 2^1, 2^2, 3, 4^1, 4^2 or 5, according to which of the seven groups it comes from. For example, the Petersen graph is 3-regular, with automorphism group S_5 of order 120 and type 3, but also A_5 acts regularly on its 2-arcs, with type 2^1 (since A_5 contains involutions that reverse an edge). Another way of saying this is that there exists an epimorphism $\psi\colon G_3 \to S_5$ (with torsion-free kernel), and the restriction of ψ to the subgroup G_2^1 (of index 2 in G_3) maps G_2^1 to A_5. On the other hand, the Sextet graph $S(17)$ constructed in [2] has automorphism group $\mathrm{PSL}(2, 17)$ of order 2448 and type 4^1, but the simple group $\mathrm{PSL}(2, 17)$ contains no proper subgroup of index up to 16, and hence contains no other subgroup acting arc-transitively on $S(17)$. In this case, there is an epimorphism $\theta\colon G_4^1 \to \mathrm{PSL}(2, 17)$ (with torsion-free kernel), but the restriction of θ to the subgroup G_1 (of index 8 in G_4^1) maps G_1 onto $\mathrm{PSL}(2, 17)$.

In this paper we provide a more detailed classification of finite symmetric cubic graphs, by determining exactly which combinations of types are realisable for arc-transitive subgroups of the full automorphism group. Some combinations are easily eliminated by existing theory (such as in [11, 15, 17]), and others will be eliminated by elementary extensions of that theory, in Section 2. The remaining combinations give 17 classes of finite symmetric cubic graph. We provide further general background to some of these in Section 3, and then in Section 4 we give detailed information about each one.

For each of the 17 classes, we give at least one representative, including the absolute smallest in 14 cases, and the smallest known to us in the other three. Also we prove that each class is infinite — in fact in two cases, every graph in the class is a cover of the smallest representative — and we consider the question of whether the graphs in each class are bipartite.

The complete classification is summarised in Section 5, and the class of every finite symmetric graph on up to 768 vertices is given in a table (showing other information about each graph) in an Appendix. We gratefully acknowledge the assistance of the Magma system [3] in the investigations published here.

2 Non-admissible combinations

The following relationships between the seven groups G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2 and G_5 are known, and obtainable from the theory developed in [11, 15, 17].

Proposition 2.1

(a) *In the group G_2^1, the subgroup generated by h and a has index 2 and is isomorphic to G_1;*

(b) *The group G_2^2 contains no subgroup isomorphic to G_1;*

(c) *In the group G_3, the subgroup generated by h, a and pq has index 2 and is isomorphic to G_2^1, the subgroup generated by h, ap and pq has index 2 and is isomorphic to G_2^2, and the subgroup generated by h and a has index 4 and is isomorphic to G_1;*

(d) *In the group G_4^1, the subgroup generated by h and a has index 8 and is isomorphic to G_1, but there are no subgroups of index 2 isomorphic to G_3 and no subgroups of index 4 isomorphic to G_2^1 or G_2^2;*

(e) *In the group G_4^2, there are no subgroups of index up to 8 isomorphic to G_1, G_2^1, G_2^2 or G_3;*

(f) *In the group G_5, the subgroup generated by hpq, a and pq has index 2 and is isomorphic to G_4^1, the subgroup generated by hpq, ap and pq has index 2 and is isomorphic to G_4^2, and the subgroup generated by h and a has index 16 and is isomorphic to G_1, but there are no subgroups of index up to 8 isomorphic to G_2^1, G_2^2 or G_3.*

As a consequence of these relationships, we know the following:

Corollary 2.2 *Let G be an arc-transitive group of automorphisms of a finite symmetric cubic graph Γ. Then*

- *if G has type 2^2, then G contains no subgroup of type 1;*
- *if G has type 4^1, then G contains no subgroup of type 2^1, 2^2 or 3;*
- *if G has type 4^2, then G contains no subgroup of type 1, 2^1, 2^2 or 3;*
- *if G has type 5, then G contains no subgroup of type 2^1, 2^2 or 3.*

This puts a severe restriction on the combinations of types of that are realisable. For example, no such graph can have arc-transitive subgroups of both the types 2^2 and 4^1; other such conditions are described in detail in [15, Section 6].

Once combinations like $\{2^2, 4^1\}$ are eliminated, we are left with just 23 possibilities to consider:

1-arc-regular Γ :	$\{1\}$ only
2-arc-regular Γ :	$\{1, 2^1\}$, $\{2^1\}$, $\{2^2\}$
3-arc-regular Γ :	$\{1, 2^1, 2^2, 3\}, \{1, 2^1, 3\}, \{1, 2^2, 3\}, \{2^1, 2^2, 3\}, \{1, 3\}, \{2^1, 3\}, \{2^2, 3\}, \{3\}$
4-arc-regular Γ :	$\{1, 4^1\}$, $\{4^1\}$, $\{4^2\}$
5-arc-regular Γ :	$\{1, 4^1, 4^2, 5\}, \{1, 4^1, 5\}, \{1, 4^2, 5\}, \{4^1, 4^2, 5\}, \{1, 5\}$, $\{4^1, 5\}$, $\{4^2, 5\}, \{5\}$.

But further, we have:

Proposition 2.3 *If the automorphism group of a 3-arc-regular finite cubic graph has an arc-transitive subgroup of type* 1 *(and index 4), then it also has arc-transitive subgroups of types* 2^1 *and* 2^2 *(and index 2). Similarly, if the automorphism group of a 5-arc-regular finite cubic graph has an arc-transitive subgroup of type* 1 *(and index 16), then it also has arc-transitive subgroups of types* 4^1 *and* 4^2 *(and index 2).*

Proof. The first part of this was proved in [15, Proposition 26], but can also be explained as follows. Suppose Γ is a 3-arc-regular finite cubic graph, and $\psi : G_3 \to \mathrm{Aut}(\Gamma)$ is an epimorphism with the property that the restriction of ψ to the subgroup M generated by h and a (of index 4 in G_3) maps M to a subgroup G of index 4 in $\mathrm{Aut}(\Gamma)$. Then the action of $\mathrm{Aut}(\Gamma)$ by right multiplication on right cosets of G is equivalent to the corresponding action of G_3 on cosets of M, namely the following:

$$\begin{array}{lcllcl} h & \mapsto & (M)(Mp)(Mq)(Mpq), & \qquad a & \mapsto & (M)(Mp, Mq)(Mpq), \\ p & \mapsto & (M, Mp)(Mq, Mpq), & \qquad q & \mapsto & (M, Mq)(Mp, Mpq). \end{array}$$

The kernel of this action is the subgroup K generated by h and aha, of index 8 in G_3, and the quotient G_3/K is dihedral. In particular, and since $aha = aphpa = (ap)h(ap)^{-1}$, the subgroup K is contained in the two subgroups $\langle h, a, pq\rangle$ and $\langle h, ap, pq\rangle$ given in Proposition 2.1(c) as isomorphic to G_2^1 and G_2^2 respectively, and these must be taken by ψ to subgroups of index 2 in $\mathrm{Aut}(\Gamma)$. Thus $\mathrm{Aut}(\Gamma)$ contains arc-transitive subgroups of types 2^1 and 2^2.

