Regular maps with almost Sylow-cyclic automorphism groups, and classification of regular maps with Euler characteristic $-p^2$

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December 24, 2007

Abstract

A regular map $\mathcal{M}$ is a cellular decomposition of a surface such that its automorphism group $\mathrm{Aut}(\mathcal{M})$ acts transitively on the flags of $\mathcal{M}$. It can be shown that if a Sylow subgroup $P \leq \mathrm{Aut}(\mathcal{M})$ has order coprime to the Euler characteristic of the supporting surface, then P is cyclic or dihedral. This observation motivates the topic of the current paper, where we study regular maps whose automorphism groups have the property that all their Sylow subgroups contain a cyclic subgroup of index at most 2. The main result of the paper is a complete classification of such maps. As an application, we show that no regular maps of Euler characteristic $-p^2$ exist for p a prime greater than 7.

1 Introduction

A *map* is an embedding of a finite connected (not necessarily simple) graph or multigraph into a *surface* X (a compact real 2-dimensional manifold without boundary) such that the graph separates X into simply-connected regions, called the *faces* of the map. An *automorphism* of a map is an incidence preserving permutation of vertices, edges and faces of the map induced by a homeomorphism of the underlying surface. It is well known that a map with E edges can have at most $4E$ automorphisms (where, if the surface is orientable, exactly half of them are induced by orientation preserving homeomorphisms). Maps that meet this upper bound are called *regular*. Thus regular maps are, in some sense, the maps with highest possible level of symmetry.

The study of regular maps has a very long history which goes back to Pythagoreans and their work on regular polyhedra. One of the central themes in the theory of regular maps is the problem of their classification. This problem is usually approached from one of the following three viewpoints (which we shall refere to as graph-theoretical, group-theoretical, and topological). From the graph-theoretical viewpoint, one would usually like

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to determine all regular maps whose underlying graph has a specified graph theoretical property (such as prescribed valency and face-length, for example). Similarly, from a group theoretical viewpoint, the task is to determine all regular maps whose automorphism group has a given group theoretical property (such as solvability). Finally, from the topological point of view, the central problem is to classify all regular maps on a given surface X.

Until recently, the latter problem was resolved only for a finite number of surfaces, mainly of large Euler characteristic (and thus small genus). (For the complete list of regular maps on surfaces with Euler characteristic at least -28, see [4].) The first paper settling this problem for an infinite family of surfaces is a recent paper of D'Azevedo, Nedela, and the third author [1], where all regular maps on surfaces with Euler characteristic $\chi = -p$, for an odd prime $p \geq 29$, were determined. An interesting by-product of their result is the fact that the automorphism group G of such a regular map has the property that every Sylow subgroup of odd order in G is cyclic and every Sylow 2-subgroup in G is dihedral. This property can be generalized. We will say that a finite group G is *almost Sylow-cyclic* if every Sylow subgroup of odd order in G is cyclic and the Sylow 2-subgroup of G is either trivial or contains a cylic subgroup of index 2.

It is interesting to note that almost Sylow-cyclic groups occur in many different group-theoretical settings, and have attracted attention of several authors — starting with Hölder (1895) who proved that every group with all Sylow subgroups cyclic is solvable, and culminating in the work of Suzuki and Wong [18, 22], who completed the classification of all almost Sylow-cyclic groups.

In this paper, we classify all regular maps whose automorphism group is almost Sylow-cyclic. As a non-trivial application of this result we show that no regular maps of Euler characteristic $-p^2$ exist for p a prime greater than 7.

2 Preliminaries

Let $\mathcal{M}$ be a regular map on surface X, and let $G = \mathrm{Aut}(\mathcal{M})$ be its automorphism group. Then G acts transitively on the vertices of $\mathcal{M}$, as well as on the faces of $\mathcal{M}$. Hence, every vertex of $\mathcal{M}$ has the same *valency* m (the number of incident edges — where the loops are counted twice), and every face has the same *length* n (the number of edges on the boundary of the face). Such a map is then said to be of *type* $\{m, n\}$. If V, E, and F denote the numbers of vertices, edges, and faces of $\mathcal{M}$, then a simple counting argument shows that $2E = mV = nF$. Then since (by definition) the order of G is $4E$, it follows that $|G| = 4E = 2mV = 2nF$. Hence, for the Euler characteristic χ of X we find that

$$\chi = V - E + F = |G|(\frac{1}{2m} - \frac{1}{4} + \frac{1}{2n}) = -|G|\frac{mn - 2m - 2n}{4mn}. \quad (1)$$

If $\chi \neq 0$ (that is, whenever X is not the torus or the Klein bottle), then (1) gives the following formula for the order of G:

$$|G| = -\lambda(m, n)\chi, \quad \text{where} \quad \lambda(m, n) = \frac{4mn}{mn - 2m - 2n}. \quad (2)$$

For the surfaces with negative Euler characteristic, Hurwitz's theorem (see for example [19]) and Singerman's non-orientable analogue thereof (see [17]) provide an upper bound of 84 for the parameter $\lambda(m,n)$. Hence, if X is not the sphere, the torus, the projective plane, or the Klein bottle, then $\lambda(m,n) \le 84$ and $|G| \le -84\chi$.

Every regular map $\mathcal{M}$ on a surface X can be made *geometric* in the following sense (see [8] for details): There exists a Riemannian metric of constant curvature on X with respect to which the edges of $\mathcal{M}$ are geodesic arcs of equal length and the interior angle at every vertex in every face is $2\pi/m$, where m is the valency of the map. In this setting, the automorphisms of $\mathcal{M}$ are induced by isometries of X. Moreover, the automorphism group $G = \mathrm{Aut}(\mathcal{M})$ is generated by three involutions a, b, c with the following geometrical meaning: Let (v, e, f) be an incident vertex-edge-face triple of $\mathcal{M}$, let B and C be the midpoints of the edge e, and the face f, respectively, and let Δ be a triangle contained in f whose vertices are v, B and C. Then a, b and c may be thought of as the reflections about the sides vC, vB, and BC of the triangle Δ, respectively. In this geometrical setting, the automorphism $\sigma = ab$ corresponds to a one-step rotation around v, and $\rho = ac$ corresponds to a one-step rotation of f around its centre. Hence, the orders of σ and ρ are precisely the valency m and the face-length n of the map $\mathcal{M}$, respectively. On the other hand, the automorphism bc has the effect of switching the two end-points of e and the two faces incident to e. In particular, $(bc)^2 = 1$, and so b and c commute. Also, the vertex-stabiliser G_v, the edge-stabiliser G_e, and the face-stabiliser G_f are the groups $G_v = \langle \sigma, a\rangle = \langle a, b\rangle \cong D_m$, $G_e = \langle b, c\rangle \cong C_2 \times C_2$, and $G_f = \langle \rho, a\rangle = \langle a, c\rangle \cong D_n$ (where we use the symbol D_t to denote the dihedral group of order $2t$). The above shows that the automorphism group G of a map with type $\{m, n\}$ has a presentation of the form

$$G = \langle\, a, b, c \mid a^2 = b^2 = c^2 = (ab)^m = (ac)^n = (bc)^2 = \ldots = 1 \,\rangle \tag{3}$$

where the above relations between the generators are fulfilled (in the sense that they indicate the true orders of the elements), rather than just satisfied. (Note that $a \notin \langle b, c\rangle$ unless $G \cong C_2 \times C_2$ and the map has just one edge.) Equivalently, there exists an epimorphism from the *extended triangle group* $\Delta^*(2, m, n) = \langle\, x, y, z \mid x^2 = y^2 = z^2 = (xy)^2 = (yz)^m = (zx)^n = 1 \,\rangle$ onto the group G, the kernel of which intersects each of the groups $\langle x, y\rangle$, $\langle y, z\rangle$, $\langle z, x\rangle$ trivially. Such an epimorphism will be called *smooth*, and the corresponding image G, called a *smooth quotient* of $\Delta^*(2, m, n)$.

Conversely, suppose that G is an abstract finite group with a presentation of the form (3). Then there exists a regular map $\mathcal{M} = \mathcal{M}(G, a, b, c)$ such that $G = \mathrm{Aut}(\mathcal{M})$ and a, b, c have the same geometrical interpretation as described above. In this correspondence, two maps $\mathcal{M}_i = \mathcal{M}(G_i, a_i, b_i, c_i)$, for $i = 1, 2$, are isomorphic if and only if there exists a group isomorphism $f\colon G_1 \to G_2$ mapping the ordered triple (a_1, b_1, c_1) to the ordered triple (a_2, b_2, c_2). This allows one to translate the entire theory of regular maps into a purely group-theoretical language, where the notion of a regular map is substituted by the concept of a *regular map group*, defined as follows:

Definition 2.1 A *regular map group* is a quadruple (G, a, b, c) where G is an abstract finite group generated by involutions a, b, c such that $bc = cb$ and $b \neq c$. Two regular map groups (G_i, a_i, b_i, c_i), for $i = 1, 2$, are *isomorphic* if there exists a group isomorphism between G_1 and G_2 mapping the ordered triple (a_1, b_1, c_1) to the ordered triple (a_2, b_2, c_2).

We define the *valency, face-length* and the *characteristic* of a regular map group $\mathcal{G} = (G, a, b, c)$ to be that of the corresponding map $\mathcal{M}(\mathcal{G})$. That is, the valency m and the face-length n of $\mathcal{G}$ are simply the orders of the elements ab and ac, respectively, while the characteristic χ can be computed using formula (1).

Clearly, if (G, a, b, c) is a regular map group, then G is generated by the involution a and the group $H = \langle b, c\rangle \cong C_2 \times C_2$. Conversely, if a finite group G is generated by an involution a and a group $H \cong C_2 \times C_2$, then for every ordered pair (b, c) of distinct non-trivial elements of H, the quadruple (G, a, b, c) is a regular map group. Since there are six choices for the ordered pair (b, c) in H, each triple (G, H, a) gives rise to six regular map groups (G, a, b, c) such that $H = \langle b, c\rangle$; note that some of them might be isomorphic. These six groups give rise to regular maps with the same *medial graph* — the graph whose vertices are edges of the map, with two such edges adjacent whenever they are consecutive on a boundary of a face — and so we say that they are *medially equivalent*. More precisely:

Definition 2.2 Two regular map groups $\mathcal{G}_1$ and $\mathcal{G}_2$ are *medially equivalent* whenever there exist regular map groups $\mathcal{G}_1' = (G, a, b_1, c_1)$ and $\mathcal{G}_2' = (G, a, b_2, c_2)$ such that $\langle b_1, c_1\rangle = \langle b_2, c_2\rangle$ and $\mathcal{G}_i \cong \mathcal{G}_i'$ for $i = 1, 2$.