The case in which Γ is 5-arc-regular is analogous to this one, except the subgroup M generated by h and a has index 16 in G_5, and its core K in G_5 is the subgroup generated by all conjugates of the element $(ha)^4(h^{-1}a)^4ha(h^{-1}a)^2(ha)^2h^{-1}a$ and has index 112896. (These facts can be verified with the help of Magma [3].) Since K is contained in the two subgroups $\langle hpq, a, pq\rangle \cong G_4^1$ and $\langle hpq, ap, pq\rangle \cong G_4^2$ given in Proposition 2.1(f), it follows that $\mathrm{Aut}(\Gamma)$ contains arc-transitive subgroups of types 4^1 and 4^2. $\square$

This eliminates the possibility of the combinations $\{1, 2^1, 3\}, \{1, 2^2, 3\}$ and $\{1, 3\}$ for 3-regular cubic graphs, and $\{1, 4^1, 5\}, \{1, 4^2, 5\}$ and $\{1, 5\}$ for 5-regular cubic graphs, leaving just 17 combinations that will all be shown to be realisable in Section 4.

3 Further background on admissible combinations

Many of the possible combinations of types of actions can be seen in the automorphism groups of symmetric cubic graphs of small order. A census of these, including most but not all examples on up to 512 vertices, was compiled by Foster [4], and a complete list of all on up to 768 vertices was obtained systematically by Conder & Dobcsányi [10]. In what follows, we will refer to graphs in the list of all examples of order up to 768 using names consistent with those in [4, 10]; for example, F234B is Wong's graph, which is the second of the two symmetric cubic graphs of order 234. We have used MAGMA [3] to help determine the types of action admitted by all the graphs in this list, and the results are given in a table in the Appendix.

Before proceeding further with the classification, we give some general properties of graphs admitting some of the type combinations.

Proposition 3.1 *Every symmetric cubic graph admitting actions of types* 2^1, 2^2 *and* 3 *is bipartite. On the other hand, every* 3*-regular cubic graph admitting an action of just one of the types* 2^1 *and* 2^2 *is non-bipartite.*

PROOF. The first part of this was proved in [15, Proposition 26] . The group G_3 has exactly three subgroups of index 2, namely $K = \langle h, a, pq \rangle \cong G_2^1$ and $L = \langle h, ap, pq \rangle \cong G_2^2$ and $M = \langle h, p, q, aha \rangle$, and any two of these intersect in the normal subgroup $\langle h, pq, aha \rangle$, of index 4 in G_3 (with quotient $\cong \langle a, p \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$). Now suppose $\psi : G_3 \to G$ is the epimorphism associated with a 3-regular action of a group G on a cubic graph Γ. If $K \cong G_2^1$ and $L \cong G_2^2$ are both mapped to subgroups of index 2 in G, then these must be different (for otherwise they both contain the ψ-images of all four of h, a, $p = a(ap)$ and $q = p(pq)$), and so the third subgroup M is also mapped to a subgroup of index 2, in which case Γ is bipartite. Similarly, if just one of the two subgroups $K \cong G_2^1$ and $L \cong G_2^2$ is mapped to a subgroup of index 2 in G, then M is not, so Γ is non-bipartite. □

Proposition 3.2 [15, Proposition 29] *Every symmetric cubic graph admitting actions of types* 1 *and* 4^1 *is a cover of the Heawood graph (the incidence graph of a projective plane of order* 2*), and in particular, is bipartite.*

PROOF. Let $\psi \colon G_4^1 \to G$ be the epimorphism associated with an action of a group G of type 4^1 on a cubic graph Γ. If ψ maps the subgroup $\langle h, a \rangle \cong G_1$ to a subgroup of index 8 in G, then the natural permutation representation of G on cosets of this index 8 subgroup is equivalent to the representation of G_4^1 on cosets of $\langle h, a \rangle$, which gives $\mathrm{PGL}(2, 7)$ as a quotient. Hence $\ker \psi$ is contained in the kernel of the latter representation, and it follows that Γ is a cover of the Heawood graph. Since the latter is bipartite, so is Γ. □

Proposition 3.3 *Every symmetric cubic graph admitting actions of types* 4^1, 4^2 *and* 5 *is bipartite. On the other hand, every* 5*-regular cubic graph admitting an action of just one of the types* 4^1 *and* 4^2 *is non-bipartite.*

Proof. This is entirely similar to the proof of Proposition 3.1. The group G_5 has exactly three subgroups of index 2, namely $K = \langle h, a, pq\rangle \cong G_4^1$ and $L = \langle h, ap, pq\rangle \cong G_4^2$, and $M = \langle h, p, q, r, s, aha\rangle$, with any two of these intersecting in the normal subgroup $\langle h, pq, aha\rangle$ of index 4 in G_5 (with quotient $\cong \langle a, p\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$). It follows that if $\psi\colon G_5 \to G$ is the epimorphism associated with a 5-regular action of a group G on a cubic graph Γ, and $K \cong G_4^1$ and $L \cong G_4^2$ are both mapped to subgroups of index 2 in G, then so is M, so Γ is bipartite. On the other hand, if just one of the two subgroups $K \cong G_4^1$ and $L \cong G_4^2$ is mapped to a subgroup of index 2 in G, then M cannot be, so Γ is non-bipartite. $\square$

Proposition 3.4 *Every* 5*-regular cubic graph admitting actions of types* 1 *and* 5 *is a cover of the Biggs-Conway graph (of order* 2352*), and in particular, is bipartite.*

This was recognised by Djoković and Miller in [15, Proposition 30], although the Biggs-Conway graph itself was not identified until later in [1]. The automorphism group of the Biggs-Conway graph is isomorphic to a subgroup of index 2 in the wreath product $\mathrm{PGL}(2,7) \wr C_2$, of order 112896. This group has a faithful permutation representation of degree 16 that is equivalent to the action of the group G_5 on the cosets of subgroup $\langle h, a\rangle \cong G_1$, with its three subgroups of index 2 being equivalent to the images of the three subgroups K, L and M considered in the proof of Proposition 3.3.

Proof of 3.4. If $\psi\colon G_5 \to G$ is the epimorphism associated with a 5-regular action of a group G on on a cubic graph Γ, and ψ maps the subgroup $\langle h, a\rangle \cong G_1$ to a subgroup of index 16 in G, then the natural permutation representation of G on cosets of this index 16 subgroup is equivalent to the representation of G_5 on cosets of $\langle h, a\rangle$, which gives the automorphism group of the Bigg-Conway graph as a quotient. Hence $\ker\psi$ is contained in the kernel of the latter representation, and it follows that Γ is a cover of the Bigg-Conway graph. Since the latter is bipartite, so is Γ. $\square$

4 Admissible combinations of actions

We consider each of the possible type combinations in turn.

4.1 Type 1 only

Finite cubic graphs admitting arc-transitive group actions of type 1 only are precisely those which are 1-regular. All examples are the underlying graphs of orientably-regular but *chiral* (irreflexible) 3-valent maps (see [9, 14] or related articles on regular maps).

The smallest is F026, with automorphism group a semi-direct product of Z_{13} by Z_6 (of order 78), but there are many others, and not every example is a cover of this one. Examples like F026 (of girth 6) are bipartite, but others like F448A (of girth 7) are not.