We remark that this notion of medial equivalence corresponds to the notion of outer automorphism, as defined in [13].

Suppose now that we want to determine all regular maps $\mathcal{M}$ (up to isomorphism) for which the automorphism group $\mathrm{Aut}(\mathcal{M})$ has a given group-theoretical property $\mathcal{P}$. It follows from the discussion above that it suffices to find one regular map group $\mathcal{G}$ from each class of medially equivalent groups, and then consider the six groups which are medially equivalent to $\mathcal{G}$. If all the (abstract) groups G having the property $\mathcal{P}$ are known, then this can be achieved simply by determining all possible (essentially different) ways in which each of these groups G can be generated by a subgroup $H \cong C_2 \times C_2$ and an involution a; and then, for each such triple (G, H, a), consider the six medially equivalent regular map groups associated with (G, H, a).

3 Regular maps of odd Euler characteristic, and amost Sylow-cyclic groups

We begin this section with observations that will be critical to what follows. For a prime p and positive integers s and n, we write $p^s \,||\, n$ whenever p^s divides n but p^{s+1} does not.

Lemma 3.1 *Let $\mathcal{M}$ be a regular map of type $\{m, n\}$ and Euler characteristic χ. Then $|\mathrm{Aut}(\mathcal{M})|$ is divisible by 4 and $2\,\mathrm{lcm}(m, n)$, and*

$$\frac{|\mathrm{Aut}(\mathcal{M})|}{2\,\mathrm{lcm}(m,n)} = \frac{-2\gcd(m,n)\chi}{mn - 2m - 2n} = \frac{-2\chi}{\mathrm{lcm}(m,n) - 2\frac{m+n}{\gcd(m,n)}}.$$

Proof. Suppose $\mathcal{M} = \mathcal{M}(G, a, b, c)$. Then $|G|$ is divisible by the orders of the subgroups $\langle b, c\rangle$, $\langle a, b\rangle$, and $\langle a, c\rangle$, which are 4, $2m$, and $2n$, respectively. Hence, the order of G is divisible by 4, as well as by $\mathrm{lcm}(2m, 2n) = 2\,\mathrm{lcm}(m, n)$. The rest then follows directly from (2). ■

Lemma 3.2 *Let $\mathcal{M}$ be a regular map of type $\{m, n\}$ and let $G = \mathrm{Aut}(\mathcal{M})$. If p is a prime divisor of $|G|$ coprime to χ, then the Sylow p-subgroup of G is cyclic (if p is odd) or dihedral (if $p = 2$).*

PROOF. Let p be a prime divisor of $|G|$ coprime to χ, let s be such that $p^s \,||\, |G|$, and let P be a Sylow p-subgroup of G. Suppose first that p is coprime to $\frac{|G|}{2\,\mathrm{lcm}(m,n)}$. Then $p^s \,||\, 2\,\mathrm{lcm}(m, n)$, and so $p^s \,||\, 2m$ or $p^s \,||\, 2n$. Since $2m$ and $2n$ are the orders of a vertex-stabilser G_v and a face-stabiliser G_f in G, respectively, this shows that P is conjugate in G to a Sylow p-subgroup of G_v or G_f. Since G_v and G_f are dihedral groups, however, this implies that P is cyclic (if p is odd) or P is dihedral (if $p = 2$).

Hence we may assume that p divides $\frac{|G|}{2\,\mathrm{lcm}(m,n)}$. Now by Lemma 3.1, p divides 2χ, and since p is coprime to χ, it follows that $p = 2$ and χ is odd. Moreover, since $p = 2$ divides the fraction on the right-hand side of the formula in Lemma 3.1, we see that $\mathrm{lcm}(m, n) - 2\frac{m+n}{\gcd(m,n)}$ divides χ, which implies that $\mathrm{lcm}(m, n)$ is odd, and so both m and n are odd. But then, by (2), $4 \,||\, |G|$, and so $P \cong C_2 \times C_2 \cong D_2$ in that case. ■

An immediate but interesting consequence of this result is the following:

Corollary 3.3 *Let $\mathcal{M}$ be a regular map of type $\{m, n\}$ and Euler characteristic χ. If $-\chi$ is a product of distinct odd primes coprime to both m and n, then $\mathrm{Aut}(\mathcal{M})$ is almost Sylow-cyclic.*

PROOF. By Lemma 3.2, the Sylow 2-subgroup of G is dihedral. Now, let p be an odd prime divisor of G and let P be a Sylow p-subgroup of G. We need to show that P is cyclic. If $p \,||\, |G|$, then the order of P is p, and P is cyclic. Hence we may assume that $p^2 \mid |G|$. If p is coprime to χ, then P is cyclic by Lemma 3.2. Suppose now that $p \mid \chi$. Then by (2), we have $\frac{|G|}{p} = \frac{4mn\chi}{(mn-2m-2n)p}$. Since χ is square-free and since p divides $|G|/p$, this implies that p divides mn. On the other hand, by assumption, p is coprime to both m and n. This contradiction shows that P is cyclic. ■

A deeper analysis, based on Lemma 3.1, leads to the following result, which justifies the study of regular maps with almost Sylow-cyclic automorphism groups.

Theorem 3.4 *Let $\mathcal{M}$ be a regular map of type $\{m, n\}$ and Euler characteristic $-p^2$ where p is an odd prime other than 3. Then $\mathrm{Aut}(\mathcal{M})$ is almost Sylow-cyclic.*

PROOF. Let $G = \mathrm{Aut}(\mathcal{M})$. In view of Lemma 3.2, G will be a almost Sylow-cyclic whenever p is coprime to $|G|$. Also, it suffices to show that the Sylow p-subgroup P of G is cyclic.

Assume from now on that P is not cyclic. Then p^2 divides $|G|$, and so $\frac{|G|}{4p^2}$ is an integer. By (2), we find that

$$k(m, n) = \frac{mn}{mn - 2m - 2n}$$

is a positive integer (not greater than 21, by Singerman's analogue of Hurwitz's bound). It is well known that there are only finitely many pairs $\{m, n\}$ for which $k(m, n)$ is a positive integer. In fact, it is easy to see that these are precisely the pairs given in the table below:

$\{m,n\}$	$k(m,n)$	$\|G\|$
$\{3,7\}$	21	$84p^2$
$\{3,8\}$	12	$48p^2$
$\{3,9\}$	9	$36p^2$
$\{3,12\}$	6	$24p^2$
$\{3,15\}$	5	$20p^2$

$\{m,n\}$	$k(m,n)$	$\|G\|$
$\{3,24\}$	4	$16p^2$
$\{4,5\}$	10	$40p^2$
$\{4,6\}$	6	$24p^2$
$\{4,8\}$	4	$16p^2$
$\{4,12\}$	3	$12p^2$

$\{m,n\}$	$k(m,n)$	$\|G\|$
$\{5,5\}$	5	$20p^2$
$\{5,20\}$	2	$8p^2$
$\{6,6\}$	3	$12p^2$
$\{6,12\}$	2	$8p^2$
$\{8,8\}$	2	$8p^2$

Suppose first that $p = 5$. Then $\mathcal{M}$ is a regular map on the non-orientable surface of Euler characteristic -25, and hence of genus 27. It is known (see [4]), however, that this surface supports no regular maps. Hence, we may assume henceforth that $p \geq 7$.

Suppose now that p divides m or n. Then $p = 7$, and $\{m,n\} = \{3,7\}$, and so $|G| = 12 \cdot 7^3$. Hence, in view of Sylow's theorem, the Sylow 7-subgroup of G is normal in G, implying that the extended triangle group $\Delta^*(2,3,7)$ has a quotient of order 12. But every quotient of $\Delta^*(2,3,7)$ other than the trivial group and C_2 must have order at least 168. This contradiction implies that p is coprime to both m and n. In particular, $|P| = p^2$, and so $P \cong C_p \times C_p$.

If P is normal in G, let a, b, c be the generating involutions of G such that (G, a, b, c) is a regular map group giving rise to the regular map $\mathcal{M}$. Consider the quotient group $\bar{G} = G/P$. Since the order of P is coprime to the subgroup orders $|\langle b, c\rangle| = 4$, $|\langle a, b\rangle| = 2m$, and $|\langle a, c\rangle| = 2n$, the images $\bar{a}$, $\bar{b}$, $\bar{c}$, of a, b, c in $\bar{G}$ give rise to the regular map group $(\bar{G}, \bar{a}, \bar{b}, \bar{c})$ of the same type as $\mathcal{M}$, and thus of Euler characteristic -1 and genus 3. It is well known, however, that there is no regular map on the surface of characteristic -1 (see also [4]). Hence P is not normal in G. Using Sylow's theorem, a straightforward calculation shows that the type $\{m,n\}$ and prime p must be as in one of the lines in the table below:

$\{m,n\}$	p	$\|G\|$
$\{3,7\}$	$11, 13, 41, 83$	$84p^2$
$\{3,8\}$	$7, 11, 23, 47$	$48p^2$
$\{3,9\}$	$7, 11, 17$	$36p^2$
$\{3,12\}$	$7, 11, 23$	$24p^2$
$\{3,15\}$	19	$20p^2$

$\{m,n\}$	p	$\|G\|$
$\{3,24\}$	7	$16p^2$
$\{4,5\}$	$7, 13, 19$	$40p^2$
$\{4,6\}$	$7, 11, 23$	$24p^2$
$\{4,8\}$	7	$16p^2$
$\{4,12\}$	11	$12p^2$

$\{m,n\}$	p	$\|G\|$
$\{5,5\}$	19	$20p^2$
$\{5,20\}$	7	$8p^2$
$\{6,6\}$	11	$12p^2$
$\{6,12\}$	7	$8p^2$
$\{8,8\}$	7	$8p^2$

Suppose now that (m, n, p) is one of the admissible triples from this table. Consider the transitive action of the group G on the cosets $G/P = \{Pg : g \in G\}$ by right multiplication, and let C be the kernel of this action, namely the core of P in G. In particular, since P is not normal in G, we know that C is a proper subgroup of P, and therefore has order 1 or p. Since the index of P in G is $\lambda(m,n)$, the degree of the induced permutation group G/C is $R = \lambda(m,n) = 4k(m,n)$. Hence the order of G/C is either Rp (if $|C| = p$), or Rp^2 (if $|C| = 1$).