There are infinitely many graphs in this class. In fact there are at least two different kinds of families of examples. On one hand, there is an infinite family of examples of girth 6 with soluble automorphism groups, obtainable by adding $(ha)^6 = 1$ plus extra relations to the group G_1; see [19]. On the other hand, it can be shown using coset graphs that for every integer $k \geq 7$, all but finitely many of the alternating groups A_n can be generated by two elements x and y such that x, y and xy have orders 2, 3 and k respectively, with the additional property that there exists no group automorphism taking x and y to x^{-1} and y^{-1} respectively. This theorem (which will be published elsewhere by the first author in a paper on chiral maps) implies that there exist infinite many 1-regular finite cubic graphs of girth k for every $k \geq 7$, with alternating (and therefore insoluble) automorphism groups. Actions of type 2^1 and 3 are avoided because of chirality — a lack of mirror symmetry — and actions of type 4^1 and 5 are avoided because $\mathrm{PSL}(2, 7)$ is not involved as a composition factor in the automorphism group.

4.2 Types 1 and 2^1 only

All graphs in this class are 2-regular, and have a 1-regular (but no 3-regular) group of automorphisms. As such, they are all underlying graphs of 3-valent regular maps that are both reflexible and orientable, but not isomorphic to their Petrie duals; see [16].

The smallest is the complete graph K_4, but not every example is a cover of this one. Examples like the 3-dimensional cube graph F008 (of girth 4) and F050 (of girth 6) are bipartite, while K_4 (of girth 3) and F056B (of girth 7) others like these are not.

There are infinitely many graphs in this class. Their automorphism groups are the images of non-degenerate epimorphisms from $G_2^1 \cong \mathrm{PGL}(2, \mathbb{Z})$ with the property that the subgroup $G_1 = \langle h, a \rangle \cong \mathrm{PSL}(2, \mathbb{Z})$ maps to a subgroup of index 2, and there exists no group automorphism taking the images of h, p and a to the images of h, p and ap respectively. As the latter condition can be checked by considering the images of $(ha)^2$ and $(hap)^2 = hah^{-1}a$, it is relatively easy to verify when $(ha)^2$ and $hah^{-1}a$ map to elements of different orders. Examples include the symmetric groups S_n for all but finitely many n (see [5]), and $\mathrm{PGL}(2, q)$ or $\mathrm{PSL}(2, q) \times Z_2$ for certain prime-powers q (see [6]). Indeed for any integer $k \geq 6$, this class contains infinitely many graphs of girth k, by residual finiteness of the $(2, 3, k)$ triangle group $\langle\, a, h \mid a^2, h^3, (ah)^k \,\rangle$.

4.3 Type 2^1 only

In this class, the graphs are 2-regular but admit no 1-regular group of automorphisms. As such, they are all underlying graphs of *non-orientable* 3-valent regular maps (see [9]).

The smallest is the graph F084, but not every example is a cover of this one. Examples like F120A (of girth 8) are bipartite, but F084 (of girth 7) and others like it are not.

Again there are infinitely graphs in this class. Their automorphism groups are the images of non-degenerate epimorphisms from $G_2^1 \cong \mathrm{PGL}(2,\mathbb{Z})$ under which $G_1 = \langle h, a\rangle \cong \mathrm{PSL}(2,\mathbb{Z})$ does *not* map to a subgroup of index 2, and there exists no group automorphism taking the images of h, p and a to the images of h, p and ap respectively. Examples include simple quotients of G_2^1 such as the alternating groups A_n for all but finitely many n (see [5]), and $\mathrm{PSL}(2,p)$ for certain primes p (see [6]). Note that simplicity implies that there will be no subgroup of type 1, but does not rule out the possibility of the graph being 3-arc-regular (as the Petersen graph shows in §4.7 below).

4.4 Type 2^2 only

This class consists of 2-regular cubic graphs admitting an action of type 2^2. The first known example of such a graph was given by Conder and Lorimer in [11], in answer to a question raised by Djoković and Miller in [15, Problem 2]. This example had automorphism group S_{11}. The smallest example was found to be F448C, of order 448, in [10]. Clearly not every example is a cover of this smallest one. Both F448C and the earlier example are bipartite, but many others are not. An infinite family of non-bipartite examples (with automorphism groups A_{6k+3} for all $k > 3$) was constructed in [12, Example 4.1].

4.5 Types 1, 2^1, 2^2 and 3

Graphs in this class are 3-regular cubic graphs admitting an action of type 1 (and therefore actions of types 2^1 and 2^2, by Proposition 2.3). As such, they are the underlying graphs of reflexible orientable 3-valent regular maps that are isomorphic to their Petrie duals.

The smallest example is the complete bipartite graph $K_{3,3}$, but there are many others, such as F040, which are not covers of this one. All examples are bipartite, by [15, Proposition 26] or Proposition 3.1.

Again there are infinitely graphs in this class; in fact there are infinitely many that are covers of $K_{3,3}$. To see this, note that the kernel of the epimorphism $\psi\colon G_3 \to \mathrm{Aut}(K_{3,3})$ is a normal subgroup K of index 72 in G_3, generated by conjugates of the element $(ha)^2(h^{-1}a)^2$. The Reidemeister-Schreier process (implemented as the `Rewrite` command in MAGMA [3]) can be used to show that the abelianisation $K/[K,K]$ of the subgroup K is isomorphic to $\mathbb{Z}^4$ (free abelian of rank 4). It follows that for every positive integer k, the group G_3 contains a normal subgroup $N_k = [K,K]K^k$ of index k^4 in K, with quotient G_3/N_k of order $72k^4$, which is then the automorphism group of a 3-regular cubic graph of order $6k^4$ that is a cover of $K_{3,3}$ and admits actions of all four types 1, 2^1, 2^2 and 3.

4.6 Types 2^1, 2^2 and 3 only

Graphs in this class are 3-regular and admit actions of types 2^1 and 2^2, but not of type 1. All such graphs are bipartite, by [15, Proposition 26] or Proposition 3.1. The smallest is F020B, but there are many others, such as F056C, which are not covers of this one.

On the other hand, the class does contain infinitely many covers of F020B. This follows from the fact that the kernel of the epimorphism $\psi\colon G_3 \to \mathrm{Aut}(\mathrm{F020B})$ is a normal subgroup K of index 240 in G_3, generated by conjugates of the element $pq(ha)^2(h^{-1}a)^2(ha)^2$, with abelianisation $K/[K,K]$ isomorphic to $\mathbb{Z}^{11}$ (free abelian of rank 11). For every positive integer k, the group G_3 therefore contains a normal subgroup $N_k = [K,K]K^k$ of index k^{11} in K, with quotient G_3/N_k of order $240k^{11}$, which is then the automorphism group of a 3-regular cubic graph of order $20k^{11}$ that is a cover of F020B and admits arc-transitive group actions of types 2^1, 2^2 and 3. Moreover, if k is odd, then the subgroup $\langle h, a\rangle \cong G_1$ of G_3 must map to the same subgroup as $\langle h, a, pq\rangle \cong G_2^1$ (of index 2 in the automorphism group), so this graph will not admit an arc-transitive group action of type 1.

4.7 Types 2^1 and 3 only

Graphs in this class are 3-regular and admit actions of type 2^1, but not of type 1 or 2^2. All such graphs are non-bipartite, by Proposition 3.1. The smallest example is the Petersen graph F010 (see [18]), with automorphism group S_5, but there are others that are not covers of this one (such as F570A, the automorphism group of which is $\mathrm{PGL}(2,19)$).