Further, the pre-image $\tilde{P}$ of P in the extended triangle group $\Delta = \Delta^*(2,m,n)$ has index R in Δ, the pre-image $\tilde{C}$ of C in Δ is the core $\tilde{C}$ of $\tilde{P}$ in Δ, with $\tilde{P}/\tilde{C} \cong P/C$, and $\Delta/\tilde{C}$ (viewed as a permutation group on $\Delta/\tilde{P}$) is permutation isomorphic to the group G/C (acting on G/P). Since G is a smooth quotient of Δ and since p is coprime to each of $2, m$ and n, also the group $\Delta/\tilde{C}$ is a smooth quotient of Δ.

We have thus shown that the extended triangle group $\Delta = \Delta^*(2, m, n)$ has a subgroup $\tilde{P}$ of index $\lambda(m, n)$ whose core $\tilde{C}$ intersects trivially each of the subgroups $\langle x, y\rangle, \langle y, z\rangle, \langle z, x\rangle$ of Δ, and has index p or p^2 in $\tilde{P}$. Since the index $\lambda(m, n)$ is bounded above by 84, all such subgroups $\tilde{P}$, along with the corresponding permutation groups $\Delta/\tilde{C}$, can be easily found using the Low Index Subgroups Algorithm, implemented in, say, Magma [2]. It transpires that there are (up to isomorphism) only three such possiblities for the triples $(\Delta, \tilde{P}, \Delta/\tilde{C})$; see below. In all three cases we have $[\tilde{P} : \tilde{C}] = p$, and so $[P : C] = p$. Since P is a split extension of C, a theorem of Gaschütz (see [10] or [16]) implies that G splits over C.

In the first case, we have $\{m, n\} = \{3, 7\}$, $p = 13$, $[\tilde{P} : \tilde{C}] = 13$ and $\Delta/\tilde{C} \cong \mathrm{PSL}(2, 13)$. Hence G is a split extension of $C = C_{13}$ by $\mathrm{PSL}(2, 13)$. Since $\mathrm{PSL}(2, 13)$ is simple, C is central in G, and so $G \cong C_{13} \times \mathrm{PSL}(2, 13)$. But this contradicts the fact that G is generated by involutions.

In the other two cases, we have $\{m, n\} = \{3, 8\}$, $p = 7$, $[\tilde{P} : \tilde{C}] = 7$, and $\Delta/\tilde{C} \cong \mathrm{PGL}(2, 7)$. Hence G is a split extension of C_7 by $\mathrm{PGL}(2, 7)$. Since C_7 is not generated by involutions, G is not a direct product of these two groups. Hence $G \cong C_7 \rtimes_\varphi \mathrm{PGL}(2, 7)$ where the kernel of φ is $\mathrm{PSL}(2, 7)$ and $\varphi(x)$ is the automorphism of C_7 of order 2, for each $x \in \mathrm{PGL}(2, 7) \setminus \mathrm{PSL}(2, 7)$. It remains to see that such a group G is not a smooth quotient of an extended triangle group (that is, it is not generated by three involution, two of which commute). This can be shown in a way similar to the proof of Lemma 5.4, or checked easily with the help of Magma [2]. ■

4 Classification of regular maps with solvable almost Sylow-cyclic automorphism groups

Solvable almost Sylow-cyclic groups were completely classified by Zassenhaus [23]. It transpires that it is more convenient for our purposes to use the following formulation of Zassenhaus' result, proved in [21]. Note that we added an extra condition on m being odd in row 4 of Table 1, which follows from the condition on the Sylow 2-subgroup.

Theorem 4.1 ([21, Theorem 6.1.10]) *A finite group G is a solvable almost Sylow-cyclic group if and only if G is isomorphic to one of the groups listed in Table* 1.

Table 1: Solvable almost Sylow-cyclic groups

Class	Generators	Relations	Conditions	Order
1	A, B	$A^m = B^n = 1$, $BAB^{-1} = A^r$	$m, n, r \geq 1$, $\gcd(m, n) = 1$, $r = 1$ or $\gcd(r-1, m) = 1$, $r^n \equiv 1 \pmod{m}$	mn
2	A, B, R	as in (1); also $R^2 = 1$, $RAR^{-1} = A^l$, $RBR^{-1} = B^k$	as in (1); also n even (and hence m odd), $k^2 \equiv 1 \pmod{n}$, $r^{k-1} \equiv l^2 \equiv 1 \pmod{m}$	$2mn$
3	A, B, R	as in (1); also $R^2 = B^{\frac{n}{2}}$, $RAR^{-1} = A^l$, $RBR^{-1} = B^k$	as in (2); also $n \equiv 0 \pmod{4}$	$2mn$
4	A, B, P, Q	as in (1); also $P^2 = Q^2 = 1$, $[P, Q] = 1$, $[A, P] = [A, Q] = 1$, $BPB^{-1} = Q$, $BQB^{-1} = PQ$	as in (1); also m, n odd, $n \equiv 0 \pmod{3}$	$4mn$
5	A, B, P, Q	as in (1); also $P^4 = 1, P^2 = Q^2$, $PQP^{-1} = Q^{-1}$, $[A, P] = [A, Q] = 1$, $BPB^{-1} = Q$, $BQB^{-1} = PQ$	as in (4)	$8mn$
6	A, B, P, Q, R	as in (4) and (2); also $[R, P] = 1$, $RQR^{-1} = PQ$	as in (4); also $k^2 \equiv 1 \pmod{n}$, $k \equiv -1 \pmod{3}$ $r^{k-1} \equiv l^2 \equiv 1 \pmod{m}$	$8mn$
7	A, B, P, Q, R	as in (5) and (2); also $RQR^{-1} = Q^{-1}$, $RPR^{-1} = QP$	as in (6)	$16mn$
8	A, B, P, Q, R	as in (5); also $R^2 = P^2$, $RQR^{-1} = Q^{-1}$, $RPR^{-1} = QP$, $RAR^{-1} = A^l$, $RBR^{-1} = B^k$	as in (6)	$16mn$

Theorem 4.1 immediately implies the only finite groups with the property that all their Sylow subgroups are cyclic are those in the first row of Table 1. Since this result was already known to Burnside, we state it separately:

Lemma 4.2 *The Sylow subgroups of a finite group G are all cyclic if and only if G is a semidirect product of two cyclic groups of coprime orders.*

We shall now determine which of the groups in Table 1 give rise to regular map groups. To this end, we first prove a few auxiliary results. The first lemma is a useful observation, which will be used in the sequel several times. The proof is straightforward and omitted.

Lemma 4.3 *Let m, n, and r be positive integers such that $r^2 \equiv 1 \pmod m$, and let $G = \langle\, x, y \mid x^m, y^n, yxy^{-1} = x^r \,\rangle$. Then an element $g \in \langle x, y\rangle \setminus \langle y\rangle$ is an involution if and only if m is even and $g = x^{\frac{m}{2}}$, or n is even and $g = x^i y^{\frac{n}{2}}$ where either* (i) *$n \equiv 0 \pmod 4$, m is even, and $i = \frac{m}{2}$, or* (ii) *$n \equiv 2 \pmod 4$ and $i(1 + r) \equiv 0 \pmod m$.* ■

The following lemma deals with the groups in Class 1 of Table 1. Note that these groups are precisely semidirect products $C_m \rtimes C_n$ of two cyclic group of coprime orders m and n.

Lemma 4.4 *Let $G \cong C_m \rtimes C_n$ for some positive integers $m \geq 1$ and $n \geq 2$. Then G is generated by involutions if and only if $n = 2$ and G is isomorphic to the dihedral group D_m.*

Proof. If $G \cong D_m$, then G is generated by two involutions, proving one direction of the lemma. Assume now that G is generated by involutions. We need to prove that $n = 2$ and $G \cong D_m$. Observe first that since G is generated by involutions, so is also every quotient group of G. By assumption, $G = \langle\, x, y \mid x^m, y^n, yxy^{-1} = x^r \,\rangle$ for some $r \in \{1, \ldots, m-1\}$, $r^n \equiv 1 \pmod m$. Since $\langle x\rangle$ is normal in G and since $G/\langle x\rangle \cong C_n$, it follows that $n = 2$, and so $r^2 \equiv 1 \pmod m$. If $r = 1$, then $G \cong C_m \times C_2$, and by the same argument, $m = 2$, implying that $G \cong C_2 \times C_2 \cong D_2$. Now suppose that $r \neq 1$, and consider the quotient projection $\varphi\colon G \to G/\langle x^{r-1}\rangle$. Clearly $G/\langle x^{r-1}\rangle = \langle \varphi(x), \varphi(y)\rangle$. Since $yxy^{-1} = x^r$, we find that $[\varphi(x), \varphi(y)] = \varphi(yxy^{-1}x^{-1}) = \varphi(x^{r-1}) = 1$ and hence $G/\langle x^{r-1}\rangle$ is abelian. Since it is generated by involutions, $G/\langle x^{r-1}\rangle$ is elementary abelian (of order 2 or 4). In particular, $\varphi(x)^2 = 1$, and so $x^2 \in \langle x^{r-1}\rangle$. But then $2 = \alpha(r-1) + \beta m$ for some integers α and β, showing that $\gcd(m, r-1) \in \{1, 2\}$. Since m divides $(r-1)(r+1)$, it follows that m divides $2(r+1)$. On the other hand, $r \leq m-1$, hence either $r = m-1$, or m is even and $r = \frac{m}{2} - 1$. In the former case, $G \cong D_m$, as claimed. We may therefore assume that $r = \frac{m}{2} - 1$. What remains to show is that this contradicts our assumption that G is generated by involutions.