Again the class is infinite. The kernel of the epimorphism $\psi\colon G_3 \to \mathrm{Aut}(\mathrm{F010})$ is a normal subgroup K of index 120 in G_3, generated by conjugates of the element $(ha)^5$. The abelianisation $K/[K,K]$ is isomorphic to $\mathbb{Z}^6$, so for every positive integer k, the group G_3 has a quotient of order $120k^6$ that is the automorphism group of a 3-regular cubic graph of order $10k^6$ and contains a subgroup of type 2^1. If k is odd, then the subgroup $\langle h, a\rangle \cong G_1$ of G_3 must map to the same subgroup as $\langle h, a, pq\rangle \cong G_2^1$ (of index 2 in the automorphism group), while the subgroup $\langle h, ap, pq\rangle \cong G_2^2$ maps onto the full automorphism group, so this graph will not admit an arc-transitive group action of type 1 or 2^2.

4.8 Types 2^2 and 3 only

This case is similar to the previous one, but with the roles of 2^1 and 2^2 reversed. Again all graphs in this class are non-bipartite, by Proposition 3.1. The smallest example is the Coxeter graph F028 (see [13]), with automorphism group $\mathrm{PGL}(2,7)$, but there are others that are not covers of this one (such as F408B, the automorphism group of which is $\mathrm{PGL}(2,17)$).

The class is infinite, for one reason because the epimorphism $\psi\colon G_3 \to \mathrm{Aut}(\mathrm{F028})$ has kernel K of index 336 in G_3, generated by conjugates of $pha(h^{-1}a)^2(ha)^2(h^{-1}a)^2$, with abelianisation $K/[K,K] \cong \mathbb{Z}^{15}$. Accordingly, for every positive integer k, the group G_3 has a quotient G_3/N_k of order $336k^{15}$ that is the automorphism group of a 3-regular cubic graph of order $28k^{15}$ and contains a subgroup of type 2^2; and moreover if k is odd, then the subgroup $\langle h, a\rangle \cong G_1$ must map to the full automorphism group, so this graph will not admit an arc-transitive group action of type 1 or 2^1.

4.9 Type 3 only

Djoković and Miller found an example in this class in [15, Section 16], having 182 vertices; this is F182D. The smallest example, however, is F110, with automorphism group $\mathrm{PGL}(2,11)$. Clearly not every example is a cover of F110. The two smallest examples are bipartite, but a third small example F506A is not. As in the previous four cases, this class contains infinitely many covers of its smallest member. The kernel of the epimorphism $\psi : G_3 \to \mathrm{Aut}(\mathrm{F110})$ is a normal subgroup K of index 1320 in G_3, generated by conjugates of $q(ha)^2(h^{-1}a(ha)^3)^2$, with abelianisation $K/[K,K]$ isomorphic to $\mathbb{Z}^{56}$. Hence for every positive integer k, the group G_3 has a quotient G_3/N_k of order $1320k^{56}$ that is the automorphism group of a 3-regular cubic graph of order $110k^{56}$, and if k is odd, then this graph admits no arc-transitive group action of type 1, 2^1 or 2^2.

4.10 Types 1 and 4^1 only

As observed by Djoković and Miller in [15, Proposition 29] (and again in Proposition 3.2 above), there is a unique minimal finite cubic graph admitting actions of types 1 and 4^1, namely the Heawood graph F014 (the incidence graph of a projective plane of order 2). Every graph in this class is a cover of the Heawood graph, and in particular, is bipartite. Furthermore, we can show that there are infinitely many such covers. The kernel of the epimorphism $\psi\colon G_4^1 \to \mathrm{Aut}(\mathrm{F014})$ is a normal subgroup K of index 336 in G_4^1, generated by conjugates of $p(ha)^3(h^{-1}a)^3$, and the abelianisation $K/[K,K]$ is isomorphic to $\mathbb{Z}^8$. Hence for every positive integer k, the group G_4^1 has a quotient G_4^1/N_k of order $336k^8$ that is the automorphism group of a 4-regular cubic graph of order $14k^8$ admitting arc-transitive group actions of both types 1 and 4^1 (but not of type 5, since F014 is not 5-regular).

4.11 Type 4^1 only

The smallest representative of this class is the Sextet graph $S(17)$ of order 102, labelled F102 in the Foster census, and one of an infinite family of symmetric cubic graphs constructed in [2]. Not every graph in this class is a cover of this smallest one. Examples like F506B (of girth 14) and F650B (of girth 12) are bipartite, but F102 (of girth 9) and others like it are not. This class contains infinitely many of the Sextet graphs $S(p)$, namely all of those for prime $p \equiv \pm 1 \bmod 16$ (in which case $S(p)$ is non-bipartite, with automorphism group $\mathrm{PSL}(2,p)$), and all of those for prime $p \equiv \pm 7 \bmod 16$ but $p \neq 7$ (in which case $S(p)$ is bipartite, with automorphism group $\mathrm{PGL}(2,p)$). Also this class contains infinitely many covers $S(17)$, because the kernel of the epimorphism $\psi : G_4^1 \to \mathrm{Aut}(S(17))$ is a normal subgroup K of index 2448 in G_4^1, generated by conjugates of the element $(ha)^9$, with abelianisation $K/[K,K] \cong \mathbb{Z}^{52}$.

4.12 Type 4^2 only

This class consists of 4-regular cubic graphs admitting an action of type 4^2. The first known example of such a graph was a (non-bipartite) graph with automorphism group A_{29}, produced by Conder & Lorimer [11] in answer to a question raised by Djoković and Miller in [15, Problem 3]. Also an infinite family of (bipartite) examples with automorphism groups S_{36k} for all $k > 4$ was subsequently constructed in [12, Example 4.2].

The smallest example we have been able to find is a graph of order 27,634,932, which is a 3^{10}-fold cover of F468 (which is itself a double cover of Wong's graph F234B). The automorphism group of this graph is an extension of an elementary abelian group of order 3^{10} by $\mathrm{Aut}(\mathrm{PSL}(3,3))$, and is isomorphic to the permutation group induced by G_4^2 on the cosets of its subgroup of index 39 generated by the elements p, q, r, a, $(ha)^3(h^{-1}a)^2hah$, $(ha)^5(h^{-1}a)^3h$ and $(ha)^4h^{-1}aha(h^{-1}a)^2hah^{-1}$. We do not know for sure if this is the smallest 4-regular cubic graph of type 4^2.

4.13 Types 1, 4^1, 4^2 and 5

As observed in [15, Proposition 30] by Djoković and Miller, and repeated above in Proposition 3.4, there is a unique minimal finite cubic graph admitting actions of types 1 and 5, namely the Biggs-Conway graph [1]. The automorphism group of this graph contains subgroups of index 2 that are the images of the subgroups $\langle h, a, pq \rangle \cong G_4^1$ and $\langle h, ap, pq \rangle \cong G_4^2$ under an epimorphism from G_5, as well as a subgroup of index 16 which is the image of the subgroup $\langle h, a \rangle \cong G_1$. Accordingly, the graph admits arc-transitive actions of types 1, 4^1, 4^2 and 5. Every finite cubic graph admitting actions of all four of these types is a cover of the Biggs-Conway graph, and in particular, must be bipartite.