Observe first that in this case, since $r^2 \equiv 1 \pmod m$, m is divisible by 4, and so r is odd. We shall now determine the set of all involutions in G. Let $z \in G$ be an involution. If $z \in \langle x\rangle$, then $z = x^{\frac{m}{2}} \in \langle x^2\rangle$. On the other hand, if $z \notin \langle x\rangle$, then $z = x^i y$ for some $i \in \{0, \ldots, m-1\}$ such that $(x^i y)^2 = x^{i\frac{m}{2}} = 1$. Therefore, $z = x^{2j} y$ for some integer j. We have thus shown that every involution of G is contained in $\langle x^2, y\rangle$, which is a proper subgroup of G. This, however, contradicts the assumption that G is generated by involutions, and completes the proof. ■

The next lemma deals with the groups in Class 2 of Table 1. Note that these groups are isomorphic to a semidirect product of three cyclic groups: $C_m \rtimes (C_n \rtimes C_2)$. The result below is slightly more general than needed for the purpose of this paper.

Lemma 4.5 *Let m, n, t be positive integers such that $n, t \geq 2$, and let $G \cong C_m \rtimes (C_n \rtimes C_t)$. If G is generated by involutions, then $t = 2$ and one of the following occurs:*

(i) $m = 2$ *and* $G \cong \langle\, A, B, R \mid A^2, B^n, R^2, [A, B], [A, R], (BR)^2 \,\rangle \cong C_2 \times D_n$*;*

(ii) $G \cong \langle\, A, B, R \mid A^m, B^n, R^2, [A, B], (AR)^2, (BR)^2 \,\rangle \cong (C_m \times C_n) \rtimes C_2$*;*

(iii) *n is even, $m \geq 3$, and for some integers r and ℓ such that $r^2 \equiv \ell^2 \equiv 1 \pmod{m}$, we have* $G \cong \langle\, A, B, R \mid A^m, B^n, R^2, BAB^{-1}A^{-r}, RAR^{-1}A^{-\ell}, (BR)^2 \,\rangle \cong C_m \rtimes D_n$*.*

If, in addition, G is almost Sylow-cyclic, then $\gcd(m, n) = 1$*, and either $G \cong D_{mn}$ or G is as in* (iii) *above.*

Proof. Suppose that G is generated by involutions. Since $G \cong C_m \rtimes (C_n \rtimes C_t)$, there exist integers r, k, ℓ such that

$$G = \langle\, A, B, R \mid A^m, B^n, R^t, BAB^{-1}A^{-r}, RAR^{-1}A^{-\ell}, RBR^{-1}B^{-k} \,\rangle$$

where $r^n \equiv \ell^t \equiv 1 \pmod{m}$ and $k^t \equiv 1 \pmod{n}$. Since $G/\langle A\rangle \cong C_n \rtimes C_t$, it follows from Lemma 4.4 that $t = 2$ and $RBR^{-1} = B^{-1}$, so that $k \equiv -1 \pmod{m}$. Also, since $R^2 = 1$, it follows that $\ell^2 \equiv 1 \pmod{m}$.

If $m = 1$, then $G \cong \langle B, R\rangle \cong D_n$, and so (ii) holds. If $m = 2$, then A lies in the centre of G, and so (i) holds. We may therefore assume for the rest of the proof that $m \geq 3$. To finish the first part of the proof, it remains to show that $r^2 \equiv 1 \pmod{m}$ in this case.

Since $\langle A\rangle$ is normal in G, the group $\langle B, R\rangle$ acts on $\langle A\rangle$ by conjugation. Let K be the kernel of this action (that is, the centraliser of A in $\langle B, R\rangle$). Then $Q = \langle B, R\rangle/K$ is isomorphic to a subgroup of the automorphism group of C_m, and so Q is abelian. On the other hand, Q is is a quotient of the dihedral group $\langle B, R\rangle$, so is generated by one or two involutions, implying that Q is isomorphic to C_2 or $C_2 \times C_2$. Furthermore, K is normal in G and $G/K \cong C_m \rtimes Q$. In view of Lemma 4.4, it follows that that either $Q \cong C_2 \times C_2$, or $Q \cong C_2$. (Note that $Q = 1$ would imply $m = 2$.) If $Q \cong C_2$, then K has index 2 in $\langle B, R\rangle \cong D_n$, and so K is one of the groups $\langle B\rangle$, $\langle B^2, R\rangle$, or $\langle B^2, BR\rangle$, with the latter two possibilities occuring only if n is even. If $Q \cong C_2 \times C_2$, then $K = \langle B^2\rangle$. In all these cases we find that $B^2 \in K$, and so $r^2 \equiv 1 \pmod{m}$, as required.

Suppose now, in addition, that G is a almost Sylow-cyclic.

If (i) holds, that is, if $m = 2$ and G is isomorphic to $C_2 \times D_n$, then every Sylow 2-subgroup P of G is isomorphic to $C_2 \times D_{2^j}$ where 2^j is the highest power of 2 dividing n. But since P has a cyclic subgroup of index 2, we see that $j = 0$, and therefore n is odd. In particular, $G \cong C_2 \times D_n \cong D_{mn}$ and $\gcd(m, n) = 1$.

If (ii) holds, then consider a Sylow p-subgroup P of G for any odd prime divisor p of mn. If p divides both m and n, then P contains the subgroup $\langle A^{\frac{m}{p}}, B^{\frac{n}{p}}\rangle \cong C_p \times C_p$, contradicting the assumption that G is almost Sylow-cyclic. Hence the only common prime divisor of m and n could be 2. If, however, m and n are both even, then $\langle A^{\frac{m}{2}}, B^{\frac{n}{2}}, R\rangle$ is a subgroup of G isomorphic to $C_2 \times C_2 \times C_2$, again contradicting the assumption that a Sylow 2-subgroup of G contains a cyclic subgroup of index 2. Hence $\gcd(m, n) = 1$. But then $\langle AB\rangle$ generates a cyclic subgroup of index 2 in G, and since $R(AB)R^{-1} = (AB)^{-1}$, it follows that $G \cong D_{mn}$.

Assume now that (iii) occurs. If m is even, then one Sylow 2-subgroup of G contains an elementary abelian subgroup $\langle A^{\frac{m}{2}}, B^{\frac{n}{2}}, R\rangle$, contradicting the assumption that G is almost

Sylow-cyclic. Hence m is odd. If there exists an odd prime p dividing both m and n, then the Sylow p-subgroup of $\langle A, B^2 \rangle \cong C_m \times C_{\frac{n}{2}}$ could not be cyclic, a contradiction. Therefore $\gcd(m, n) = 1$. ■

The following lemma will be frequently used later.

Lemma 4.6 *Let (G, a, b, c) be a regular map group such that every Sylow 2-subgroup of G contains a cyclic subgroup of index 2. If the order of the centre of G is even, then $G \cong D_n$ for some even positive integer n.*

PROOF. Since G contains a subgroup $\langle b, c \rangle$ of order 4, it suffices to show that G is a dihedral group. Let Z be the centre of G, and suppose that $|Z|$ is even. Then Z contains an involution x. Since $\langle x \rangle$ is normal in G, it is contained in every Sylow 2-subgroup of G; in particular, $\langle x \rangle$ lies in a Sylow subgroup P containing $\langle b, c \rangle \cong C_2 \times C_2$. If $x \notin \langle b, c \rangle$, then P contains an elementary abelian subgroup $\langle b, c, x \rangle$ of order 8, contradicting the assumption that P contains a cyclic subgroup of index 2. Hence $x \in \langle b, c \rangle$. We may assume without loss of generality that $b \neq x$ (by swapping the roles of b and c if necessary). Then x is either c or bc. In any case, $G = \langle a, b, x \rangle$, and so, since x is central, $\langle x \rangle$ and $\langle a, b \rangle$ are normal subgroups of G. If $x \in \langle a, b \rangle$, then $G = \langle a, b, x \rangle = \langle a, b \rangle$ is generated by two involutions, and is therefore dihedral. On the other hand, if $x \notin \langle a, b \rangle$, then $G \cong \langle x \rangle \times \langle a, b \rangle \cong C_2 \times D_m$ where m is the order of ab. If m is even, then D_m contains a copy of $C_2 \times C_2$, which generates with x an elementary abelian group of order 8, contradicting our assumption. Hence m is odd. But then $C_2 \times D_m \cong D_{2m}$, and so $G \cong D_{2m}$. ■

We are now ready to prove the main result of this subsection: the classification of regular maps whose automorphism group is a solvable almost Sylow-cyclic group. For the sake of simplicity we shall leave out the three regular maps with just one edge: the loop on a sphere, K_2 on the sphere, and the loop on the projective plane.

Theorem 4.7 *Let $\mathcal{M}$ be a regular map with at least two edges. Then* $\mathrm{Aut}(\mathcal{M})$ *is a solvable almost Sylow-cyclic group if and only if it is isomorphic to one of the following groups:*

(i) $G_1(N) = \langle\, x, z \mid x^2, z^N, (zx)^2 \,\rangle \cong D_N$, *with N even and $N \geq 4$;*

(ii) $G_2(N, m) = \langle\, x, y, z \mid x^2, y^2, z^N, [x, y], (xz)^2, yzy^{-1} = z^s \,\rangle$, *with* N *odd,* $m \mid N$, $m \notin \{1, N\}$, $\gcd(m, \frac{N}{m}) = 1$, $s \equiv 1 \pmod{m}$ *and* $s \equiv -1 \pmod{\frac{N}{m}}$*;*

(iii) $G_3(N) = \langle\, x, y, z, t \mid x^2, y^2, z^N, t^2, [x, y], zxz^{-1} = y, zyz^{-1} = xy, [x, t], tyt^{-1} = xy, (tz)^2 \,\rangle$, *with N odd, and $N \equiv 0 \pmod{3}$;*

and $\mathcal{M}$ or the dual $\mathcal{M}^*$ is isomorphic to one of the maps $\mathcal{M}(G, a, b, c)$ given in Table 2. A map from Table 2 is self-dual if and only if it has the same number of vertices as faces.

Table 2: Regular maps $\mathcal{M} = \mathcal{M}(G, a, b, c)$ for which G is a solvable almost Sylow-cyclic group

	G (a,b,c)	(V,E,F)	Type	Characteristic / Orientable?	Conditions
1	$G_1(N) \cong D_N$ $(zx, z^{\frac{N}{2}}, x)$	$(\frac{N}{2}, \frac{N}{2}, 1)$	$\{2, N\}$	1 No	N even $N \geq 4$
2	$G_1(N) \cong D_N$ $(zx, z^{\frac{N}{2}}, z^{\frac{N}{2}}x)$	$(\frac{N}{2}, \frac{N}{2}, 2)$	$\{2, \frac{N}{2}\}$	2 Yes	As in (1)
3	$G_1(N) \cong D_N$ $(zx, x, z^{\frac{N}{2}}x)$	$(1, \frac{N}{2}, 2)$	$\{N, \frac{N}{2}\}$	$3 - \frac{N}{2}$ Yes	As in (1); also $N \equiv 2\ (4)$
4	$G_1(N) \cong D_N$ $(zx, x, z^{\frac{N}{2}}x)$	$(1, \frac{N}{2}, 1)$	$\{N, N\}$	$2 - \frac{N}{2}$ Yes	As in (1); also $N \equiv 0\ (4)$
5	$G_2(N, m)$ (zx, x, y)	$(2, N, m)$	$\{N, 2\frac{N}{m}\}$	$m + 2 - N$ Yes	N odd, $m \mid N$, $m \notin \{1, N\}$, $\gcd(m, \frac{N}{m}) = 1$
6	$G_2(N, m)$ (zx, y, xy)	$(m, N, \frac{N}{m})$	$\{2\frac{N}{m}, 2m\}$	$\frac{N}{m} + m - N$ No	As in (5); also $m^2 < N$
7	$G_3(N)$ (zt, xt, t)	$(4, 2N, 4)$	$\{N, N\}$	$8 - 2N$ Yes	N odd, $N \equiv 0\ (3)$
8	$G_3(N)$ (zt, x, t)	$(N, 2N, 4)$	$\{4, N\}$	$4 - N$ No	As in (7)

PROOF. Suppose first that (G, a, b, c) is one of the quadruples from the second coloumn of Table 2. Then a, b, c are involutions which generate G and satisfy $bc = cb$. Hence (G, a, b, c) is a regular map group. Observe that $G \cong D_N$ (in Rows (1)–(4)), or $G \cong C_N \rtimes (C_2 \times C_2)$ with N odd (in Rows (5) and (6)), or $G \cong (C_2 \times C_2) \rtimes D_N$ with N odd (in Rows (7) and (8)). Hence every odd order Sylow subgroup of G is cyclic in each of the above cases, and Sylow 2-subgroups are dihedral in Rows (1)-(4) and isomorphic to $C_2 \times C_2$ in Rows (5) and (6). Similarly, if G is as in Row (7) or (8), the Sylow 2-subgroup of G is $\langle x, y, t\rangle$. Since $x = (ty)^2$, the latter is generated by two involutions, y and t, and is therefore isomorphic to D_4. This shows that G is solvable and almost Sylow-cyclic, as claimed. The type of the map $\mathcal{M} = \mathcal{M}(G, a, b, c)$ is $\{k, l\}$ where $k = |ab|$ and $l = |ac|$ are the orders of ab and ac. The number of edges E is, by definition, $\frac{1}{4}|G|$. Hence $E = \frac{N}{2}, N$, or $2N$, depending on whether $G \cong G_1(N)$, $G_2(N, m)$, or $G_3(N)$, respectively. From here, the number of vertices V and faces F can be computed using formulas $V = 2E/|ab|$ and $F = 2E/|ac|$. The Euler characteristic χ is then determined by $\chi = V - E + F$. The map $\mathcal{M}$ is orientable if and only if the elements ab and ac generate a proper subgroup of G. This is the case in all rows, except in (1), (6), and (8). The map can only be self-dual when $V = F$. This only occurs in Row (2) if $N = 2$, and in Rows (4) and (7). The map from Row (2) with $N = 2$ is the embedding of the dipole into a sphere, and is clearly self-dual. We shall see later that the maps from Rows (4) and (7) are also self-dual. We invite the reader to check that all the parameters given in the table are correct.

Suppose now that $\mathcal{M}$ is a regular map such that $\mathrm{Aut}(\mathcal{M})$ is a solvable almost Sylow-cyclic group. Then $\mathcal{M}$ is isomorphic to a map $\mathcal{M}(\mathcal{G})$ for some regular map group $\mathcal{G} = (G, a, b, c)$ such that G is solvable and almost Sylow-cyclic. We may therefore assume that G belongs to one of the eight classes of groups presented in Theorem 4.1. Let us first show that groups from Classes 3, 4, 5, 7 and 8 are not generated by involutions (and hence G cannot belong to any of these classes). We shall do that by finding a normal subgroup with respect to which the quotient group is not generated by involutions.

A group from Class 3 contains a normal subgroup $\langle A, B^{\frac{n}{4}}\rangle$ with respect to which the quotient is isomorphic to the quaternion group Q_8 if $k \equiv -1 \pmod 4$, or to the group $C_4 \times C_2$ if $k \equiv 1 \pmod 4$. Neither of these quotients can be generated by involutions.

A group from Class 4 or 5 has a quotient (via the normal subgroup generated by P and Q) which is isomorphic to the semidirect product $C_m \rtimes C_n$, where n is odd. By Lemma 4.4, such a group is never generated by involutions.

A group from Class 7 or 8 has a quotient Q of order 48 (via the group generated by A and $B^{\frac{n}{3}}$) which is isomorphic to the group from the same class with parameters $m = 1, n = 3, r = 1, k = -1, l = 1$. If (G, a, b, c) is a regular map group arising from such a group G, then the quotient Q is generated by the images of a, b, and c. In particular, Q is either generated by two involutions (and is therefore isomorphic to the dihedral group), or generated by three involutions, two of which commute. Neither possibility holds for this group Q, which shows that G cannot be a group from Class 7 or 8.

Assume now that G belongs to Class 1 of Table 1, that is, $G \cong C_m \rtimes C_n$, with $\gcd(m, n) = 1$. By Lemma 4.4, $n = 2$ and $G \cong D_m$. But m is coprime to n and thus odd, implying that G does not contain a pair of commuting involutions, which is a contradiction.

We may therefore assume that G belongs to Class 2 or 6 of Table 1. The rest of the proof is divided into three cases: (1) G is dihedral, (2) G is not dihedral and belongs to Class 2, and (3) G is not dihedral and belongs to Class 6. In each of these cases, we first find all pairs a, H such that $G = \langle H, a\rangle$, $a^2 = 1$ and $H \cong C_2 \times C_2$. In fact, we find one such pair from each equivalence class with respect to the relation: $(H_1, a_1) \cong (H_2, a_2)$ whenever there exists an automorphism of G mapping a_1 to a_2 and H_1 to H_2. (If $(H_1, a_1) \cong (H_2, a_2)$, then we say that the pairs (H_1, a_1), (H_2, a_2) are *congruent*.) For the purpose of this proof we shall call such a pair (H, a) a *medial* (generating G), since each medial (H, a) determines six medially equivalent regular map groups (G, a, b, c), one for each choice of generating ordered pair $(b, c) \in H$. Note that two such choices (b_1, c_1) and (b_2, c_2) will give rise to isomorphic regular map groups (G, a, b_1, c_1), (G, a, b_2, c_2) if and only if there is an automorphism of G which fixes a, and H setwise, while mapping (a_1, b_1) to (a_2, b_2). In particular, the regular map group (G, a, b, c) is self-dual (in the sense that $\mathcal{M}(G, a, b, c)$ is self-dual) if and only if an automorphism of G fixes a while swapping b with c.

CASE 1: Let $G \cong \langle\, x, z \mid x^2, z^N, (zx)^2 \,\rangle \cong D_N$ for some positive integer N. We shall first show that $N \geq 4$ and is even, and that each medial generating G is congruent to the medial $(\langle z^{\frac{N}{2}}, x\rangle, zx)$. Suppose that (H, a) is a medial generating G. Since the order of G is divisible by $|H| = 4$, we see that N is even. Also since the map has at least two edges, $|G| > 4$, and so $N \geq 4$. Further, since every copy of $C_2 \times C_2$ in D_N contains the central involution $z^{\frac{N}{2}}$, we find that H is generated by $z^{\frac{N}{2}}$ and an element in the coset $\langle z\rangle x$. But since the automorphism group of D_N acts transitively on the elements of the coset $\langle z\rangle x$, we may assume that $H = \langle z^{\frac{N}{2}}, x\rangle$. Moreover, since $G = \langle H, a\rangle$, the involution a is of the form $z^i x$ for some integer i such that $\gcd(i, \frac{N}{2}) = 1$, and so $\gcd(i, N) \in \{1, 2\}$.