Furthermore, we can show that there are infinitely many of these covers. The kernel of the epimorphism from G_5 to the automorphism group of the Biggs-Conway graph is a normal subgroup K of index 112896 in G_5, generated by conjugates of the element $(ha)^4(h^{-1}a)^4ha(h^{-1}a)^2(ha)^2h^{-1}a$, with abelianisation $K/[K,K] \cong \mathbb{Z}^{1177}$. (Note that computation of this abelianisation is not easy because of the large index $|G_5 : K|$; but it is relatively easy to see that K is contained in the subgroup $L = \langle h^{-1}aha, hah^{-1}a \rangle$, which has index 96 in G_5 and has abelianisation $L/[L,L] \cong \mathbb{Z}^2$, and then since $[K,K] \subset [L,L]$, it follows that $[K,K]$ has infinite index in L, and therefore infinite index in K.)

4.14 Types 4^1, 4^2 and 5 only

Graphs in this class are 5-regular and admit actions of types 4^1 and 4^2, but not of type 1. All such graphs are bipartite, by Proposition 3.3. The smallest is Tutte's 8-cage F030, which is the Sextet graph $S(3)$, but there are also many others, such as F468 (the double cover of Wong's graph), which are not covers of this one.

In fact this class contains the Sextet graph $S(p)$ for every prime $p \equiv \pm 3$ or ± 5 mod 16. Each such $S(p)$ is 5-regular and bipartite, with automorphism group $\mathrm{P\Gamma L}(2, p^2)$. The latter group contains three subgroups of index 2, one of which is $\mathrm{PGL}(2, p^2)$, shown in [2] to be 4-regular on $S(p)$. As $S(p)$ is bipartite, it follows from Proposition 3.3 that its automorphism group contains subgroups of both types 4^1 and 4^2 (but no 1-regular subgroup, since $S(p)$ does not cover the Biggs-Conway graph).

Also the kernel of the epimorphism $\psi\colon G_5 \to \mathrm{Aut}(S(3))$ is a normal subgroup K of index 1440 in G_5, generated by conjugates of the element $pq(ha)^4(h^{-1}a)^4$, with abelianisation $K/[K, K] \cong \mathbb{Z}^{16}$, so this class contains infinitely many covers of $S(3)$.

4.15 Types 4^1 and 5 only

For every graph Γ in this class, there is an epimorphism $\psi\colon G_5 \to \mathrm{Aut}(\Gamma)$ that maps both of the subgroups $\langle h, a\rangle \cong G_1$ and $\langle h, a, pq\rangle \cong G_4^1$ to a subgroup of index 2 in $\mathrm{Aut}(\Gamma)$, and the subgroup $\langle h, ap, pq\rangle \cong G_4^2$ onto $\mathrm{Aut}(\Gamma)$ itself. Every such graph Γ will be non-bipartite, by Proposition 3.3.

One example is a graph of order 75600 with automorphism group S_{10} shown to exist by Conder in [7], and we believe this is the smallest example.

There are, in fact, infinitely many such examples, because the kernel K of the corresponding epimorphism $\psi : G_5 \to S_{10}$ has infinite abelianisation. The precise form of the abelianisation is difficult to determine, since $|G_5 : K| = 10!$, but it is easy to verify (with the help of Magma [3] for example) that K is contained in the subgroup L of index 90 in G_5 generated by p, $ahah^{-1}a$, $h^{-1}ahah^{-1}ahpahah^{-1}ah^{-1}$ and $hahah^{-1}ahahah^{-1}apahah^{-1}ah^{-1}$, and that this has abelianisation $L/[L, L] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$. Since $[K, K] \subset [L, L]$, it follows that $[K, K]$ has infinite index in L, and therefore infinite index in K. Hence this class of graphs contains infinitely many covers of Conder's graph of order 75600.

Also there are many more examples than these: Conder showed in in [8] how to combine permutation representations for $S_4 \times Z_2$ to prove that for all but finitely many n, there exists an epimorphism $\psi\colon G_5 \to S_n$ under which the ψ-images of h and a are permutations generating A_n while those of p, q, r and s are odd.

4.16 Types 4^2 and 5 only

This case is similar to the previous one, but with the roles of 4^1 and 4^2 reversed. Again all graphs in this class are non-bipartite, by Proposition 3.3. The smallest is Wong's graph

F234B (see [22]), with automorphism group Aut(PSL$(3,3)$), of order 11232. Infinitely many of its covers belong to the same class, as the kernel of the epimorphism $\psi\colon G_5 \to$ Aut(PSL$(3,3)$) has infinite abelianisation (isomorphic to $\mathbb{Z}^{118}$). The class contains other examples as well, however, such as one with automorphism group S_{42} (obtainable as the quotient of G_5 by the core of the subgroup generated by ps, ap and $(ha)^3h^{-1}ahah^{-1}$).

4.17 Type 5 only

The first known graph in this class was an example with automorphism group A_{26}, produced by Conder & Lorimer [11] in answer to a question raised by Djoković and Miller in [15, Problem 1]. Following on from this, Conder proved in [8] that there are infinitely many such graphs, by showing that examples exist with automorphism group A_n for all but finitely many n. (The fact that A_n contains no proper subgroup of index less than n implies that there can be no 1- or 4-regular subgroup whenever $n > 16$.)

These examples are non-bipartite, but there are also infinitely many bipartite examples. The construction presented in [8] can be adapted (by the addition of single extra point) to prove that for all but finitely many n, there exists an epimorphism $\psi : G_5 \to S_n$ under which the ψ-images of h, p, q, r and s are even while the ψ-image of a is odd; this epimorphism then maps both $\langle h, a\rangle \cong G_1$ and $\langle h, ap, pq\rangle \cong G_4^2$ onto S_n. The smallest such example has automorphism group S_{20}, but we have been able to find another example, smaller than this, with a rather more interesting group, which we describe below.

The group G_5 has a subgroup of index 48 generated by the elements h, p, $ahahah^{-1}a$ and $qsah^{-1}ah^{-1}aha$, the cosets of which are permuted by G_5 (under multiplication by elements) as follows:

$h \mapsto$ (2, 3, 6)(5, 13, 14)(7, 11, 9)(8, 17, 18)(10, 20, 21)(12, 23, 24)(15, 29, 30)(16, 25, 27)(19, 33, 34)(22, 35, 37)(28, 42, 43)(31, 38, 45)(32, 46, 40)(39, 47, 44),

$p \mapsto$ (5, 8)(10, 15)(12, 16)(13, 17)(14, 18)(19, 22)(20, 29)(21, 30)(23, 25)(24, 27)(26, 41)(28, 40)(31, 39)(32, 42)(33, 35)(34, 37)(36, 48)(38, 47)(43, 46)(44, 45),

$q \mapsto$ (1, 2)(3, 6)(4, 7)(9, 11)(13, 25)(14, 24)(17, 23)(18, 27)(20, 35)(21, 34)(26, 31)(28, 32)(29, 33)(30, 37)(36, 43)(38, 44)(39, 41)(40, 42)(45, 47)(46, 48),

$r \mapsto$ (1, 3)(2, 6)(4, 11)(5, 12)(7, 9)(8, 16)(10, 19)(14, 27)(15, 22)(18, 24)(21, 37)(26, 38)(28, 36)(30, 34)(31, 44)(32, 43)(39, 45)(40, 48)(41, 47)(42, 46),

$s \mapsto$ (1, 4)(2, 7)(3, 9)(5, 10)(6, 11)(8, 15)(12, 22)(13, 21)(14, 20)(16, 19)(17, 30)(18, 29)(23, 37)(24, 35)(25, 34)(27, 33)(28, 42)(32, 40)(38, 45)(44, 47),

$a \mapsto$ (1, 5)(2, 8)(3, 10)(4, 12)(6, 15)(7, 16)(9, 19)(11, 22)(13, 26)(14, 28)(17, 31),(18, 32)(20, 36)(21, 38)(23, 39)(24, 40)(25, 41)(27, 42)(29, 43)(30, 44)(33, 46)(34, 47)(35, 48)(37, 45).

These permutations generate a group P isomorphic to the wreath product $M_{24} \wr C_2$ (of order $2|M_{24}|^2 = 119{,}876{,}641{,}829{,}683{,}200$), where M_{24} is the simple Mathieu group of degree 24. This group P has just one subgroup of index 2, namely the direct product $M_{24} \times M_{24}$, which is the image of the subgroup $\langle h, aha, p, q, r, s\rangle \cong G_4^1$. It follows that P is the automorphism group of a bipartite 5-regular cubic graph of order 2,497,430,038,118,400, but contains no subgroup of type 4^1 or 4^2.

The above example was found from a search for subgroups of small index in the group G_5 using MAGMA [3], and is the smallest example we have been able to find in this class.

5 Summary

Our findings can be summarised as follows:

Theorem 5.1 *Finite symmetric cubic graphs can be classified into* 17 *different families, according to the combinations of arc-transitive actions they admit. Information on these classes is given in Table 1. In* 14 *of the* 17 *classes, the smallest representatives have been determined, and in the other three, the smallest known examples have orders* $10! = 75600$, $3^{10}\cdot 468$ *and* $2^{17}\cdot 3^5\cdot 5^2\cdot 7^2\cdot 11^2\cdot 23^2$ *(for the type combinations* $\{4^1, 5\}$, $\{4^2\}$ *and* $\{5\}$ *respectively). There are infinitely many graphs in each class, and in two cases (the type combinations* $\{1, 4^1\}$ *and* $\{1, 4^1, 4^2, 5\}$*) every graph in the class is a cover of the smallest example.*

s	Types	Bipartite?	Smallest example	Unique minimal?
1	1	Sometimes	F026	No
2	$1,2^1$	Sometimes	F004 (K_4)	No
2	2^1	Sometimes	F084	No
2	2^2	Sometimes	F448C	No
3	$1, 2^1, 2^2, 3$	Always	F006 ($K_{3,3}$)	No
3	$2^1, 2^2, 3$	Always	F020B (GP(10,3))	No
3	$2^1, 3$	Never	F010 (Petersen)	No
3	$2^2, 3$	Never	F028 (Coxeter)	No
3	3	Sometimes	F110	No
4	$1, 4^1$	Always	F014 (Heawood)	Yes
4	4^1	Sometimes	F102 (S(17))	No
4	4^2	Sometimes	3^{10}-fold cover of F468?	No
5	$1, 4^1, 4^2, 5$	Always	Biggs-Conway graph	Yes
5	$4^1, 4^2, 5$	Always	F030 (Tutte's 8-cage)	No
5	$4^1, 5$	Never	S_{10} graph?	No
5	$4^2, 5$	Never	F234B (Wong's graph)	No
5	5	Sometimes	$M_{24} \wr C_2$ graph?	No

Table 1: The 17 families of finite symmetric cubic graphs
(classified according to the types of arc-transitive group actions admitted)

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Appendix: Arc-transitive actions on graphs of order up to 768

Graph	Order	Automs	s-trans	Girth	Diameter	Bipartite?	Types
F004	4	24	2	3	1	No	$1, 2^1$
F006	6	72	3	4	2	Yes	$1, 2^1, 2^2, 3$
F008	8	48	2	4	3	Yes	$1, 2^1$
F010	10	120	3	5	2	No	$2^1, 3$
F014	14	336	4	6	3	Yes	$1, 4^1$
F016	16	96	2	6	4	Yes	$1, 2^1$
F018	18	216	3	6	4	Yes	$1, 2^1, 2^2, 3$
F020A	20	120	2	5	5	No	$1, 2^1$
F020B	20	240	3	6	5	Yes	$2^1, 2^2, 3$
F024	24	144	2	6	4	Yes	$1, 2^1$
F026	26	78	1	6	5	Yes	1
F028	28	336	3	7	4	No	$2^2, 3$
F030	30	1440	5	8	4	Yes	$4^1, 4^2, 5$
F032	32	192	2	6	5	Yes	$1, 2^1$
F038	38	114	1	6	5	Yes	1
F040	40	480	3	8	6	Yes	$1, 2^1, 2^2, 3$
F042	42	126	1	6	6	Yes	1
F048	48	288	2	8	6	Yes	$1, 2^1$
F050	50	300	2	6	7	Yes	$1, 2^1$
F054	54	324	2	6	6	Yes	$1, 2^1$
F056A	56	168	1	6	7	Yes	1
F056B	56	336	2	7	6	No	$1, 2^1$
F056C	56	672	3	8	7	Yes	$2^1, 2^2, 3$
F060	60	360	2	9	5	No	$1, 2^1$
F062	62	186	1	6	7	Yes	1
F064	64	384	2	8	6	Yes	$1, 2^1$
F072	72	432	2	6	8	Yes	$1, 2^1$
F074	74	222	1	6	7	Yes	1
F078	78	234	1	6	8	Yes	1
F080	80	960	3	10	8	Yes	$1, 2^1, 2^2, 3$
F084	84	504	2	7	7	No	2^1
F086	86	258	1	6	9	Yes	1
F090	90	4320	5	10	8	Yes	$4^1, 4^2, 5$
F096A	96	576	2	6	8	Yes	$1, 2^1$
F096B	96	1152	3	8	7	Yes	$1, 2^1, 2^2, 3$
F098A	98	294	1	6	9	Yes	1
F098B	98	588	2	6	9	Yes	$1, 2^1$
F102	102	2448	4	9	7	No	4^1
F104	104	312	1	6	9	Yes	1
F108	108	648	2	9	7	No	$1, 2^1$
F110	110	1320	3	10	7	Yes	3
F112A	112	672	2	8	7	Yes	$1, 2^1$