What remains to show is that some automorphism of G preserves H and maps $z^i x$ to zx. To this end, note that the automorphism group of G consists of automorphisms $\Theta_{\alpha,\beta}$, for $\alpha \in \mathbb{Z}_N^*$, and $\beta \in \mathbb{Z}_N$, where $\Theta_{\alpha,\beta}$ maps z^i to $z^{\alpha i}$ and $z^i x$ to $z^{\alpha i+\beta}x$. If $\gcd(i,N)=1$, then $\Theta_{i^{-1},0}$ preserves H and maps $z^i x$ to zx, showing that (H,a) is congruent to the medial $(\langle z^{\frac{N}{2}}, x\rangle, zx)$, as claimed. We may thus assume that $\gcd(i,N)=2$. Now there exists an automorphism $\Theta_{\alpha,0}$ mapping $z^i x$ to $z^2 x$, while preserving H. If $\frac{N}{2}$ is even, then $\gcd(i,\frac{N}{2})=1$ implies that $\gcd(i,N)=1$, contradicting our assumption. On the other hand, if $\frac{N}{2}$ is odd, then $\gcd(N,\frac{N}{2}+2)=1$, and so there is an automorphism of G mapping $z^{i(\frac{N}{2}+2)}$ to z^i and $z^{i(\frac{N}{2}+2)+\frac{N}{2}}x$ to $z^i x$. Note that the latter automorphism preserves H and maps zx to z^2x. Composing the inverse of this automorphism by the one mapping $z^i x$ to $z^2 x$, we obtain an automorphism of G mapping the medial (H,a) to the medial $(\langle z^{\frac{N}{2}}, x\rangle, zx)$. This completes the proof that every medial generating the group G is congruent to the medial $(\langle z^{\frac{N}{2}}, x\rangle, zx)$.

Hence either (G,a,b,c) or its dual (G,a,c,b) is isomorphic to one of $(\langle z,x\rangle, zx, z^{\frac{N}{2}}, x)$ from Row 1, or $(\langle z,x\rangle, zx, z^{\frac{N}{2}}, z^{\frac{N}{2}}x)$ from Row 2, or $(\langle z,x\rangle, zx, x, z^{\frac{N}{2}}x)$ from Rows 3 and 4 (depending on whether $\frac{N}{2}$ is odd or even). Observe that none of these regular map groups can be self-dual, except the one from Row 4. In this case, the automorphism $\Theta_{\frac{N}{2}+1,\frac{N}{2}}$ fixes zx and swaps x with $z^{\frac{N}{2}}x$, thus giving rise to the isomorphism between the dual maps. This completes Case 1.

Case 2: Let G be a group from Class 2 of Table 1 other than a dihedral group. In this case, by Lemma 4.5, we find that

$$G = \langle\, A, B, R \mid A^k, B^n, R^2, BAB^{-1} = A^r, RAR^{-1} = A^{\ell}, (BR)^2 \,\rangle \tag{4}$$

for some positive integers k, n, r, and ℓ such that n is even, $k \geq 3$, $\gcd(k,n)=1$, and $r^2 \equiv \ell^2 \equiv 1 \pmod{k}$. Let $N = k\frac{n}{2}$. If $n \equiv 0 \pmod 4$, then G contains a central involution $B^{\frac{n}{2}}$, and by Lemma 4.6, G is a dihedral group. We may therefore assume that $n \equiv 2 \pmod 4$. In this case we see that $G \cong \langle A, B^2\rangle \rtimes \langle B^{\frac{n}{2}}, R\rangle$, and hence

$$G = \langle\, x, y, z \mid x^2, y^2, z^N, [x,y], xzx^{-1} = z^t, yzy^{-1} = z^s \,\rangle \tag{5}$$

for some integers t, s such that $t^2 \equiv s^2 \equiv 1 \pmod{N}$.

Let (H,a) be a medial generating G. Since $\langle z\rangle$ has odd order, it intersects H trivially, implying that H is a complement of $\langle z\rangle$ in G. Moreover, since the index of $\langle z\rangle$ in G is coprime to its order, any two complements of $\langle z\rangle$ are conjugate in G (as in [16, Theorem 7.42]). In particular, H is conjugate to $\langle x,y\rangle$, and since we are only concerned with the congruence class of (H,a), we may assume that $H = \langle x,y\rangle$.

Suppose now that $a = z^i x^j y^k$. Then $G = \langle H, a\rangle = \langle z^i, x, y\rangle$, and so $\gcd(i,N)=1$. Hence there exists an automorphism of $\langle z\rangle$ mapping z^i to z. Note that every automorphism of $\langle z\rangle$ extends to an automorphism of G acting on H as identity. In particular, there is an automorphism of G fixing H pointwise and mapping a to $zx^j y^k$. Hence we may assume that $a = zx^j y^k$ for some integers $j,k \in \{0,1\}$ not both equal to 0. By Lemma 4.3 it follows that $t^j r^k \equiv -1 \pmod{N}$. If $(j,k)=(1,0)$, then $a = zx$ and $t \equiv -1 \pmod{N}$. If $(j,k)=(0,1)$ (that is, if $a = zy$), then swap the roles of the generators x and y, so that again $a = zx$ and $t \equiv -1 \pmod{N}$. Finally, if $(j,k)=(1,1)$, replace the generator x by xy, and observe that that again results in having $a = zx$ and $t \equiv -1 \pmod{N}$. To summarise, we may assume without loss of generality that $t \equiv -1 \pmod{N}$ and $(H,a) = (\langle x,y\rangle, zx)$.

Now let $N = p_1^{\alpha_1} \cdots p_c^{\alpha_c}$ be a prime factorisation of N. Since $s^2 \equiv 1 \pmod{N}$, we see that $s \equiv \pm 1 \pmod{p_i^{\alpha_i}}$ for every $i \in \{1, \ldots, c\}$. Let m be the product of all $p_i^{\alpha_i}$ for which $s \equiv 1 \pmod{p_i^{\alpha_i}}$. Now $s \equiv 1 \pmod{m}$ and $s \equiv -11 \pmod{\frac{N}{m}}$. If $m = 1$ or $m = N$ then one of y and xy belongs to the centre of G, and so by Lemma 4.6, we see that $G \cong D_{2N}$, which is a contradiction. This shows that $G \cong G_2(N, m)$.

The medial $(H, a) = (\langle x, y\rangle, zx)$ now gives rise to three dual pairs of regular map groups: $(G_2(N, m), zx, x, y)$, $(G_2(N, m), zx, x, xy)$, $(G_2(N, m), zx, xy, y)$, and their respective duals. One group from the first dual pair is described in Row 5 of Table 2, and one from the third pair can be found in Row 6 of Table 2 (provided that $m^2 < N$). Now observe that $G_2(N, \frac{N}{m}) \cong \langle \bar{x}, \bar{y}, \bar{z} \mid \bar{x}^2, \bar{y}^2, \bar{z}^N, [\bar{x}, \bar{y}], (\bar{x}\bar{z})^2, \bar{y}\bar{z}\bar{y}^{-1} = \bar{z}^{-s}\rangle$. Hence there exists an isomorphism from $G_2(N, m)$ to $G_2(N, \frac{N}{m})$ which takes $z \mapsto \bar{z}$, $x \mapsto \bar{x}$, and $y \mapsto \bar{x}\bar{y}$. In view of this isomorphism, we see that a regular map group from the second pair is in fact isomorphic to a regular map group in Row 5 (or a dual thereof), and that a regular map group from the second pair is isomorphic to one from Row 6 (or a dual thereof). This completes Case 2.

Case 3: Let G be a group from Class 6 of Table 1 other than a dihedral group, so

$$\begin{aligned} G \;=\; & \langle\, P, Q, A, B, R \mid P^2, Q^2, A^m, B^n, R^2, [P, Q], [A, P], [A, Q], BPB^{-1} = Q, \\ & BQB^{-1} = PQ, [R, P], BAB^{-1} = A^r, RAR^{-1} = A^\ell, RBR^{-1} = B^k \,\rangle \end{aligned}$$

for some positive integers m, n and elements $r, \ell \in \mathbb{Z}_m^*$, $k \in \mathbb{Z}_n^*$ such that $n \equiv 1 \pmod{2}$, $n \equiv 0 \pmod{3}$, $\gcd(m, n) = 1$, $r = 1$ or $\gcd(r - 1, m) = 1$, $r^n \equiv r^{k-1} \equiv \ell^2 \equiv 1 \pmod{m}$, $k^2 \equiv 1 \pmod{n}$, $k \equiv 2 \pmod{3}$. Note that $G \cong (C_2 \times C_2) \rtimes (C_n \rtimes (C_m \rtimes C_2))$ in this case.

If m is even, then $A^{\frac{m}{2}}$ is a central involution in G, and by Lemma 4.6, G is dihedral. We may therefore assume that m is odd.

Since $\langle P, Q\rangle$ is normal in G and $G/\langle P, Q\rangle$ is isomorphic to $\langle A, B, R\rangle \cong C_m \rtimes (C_n \rtimes C_2)$, Lemma 4.5 implies that $\langle A, B, R\rangle \cong (C_m \times C_n) \rtimes C_2$, or more precisely, $r \equiv -\ell \equiv 1 \pmod{m}$ and $k \equiv -1 \pmod{n}$. If m and n have a common prime divisor p, then G contains a subgroup isomorphic to $C_p \times C_p$, which contradicts the fact that G is almost Sylow-cyclic. Hence $\gcd(m, n) = 1$, $\langle A, B, R\rangle \cong D_{mn}$, and $G \cong (C_2 \times C_2) \rtimes D_{mn}$. Note that the elements $x = P$, $y = Q$, $z = AB$, $t = R$ generate G, and satisfy

$$x^2 = y^2 = z^N = t^2 = [x, y] = 1, zxz^{-1} = y, zyz^{-1} = xy, [x, t] = 1, tyt^{-1} = xy, (tz)^2 = 1,$$

implying that $G \cong G_3(N)$ where $N = mn$ is odd and divisible by 3. Note that $\langle x, y, t\rangle \cong D_4$, which is a Sylow 2-subgroup of G.

Let (H, a) be a medial generating G. Then H is conjugate in G to a subgroup of $\langle x, y, t\rangle$. Moreover, since any copy of $C_2 \times C_2$ in $\langle x, y, t\rangle$ is conjugate in $\langle x, y, t\rangle$ either to $\langle x, y\rangle$ or $\langle x, t\rangle$, we may assume that $H = \langle x, y\rangle$ or $H = \langle x, t\rangle$. If $H = \langle x, y\rangle$, then H is normal in G, and so $G = \langle x, y, t\rangle \cong D_4$. We may therefore assume that $H = \langle x, t\rangle$.

We shall now show that there exists an automorphism of G preserving H and mapping a to zt. Observe that for every $s \in \mathbb{Z}_N^*$ such that $s \equiv 1 \pmod{3}$, the permutation fixing the elements x, y, t, and mapping z to z^s extends to an automorphism of G. Observe also that the composition of this automorphism with conjugation by t is an automorphism of G fixing x and t, swapping y and xy, and mapping z to z^{-s}. Hence, for every $s \in \mathbb{Z}_N^*$ (regardless of the congruence class of s modulo 3), there is an automorphism of G preserving H pointwise and mapping z^s to z. Suppose that $a = x^i y^j z^s t^k$ for some $i, j, k \in \mathbb{Z}_2$ and $s \in \mathbb{Z}_N$. Since $G = \langle a, t \rangle$, we have $a \notin \langle x, y \rangle$, for otherwise $G \leq \langle x, y, t \rangle \cong D_4$. Further, since a is an invloution, its image under the quotient projection $G \to G/\langle x, y \rangle$ is an involution, and so $k = 1$. Moreover, since the images of a and t generate the group $G \to G/\langle x, y \rangle \cong D_N$, we find that $s \in \mathbb{Z}_N^*$. But now there exists an automorphism of G mapping a into an element of the coset $\langle x, y \rangle zt$. Furthermore, since conjugation by x fixes x and t and maps z to xyz, there exists such an automorphism of G which maps a to zt or to xzt. But xzt is not an involution, and so we may assume without loss of generality that $a = zt$.

To finish the proof, observe that conjugation by xy fixes zt and x, and swaps the other two non-trivial elements of H. This reduces the six possibilities for the regular map groups arising from the medial (H, a) down to three, with two of them being dual to each other. These three families can be found in Rows 7 and 8. ∎

5 Classification of regular maps with non-solvable almost Sylow-cyclic automorphism groups

In this section we determine all regular maps $\mathcal{M}$ for which $\mathrm{Aut}(\mathcal{M})$ is a non-solvable almost Sylow-cyclic group. The classification of such groups G was initiated by Suzuki [18] and completed by Wong [22]. The following theorem can be deduced from their work, the result of Frobenius and Burnside on groups whose all Sylow subgroups are cyclic (see Lemma 4.2), and the fact that all non-solvable groups have even order.

Theorem 5.1 [Suzuki [18, Theorems A, B, C] and Wong [22, Theorem 2]] *Let G be a non-solvable almost Sylow-cyclic group. Then G contains a normal subgroup G_0 of index at most 2 such that*

$$G_0 \cong (C_m \rtimes C_n) \times L,$$

where $L \cong \mathrm{PSL}(2, p)$ or $L \cong \mathrm{SL}(2, p)$ for some prime $p \geq 5$, and m, n are odd integers coprime to the order of L. Moreover if the the Sylow 2-subgroup of G is dihedral, then $L \cong \mathrm{PSL}(2, p)$.

In addition, the following deep result of Gorenstein and Walter [11] will be used in our analysis.

Theorem 5.2 *Suppose G is a finite group whose Sylow 2-subgroups are dihedral, and let M be the largest normal subgroup odd order in G. Then G/M is isomorphic to a group Q for which one of the following holds:*

(1) *$Q \cong A_7$;*

(2) *$\mathrm{PSL}(2, q) \leq Q \leq \mathrm{P\Gamma L}(2, q)$ for some odd prime power q; or*

(3) *Q is isomorphic to a Sylow 2-subgroup of G.*

The above two results suggest that regular maps whose automorphism groups are isomorphic to $\mathrm{PSL}(2,p)$ or $\mathrm{PGL}(2,p)$ will play an important role in the classification we are aiming for. We thus continue with two basic constructions from [5].

For convenience we will write $n|(p\pm 1)$ whenever n divides one of $p+1$ or $p-1$. The set of non-zero squares in the prime field $\mathbb{F}_p$ will be denoted by $\mathrm{sq}(\mathbb{F}_p)$.

Construction 1. Let p be a prime, $p \geq 5$, and let a, b, c be elements of $\mathrm{PGL}(2,p)$ given by

$$a = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}, \quad c = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix},$$

and let $G = \langle a, b, c\rangle$. Then $\mathcal{M} = \mathcal{M}(G, a, b, c)$ is a regular map of type $\{p, p\}$, and $G \cong \mathrm{PSL}(2,p)$ or $\mathrm{PGL}(2,p)$, depending on whether $p \equiv 1$ or $3 \,(\mathrm{mod}\, 4)$, respectively. The map $\mathcal{M}$ is orientable if and only if $p \equiv 3 \,(\mathrm{mod}\, 4)$.

Construction 2. Let p be a prime, $p \geq 5$, and let $\bar{\mathbb{F}}_p$ be an algebraically closed field of characteristic p. Let m and n be integers such that $m|(p\pm 1)$ and either $n|(p\pm 1)$ or $n = p$. For $h \in \{m, n\}$ let ξ_h denote a primitive $(2h)$th root of unity in $\bar{\mathbb{F}}_p$ if $h \neq p$, and let $\xi_h = -1$ if $h = p$. Further, let $\omega_h = \xi_h + \xi_h^{-1}$, let $D = \omega_m^2 + \omega_n^2 - 4$, and let $\beta = -1/\sqrt{-D} \in \bar{\mathbb{F}}_p$ where $\sqrt{-D}$ is a square root of $-D$. For $D \neq 0$ let $a, b, c \in \mathrm{PSL}(2, \bar{F}_p)$ be given by

$$a = \beta \begin{bmatrix} 0 & \xi_m D \\ \xi_m^{-1} & 0 \end{bmatrix}, \quad b = \frac{\beta}{\xi_m - \xi_m^{-1}} \begin{bmatrix} D & -D\omega_n \\ \omega_n & -D \end{bmatrix}, \quad c = \beta \begin{bmatrix} 0 & D \\ 1 & 0 \end{bmatrix},$$

and let $G = \langle a, b, c\rangle$. Then $\mathcal{M} = \mathcal{M}(G, a, b, c)$ is a regular map of type $\{m, n\}$, and one of the following holds:

i) $G \cong \mathrm{PSL}(2,p)$ and $\mathcal{M}$ is non-orientable — this happens if and only if $h|\frac{p\pm 1}{2}$ for each $h \in \{m, n\} \setminus \{p\}$ and $-D \in \mathrm{sq}(\mathbb{F}_p)$;

ii) $G \cong \mathrm{PGL}(2,p)$ and $\mathcal{M}$ is orientable — this happens if and only if $h|\frac{p\pm 1}{2}$ for each $h \in \{m, n\} \setminus \{p\}$ and $-D \notin \mathrm{sq}(\mathbb{F}_p)$;

iii) $G \cong \mathrm{PGL}(2,p)$ and $\mathcal{M}$ is non-orientable — this happens if and only if for some $h \in \{m, n\}$ we have $h|(p\pm 1)$ but not $h|\frac{p\pm 1}{2}$.

It transpires from the results of [5] that every regular map $\mathcal{M}$ with $\mathrm{Aut}(\mathcal{M})$ being $\mathrm{PSL}(2,p)$ or $\mathrm{PGL}(2,p)$ for some prime $p \geq 5$ arises from Constructions 1 or 2. As the next lemma shows, some of the above maps with automorphism group isomorphic to $\mathrm{PGL}(2,p)$ admit cyclic covers with almost Sylow-cyclic automorphism groups. The conditon for existence of such a cover is that a and at least one of b and c lies outside the subgroup of index 2 isomorphic to $\mathrm{PSL}(2,p)$. It can be deduced from the results of [5] that this condition is fullfiled exactly when the map is orientable (and comes either from Construction 1 with $p \equiv 3 \,(\mathrm{mod}\, 4)$ or from Case ii) of Construction 2), or if the map arises from Case iii) of Construction 2 with either $n = p \equiv 3 \,(\mathrm{mod}\, 4)$ or $n \neq p$, at least one of m, n divides $\frac{p\pm 1}{2}$ and $-D \in \mathrm{sq}(\mathbb{F}_p)$.

Lemma 5.3 *Let p be a prime, let $t \geq 3$ be an integer coprime to $p(p^2-1)$, let $\mathcal{M} = \mathcal{M}(K, a, b, c)$ be a regular map with $K = \mathrm{PGL}(2, p)$, let S be the subgroup of index 2 in K (isomorphic to $\mathrm{PSL}(2, p)$), let M be a cyclic group of order t generated by an element z, let φ be the group homomorphism from $\mathrm{PGL}(2, p)$ to $\mathrm{Aut}(M)$ mapping S to the identity automorphism and $K \backslash S$ to the automorphism mapping $z \in M$ to z^{-1}, and let $G = M \rtimes_\varphi K$. Then (G, za, b, c) is a regular map group if and only if $a \notin S$ and $\langle b, c\rangle \not\subseteq S$.*

Proof. Assume first that $a \notin S$ and $\langle b, c\rangle \not\subseteq S$. Then $z^{\varphi(a)} = z^{-1}$, and so za is an involution. What remains to show is that $\langle za, b, c\rangle = G$. Since $\langle b, c\rangle \not\subseteq S$, at least one of the elements b and c (say b) lies outside S. But then $ab \in S$, implying that $z(ab) = (ab)z$. Since t is coprime to the order of $S \cong \mathrm{PSL}(2, p)$, we see that $(zab)^{p(p^2-1)}$ is a generator of M. Hence $z \in \langle za, b, c\rangle$, and so $\langle za, b, c\rangle = \langle z, a, b, c\rangle = G$, as required.

Conversely, suppose that (G, za, b, c) is a regular map group. Then za is an involution, and so $a \notin S$. If $H = \langle b, c\rangle \subseteq S$, then H is preserved by conjugation by z. On the other hand, conjugation by $z^{\frac{t+1}{2}}$ maps za to a. Hence the group $K = \langle a, b, c\rangle$ is conjugate to $\langle za, b, c\rangle$ in G, implying that za, b, c do not generate G, which is a contradiction. ■

A map $\mathcal{M}(G, za, b, c)$ from Lemma 5.3 will be called a *t-fold cyclic cover* of $\mathcal{M}$ and denoted by $C_t \cdot \mathcal{M}$. We are now ready to state a complete classification of regular maps with non-solvable almost Sylow-cyclic automorphism groups.