Graph	Order	Automs	s-trans	Girth	Diameter	Bipartite?	Types
F112B	112	1344	3	8	10	Yes	1, 2^1, 2^2, 3
F112C	112	336	1	10	7	Yes	1
F114	114	342	1	6	10	Yes	1
F120A	120	720	2	8	8	Yes	2^1
F120B	120	720	2	10	9	Yes	1, 2^1
F122	122	366	1	6	9	Yes	1
F126	126	378	1	6	10	Yes	1
F128A	128	768	2	6	11	Yes	1, 2^1
F128B	128	768	2	10	8	Yes	1, 2^1
F134	134	402	1	6	11	Yes	1
F144A	144	432	1	8	7	Yes	1
F144B	144	864	2	10	8	Yes	1, 2^1
F146	146	438	1	6	11	Yes	1
F150	150	900	2	6	10	Yes	1, 2^1
F152	152	456	1	6	11	Yes	1
F158	158	474	1	6	11	Yes	1
F162A	162	972	2	6	12	Yes	1, 2^1
F162B	162	486	1	12	7	Yes	1
F162C	162	1944	3	12	8	Yes	1, 2^1, 2^2, 3
F168A	168	504	1	6	12	Yes	1
F168B	168	1008	2	7	9	No	1, 2^1
F168C	168	1008	2	8	8	No	1, 2^1
F168D	168	1008	2	9	7	No	1, 2^1
F168E	168	504	1	12	7	Yes	1
F168F	168	1008	2	12	8	Yes	2^1
F182A	182	546	1	6	11	Yes	1
F182B	182	546	1	6	13	Yes	1
F182C	182	1092	2	7	8	No	2^1
F182D	182	2184	3	12	9	Yes	3
F186	186	558	1	6	12	Yes	1
F192A	192	2304	3	8	12	Yes	1, 2^1, 2^2, 3
F192B	192	1152	2	10	10	Yes	1, 2^1
F192C	192	1152	2	12	8	Yes	1, 2^1
F194	194	582	1	6	13	Yes	1
F200	200	1200	2	6	13	Yes	1, 2^1
F204	204	4896	4	12	9	Yes	4^1
F206	206	618	1	6	13	Yes	1
F208	208	624	1	10	9	Yes	1
F216A	216	1296	2	6	12	Yes	1, 2^1
F216B	216	1296	2	10	9	Yes	1, 2^1
F216C	216	1296	2	12	8	Yes	1, 2^1
F218	218	654	1	6	13	Yes	1
F220A	220	1320	2	10	9	No	1, 2^1
F220B	220	1320	2	10	9	Yes	2^1
F220C	220	2640	3	10	10	Yes	2^1, 2^2, 3

Graph	Order	Automs	s-trans	Girth	Diameter	Bipartite?	Types
F222	222	666	1	6	14	Yes	1
F224A	224	672	1	6	13	Yes	1
F224B	224	1344	2	12	9	Yes	$1,\ 2^1$
F224C	224	2688	3	12	10	Yes	$1,\ 2^1,\ 2^2,\ 3$
F234A	234	702	1	6	14	Yes	1
F234B	234	11232	5	12	8	No	$4^2,\ 5$
F240A	240	1440	2	8	10	Yes	$1,\ 2^1$
F240B	240	1440	2	9	10	No	$1,\ 2^1$
F240C	240	1440	2	10	11	Yes	$1,\ 2^1$
F242	242	1452	2	6	15	Yes	$1,\ 2^1$
F248	248	744	1	6	15	Yes	1
F250	250	1500	2	10	10	Yes	$1,\ 2^1$
F254	254	762	1	6	13	Yes	1
F256A	256	1536	2	8	10	Yes	$1,\ 2^1$
F256B	256	1536	2	10	10	Yes	$1,\ 2^1$
F256C	256	1536	2	10	11	Yes	$1,\ 2^1$
F256D	256	768	1	12	9	Yes	1
F258	258	774	1	6	14	Yes	1
F266A	266	798	1	6	15	Yes	1
F266B	266	798	1	6	15	Yes	1
F278	278	834	1	6	15	Yes	1
F288A	288	1728	2	6	16	Yes	$1,\ 2^1$
F288B	288	3456	3	12	9	Yes	$1,\ 2^1,\ 2^2,\ 3$
F294A	294	1764	2	6	14	Yes	$1,\ 2^1$
F294B	294	882	1	6	16	Yes	1
F296	296	888	1	6	15	Yes	1
F302	302	906	1	6	15	Yes	1
F304	304	912	1	10	11	Yes	1
F312A	312	936	1	6	16	Yes	1
F312B	312	936	1	12	9	Yes	1
F314	314	942	1	6	17	Yes	1
F326	326	978	1	6	17	Yes	1
F336A	336	2016	2	8	10	Yes	$1,\ 2^1$
F336B	336	2016	2	8	13	Yes	$1,\ 2^1$
F336C	336	1008	1	10	12	Yes	1
F336D	336	2016	2	12	9	Yes	$1,\ 2^1$
F336E	336	2016	2	12	12	Yes	$1,\ 2^1$
F336F	336	1008	1	12	12	Yes	1
F338A	338	1014	1	6	15	Yes	1
F338B	338	2028	2	6	17	Yes	$1,\ 2^1$
F342	342	1026	1	6	16	Yes	1
F344	344	1032	1	6	17	Yes	1
F350	350	1050	1	6	17	Yes	1
F360A	360	2160	2	8	11	Yes	2^1
F360B	360	4320	3	12	10	Yes	$1,\ 2^1,\ 2^2,\ 3$

Graph	Order	Automs	s-trans	Girth	Diameter	Bipartite?	Types
F362	362	1086	1	6	17	Yes	1
F364A	364	2184	2	7	11	No	1, 2^1
F364B	364	2184	2	7	12	No	1, 2^1
F364C	364	2184	2	7	13	No	1, 2^1
F364D	364	2184	2	12	9	No	1, 2^1
F364E	364	2184	2	12	9	Yes	2^1
F364F	364	2184	2	12	10	Yes	2^1
F364G	364	4368	3	12	12	Yes	2^1, 2^2, 3
F366	366	1098	1	6	18	Yes	1
F378A	378	1134	1	6	18	Yes	1
F378B	378	1134	1	12	10	Yes	1
F384A	384	2304	2	6	16	Yes	1, 2^1
F384B	384	2304	2	12	10	Yes	1, 2^1
F384C	384	2304	2	12	10	Yes	1, 2^1
F384D	384	4608	3	12	12	Yes	1, 2^1, 2^2, 3
F386	386	1158	1	6	17	Yes	1
F392A	392	1176	1	6	17	Yes	1
F392B	392	2352	2	6	19	Yes	1, 2^1
F398	398	1194	1	6	19	Yes	1
F400A	400	1200	1	8	10	Yes	1
F400B	400	2400	2	10	13	Yes	1, 2^1
F402	402	1206	1	6	18	Yes	1
F408A	408	2448	2	9	10	No	2^1
F408B	408	4896	3	9	10	No	2^2, 3
F416	416	1248	1	6	19	Yes	1
F422	422	1266	1	6	19	Yes	1
F432A	432	2592	2	10	12	Yes	1, 2^1
F432B	432	2592	2	10	14	Yes	1, 2^1
F432C	432	1296	1	12	10	Yes	1
F432D	432	2592	2	12	12	Yes	1, 2^1
F432E	432	1296	1	8	12	Yes	1
F434A	434	1302	1	6	17	Yes	1
F434B	434	1302	1	6	19	Yes	1
F438	438	1314	1	6	18	Yes	1
F440A	440	2640	2	10	11	Yes	1, 2^1
F440B	440	2640	2	10	12	Yes	1, 2^1
F440C	440	5280	3	12	10	Yes	1, 2^1, 2^2, 3
F446	446	1338	1	6	19	Yes	1
F448A	448	1344	1	7	11	No	1
F448B	448	1344	1	10	13	Yes	1
F448C	448	2688	2	14	10	Yes	2^2
F450	450	2700	2	6	20	Yes	1, 2^1
F456A	456	1368	1	6	20	Yes	1
F456B	456	1368	1	12	10	Yes	1
F458	458	1374	1	6	19	Yes	1

Graph	Order	Automs	s-trans	Girth	Diameter	Bipartite?	Types
F468	468	22464	5	12	13	Yes	4^1, 4^2, 5
F474	474	1422	1	6	20	Yes	1
F480A	480	2880	2	9	15	No	1, 2^1
F480B	480	2880	2	12	11	Yes	1, 2^1
F480C	480	2880	2	12	10	No	1, 2^1
F480D	480	2880	2	10	10	Yes	1, 2^1
F482	482	1446	1	6	21	Yes	1
F486A	486	2916	2	6	18	Yes	1, 2^1
F486B	486	2916	2	12	12	Yes	1, 2^1
F486C	486	5832	3	12	12	Yes	1, 2^1, 2^2, 3
F486D	486	5832	3	12	12	Yes	1, 2^1, 2^2, 3
F488	488	1464	1	6	19	Yes	1
F494A	494	1482	1	6	19	Yes	1
F494B	494	1482	1	6	21	Yes	1
F496	496	1488	1	10	15	Yes	1
F500	500	3000	2	10	12	No	1, 2^1
F504A	504	1512	1	6	20	Yes	1
F504B	504	1512	1	9	10	No	1
F504C	504	3024	2	9	12	No	1, 2^1
F504D	504	1512	1	12	12	Yes	1
F504E	504	3024	2	14	10	No	1, 2^1
F506A	506	6072	3	11	11	No	3
F506B	506	12144	4	14	10	Yes	4^1
F512A	512	3072	2	6	21	Yes	1, 2^1
F512B	512	3072	2	10	12	Yes	1, 2^1
F512C	512	3072	2	12	11	Yes	1, 2^1
F512D	512	3072	2	12	11	Yes	1, 2^1
F512E	512	1536	1	14	12	Yes	1
F512F	512	3072	2	8	12	Yes	1, 2^1
F512G	512	3072	2	12	10	Yes	1, 2^1
F518A	518	1554	1	6	21	Yes	1
F518B	518	1554	1	6	21	Yes	1
F536	536	1608	1	6	21	Yes	1
F542	542	1626	1	6	19	Yes	1
F546A	546	1638	1	6	22	Yes	1
F546B	546	1638	1	6	20	Yes	1
F554	554	1662	1	6	21	Yes	1
F558	558	1674	1	6	22	Yes	1
F566	566	1698	1	6	21	Yes	1
F570A	570	6840	3	9	11	No	2^1, 3
F570B	570	3420	2	9	11	No	2^1
F576A	576	3456	2	10	16	Yes	1, 2^1
F576B	576	1728	1	8	12	Yes	1
F576C	576	3456	2	12	12	Yes	1, 2^1
F576D	576	6912	3	12	14	Yes	1, 2^1, 2^2, 3

Graph	Order	Automs	s-trans	Girth	Diameter	Bipartite?	Types
F578	578	3468	2	6	23	Yes	1, 2^1
F582	582	1746	1	6	22	Yes	1
F584	584	1752	1	6	23	Yes	1
F592	592	1776	1	10	15	Yes	1
F600A	600	3600	2	12	12	Yes	1, 2^1
F600B	600	3600	2	6	20	Yes	1, 2^1
F602A	602	1806	1	6	21	Yes	1
F602B	602	1806	1	6	23	Yes	1
F608	608	1824	1	6	21	Yes	1
F614	614	1842	1	6	23	Yes	1
F618	618	1854	1	6	22	Yes	1
F620	620	14880	4	15	10	No	4^1
F624A	624	1872	1	14	12	Yes	1
F624B	624	1872	1	10	16	Yes	1
F626	626	1878	1	6	23	Yes	1
F632	632	1896	1	6	23	Yes	1
F640	640	7680	3	10	12	Yes	1, 2^1, 2^2, 3
F648A	648	3888	2	6	24	Yes	1, 2^1
F648B	648	3888	2	12	14	Yes	1, 2^1
F648C	648	3888	2	12	12	Yes	1, 2^1
F648D	648	1944	1	12	13	Yes	1
F648E	648	1944	1	12	12	Yes	1
F648F	648	3888	2	12	10	Yes	1, 2^1
F650A	650	1950	1	6	23	Yes	1
F650B	650	31200	5	12	11	Yes	4^1, 4^2, 5
F654	654	1962	1	6	24	Yes	1
F660	660	3960	2	10	11	No	1, 2^1
F662	662	1986	1	6	21	Yes	1
F666	666	1998	1	6	22	Yes	1
F672A	672	2016	1	12	12	Yes	1
F672B	672	2016	1	6	24	Yes	1
F672C	672	4032	2	8	12	No	1, 2^1
F672D	672	4032	2	12	12	Yes	1, 2^1
F672E	672	4032	2	14	12	Yes	1, 2^1
F672F	672	4032	2	12	13	Yes	1, 2^1
F672G	672	2016	1	12	12	Yes	1
F674	674	2022	1	6	23	Yes	1
F680A	680	8160	3	10	11	No	2^1, 3
F680B	680	4080	2	12	10	No	2^1
F686A	686	2058	1	6	25	Yes	1
F686B	686	4116	2	12	12	Yes	1, 2^1
F686C	686	2058	1	6	23	Yes	1
F688	688	2064	1	10	17	Yes	1
F698	698	2094	1	6	25	Yes	1

Graph	Order	Automs	s-trans	Girth	Diameter	Bipartite?	Types
F702A	702	2106	1	12	14	Yes	1
F702B	702	2106	1	6	24	Yes	1
F720A	720	8640	3	12	12	Yes	1, 2^1, 2^2, 3
F720B	720	4320	2	8	12	Yes	1, 2^1
F720C	720	4320	2	10	10	Yes	1, 2^1
F720D	720	2160	1	8	12	Yes	1
F720E	720	4320	2	8	16	Yes	1, 2^1
F720F	720	2160	1	10	11	Yes	1
F722A	722	2166	1	6	25	Yes	1
F722B	722	4332	2	6	25	Yes	1, 2^1
F726	726	4356	2	6	22	Yes	1, 2^1
F728A	728	2184	1	6	23	Yes	1
F728B	728	2184	1	6	25	Yes	1
F728C	728	4368	2	12	12	Yes	1, 2^1
F728D	728	4368	2	12	12	Yes	1, 2^1
F728E	728	4368	2	12	13	Yes	1, 2^1
F728F	728	4368	2	12	14	Yes	1, 2^1
F728G	728	8736	3	12	14	Yes	1, 2^1, 2^2, 3
F734	734	2202	1	6	23	Yes	1
F744A	744	2232	1	12	13	Yes	1
F744B	744	2232	1	6	24	Yes	1
F746	746	2238	1	6	25	Yes	1
F750	750	4500	2	12	12	Yes	1, 2^1
F758	758	2274	1	6	25	Yes	1
F762	762	2286	1	6	26	Yes	1
F768A	768	4608	2	10	18	Yes	1, 2^1
F768B	768	4608	2	12	11	Yes	1, 2^1
F768C	768	4608	2	12	11	Yes	1, 2^1
F768D	768	2304	1	12	12	Yes	1
F768E	768	4608	2	12	12	Yes	1, 2^1
F768F	768	4608	2	12	13	Yes	1, 2^1
F768G	768	4608	2	14	12	Yes	1, 2^1