Theorem 5.4 *Let $\mathcal{M}$ be a regular map. Then $\mathrm{Aut}(\mathcal{M})$ is a non-solvable almost Sylow-cyclic group if and only if one of the following holds:*

(i) *$\mathrm{Aut}(\mathcal{M}) \cong \mathrm{PSL}(2, p)$ or $\mathrm{PGL}(2, p)$ for $p \geq 5$, and either $\mathcal{M}$ or its dual is isomorphic to a map given in Table 3 below.*

(ii) *$\mathcal{M} \cong C_t \cdot \mathcal{M}'$ where t is an integer coprime to $|\mathrm{PGL}(2, p)|$ and $\mathcal{M}'$ or its dual is isomorphic to a map from Row 2 or Row 4 of Table 3.*

(iii) *$\mathcal{M} \cong C_t \cdot \mathcal{M}'$ where t is an integer coprime to $|\mathrm{PGL}(2, p)|$ and $\mathcal{M}'$ or its dual is isomorphic to a map from Row 5 of Table 3, with the extra condition that either $n = p \equiv 3 \pmod 4$, or $n | \frac{p\pm1}{2}$ and $-D \in \mathrm{sq}(\mathbb{F}_p)$.*

Table 3: Regular maps $\mathcal{M} = \mathcal{M}(G, a, b, c)$ with $G \cong \mathrm{PSL}(2, p)$ or $\mathrm{PGL}(2, p)$, $p \geq 5$.

	G	(a,b,c)	Type	Orientable?	Conditions
1	$\mathrm{PSL}(2,p)$	see Construction 1	$\{p,p\}$	No	$p \equiv 1\,(4)$
2	$\mathrm{PGL}(2,p)$	see Construction 1	$\{p,p\}$	Yes	$p \equiv 3\,(4)$
3	$\mathrm{PSL}(2,p)$	see Construction 2	$\{m,n\}$	No	$m \mid \frac{p\pm1}{2}$, $n \mid \frac{p\pm1}{2}$ or $n=p$, $-D \in \mathrm{sq}(\mathbb{F}_p)$
4	$\mathrm{PGL}(2,p)$	see Construction 2	$\{m,n\}$	Yes	$m \mid \frac{p\pm1}{2}$, $n \mid \frac{p\pm1}{2}$ or $n=p$, $-D \notin \mathrm{sq}(\mathbb{F}_p)$
5	$\mathrm{PGL}(2,p)$	see Construction 2	$\{m,n\}$	No	$m \mid (p\pm1)$, not $m \mid \frac{p\pm1}{2}$, $n \mid (p\pm1)$ or $n=p$

Proof. From the comments preceding Lemma 5.3 we see that all we need to show is the following: $\mathrm{Aut}(\mathcal{M})$ is a non-solvable almost Sylow-cyclic group if and only if for some prime $p \geq 5$ we have either $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{PSL}(2,p)$ or $\mathrm{PGL}(2,p)$, or $\mathcal{M} \cong C_t \cdot \mathcal{M}'$ for some regular map $\mathcal{M}'$ with $\mathrm{Aut}(M') \cong \mathrm{PGL}(2,p)$ and some $t \geq 2$ with $\gcd(t, p(p^2-1)) = 1$.

If one of the above holds, then clearly $\mathrm{Aut}(\mathcal{M})$ is almost Sylow-cyclic. Suppose now that $\mathrm{Aut}(\mathcal{M})$ is a non-solvable almost Sylow-cyclic group, and write $\mathcal{M} = \mathcal{M}(G,a,b,c)$ where $G \cong \mathrm{Aut}(\mathcal{M})$. By Lemma 3.2, it follows that the Sylow 2-subgroup of G is dihedral, and in view of Theorem 5.1, G contains a subgroup G_0 such that $|G : G_0| \leq 2$ and $G_0 \cong M \times L$ where $M \cong C_m \rtimes C_n$ and $L \cong \mathrm{PSL}(2,p)$ for some prime $p \geq 5$ and odd integers m, n coprime to the order of $\mathrm{PSL}(2,p)$.

If $|G : G_0| = 1$, then the group M is a quotient of G, and since G is generated by involutions, so is M, provided M is not trivial. The order of M is odd, however, implying that $|M| = mn = 1$. Hence (i) holds in this case.

We may assume henceforth that $|G : G_0| = 2$. Since $\gcd(|M|, |L|) = 1$, it follows that M is a Hall subgroup of G_0 and G. Furthermore, since L is simple, M is the largest normal subgroup of odd order in G_0, and therefore characteristic in G_0 and normal in G. Since $|G : G_0| = 2$, M is also the largest normal subgroup of odd order in G. Similarly, since L is simple, L is characteristic in G_0 and therefore normal in G.

By the Schur-Zassenhaus lemma, M is complemented in G, and so $G = M \rtimes K$ for some $K \leq G$. Clearly $G_0 \cap K$ is a subgroup of index 2 in K which complements M in G_0. Since M is a Hall normal subgroup of G_0, any two complements of M in G_0 are conjugate (see [16, Theorem 7.42]). But L is a *normal* complement of M in G_0, and therefore unique. In particular, $K \cap G_0 = L$.

Since M is the largest odd order normal subgroup of G, and since K is neither simple nor a 2-group, the result of Gorenstein and Walter (see Theorem 5.2) implies that $G/M \cong K \cong \mathrm{PGL}(2,p)$. If $|M| = 1$, then $G \cong \mathrm{PGL}(2,p)$, and the result follows. Hence we may assume that $|M| \geq 3$.

Since $K \cong \mathrm{PGL}(2,p)$ is a split extension of $L \cong \mathrm{PSL}(2,p)$, there exists an involution $\tau \in K \setminus L$. Hence $G = (M \times L) \rtimes \langle \tau \rangle$. But then $G/L \cong \langle M, \tau \rangle$ is a solvable almost Sylow-cyclic group whose order is twice an odd number. In particular, every Sylow subgroup of $\langle M, \tau \rangle$ is cyclic, and by Lemma 4.2, $\langle M, \tau \rangle$ is metacyclic. Moreover, since $\langle M, \tau \rangle$ is a homomorphic image of G, it is generated by involutions, and by Lemma 4.4 isomorphic to D_{mn}. In particular, $M \cong C_{mn}$, and so $G \cong C_{mn} \rtimes_\varphi \mathrm{PGL}(2,p)$, where $\ker(\varphi) = L$.

Let P be the Sylow 2-subgroup of G containing the group $H = \langle b, c \rangle$. Since P is conjugate in G to a Sylow 2-subgroup of K, we may assume that $H \le K$. Further, let $a' \in K$ and $z \in M$ be such that $a = za'$. Then (in view of the fact that the kernel M of the natural mapping $G \to G/M \cong K$ has odd order), we see that (K, a', b, c) is a regular map group. If z were not a generator of M, then $\langle z \rangle K$ would be a proper subgroup of G containing a, b and c, contradicting the fact that a, b, c generate G. But then in view of Lemma 5.3, $a' \notin L$, $\langle b, c \rangle \not\subseteq L$, and $\mathcal{M} \cong C_{mn} \cdot \mathcal{M}(K, a', b, c)$, as claimed. ■

6 Regular maps of characteristic $-p^2$

We conclude the paper with a result about regular maps of Euler characteristic $-p^2$, for p prime, as announced in the introduction.

Theorem 6.1 *Let p be an odd prime, and let $\mathcal{M}$ be a regular map of Euler characteristic $-p^2$. Then $p = 3$ and $\mathcal{M}$ is one of the regular maps of characteristic -9 described in* [4]*, or $p = 7$ and $\mathcal{M}$ is one of the regular maps of characterstic -49 metioned in* [14].

Proof. For $p = 3$ and $p = 5$, the statement of the theorem follows from [4], where all regular maps of characteristic at least -28 were determined. Hence we may assume that $p \ge 7$, and then also by Proposition 3.4, that $\mathrm{Aut}(\mathcal{M})$ is almost Sylow-cyclic.

If $\mathrm{Aut}(\mathcal{M})$ is solvable, then by Theorem 4.7, $\mathcal{M}$ has to be one of the non-orientable maps from Table 2. Clearly, $\mathcal{M}$ is not the map from Row 1. If $\mathcal{M}$ is a map from Row 6, then $-p^2 = N/m + m - N$ for odd integers m and N satisfying $m \ne 1, N$ and $\gcd(m, m/N) = 1$, implying that $p^2 = (x-1)(y-1) - 1$ where $x = N/m$, $y = m$. Since x and y are both odd, we see that $p^2 \equiv -1 \pmod 4$, which contradicts the fact that -1 is not a square in $\mathbb{Z}_4^*$. Similarly, if $\mathcal{M}$ is a map from Row 8, then $-p^2 = 4 - N$ where N is an odd integer divisible by 3, and thus $p^2 \equiv -1 \pmod 6$, which contradicts the fact that -1 is not a square in $\mathbb{Z}_6^*$.

We may therefore assume that $\mathrm{Aut}(\mathcal{M})$ is non-solvable. To make the proof as self-contained as possible, we will not use the information from [5] about the reglar maps with auomorphism groups isomorphic to $\mathrm{PGL}(2,p)$ or $\mathrm{PSL}(2,p)$, but rather only rely on the well known facts about the orders of elements in the projective linear groups.

In view of the first paragraph of the proof of Theorem 5.4, we see that for some prime $q \ge 5$, one of the following holds:

(A) $\mathrm{Aut}(\mathcal{M}) \cong \mathrm{PSL}(2,q)$, or

(B) $\mathrm{Aut}(\mathcal{M}) \cong C_t \rtimes \mathrm{PGL}(2,q)$ for some odd integer t coprime to $|\mathrm{PGL}(2,q)| = q(q^2-1)$ (possibly $t = 1$), where the subgroup $\mathrm{PSL}(2,q)$ commutes with every element in C_t, while conjugation by an element of $\mathrm{PGL}(2,q) \setminus \mathrm{PSL}(2,q)$ inverts the elements of C_t.

Furthermore, in case (B), $\mathcal{M} = \mathcal{M}(C_t \rtimes \mathrm{PGL}(2,q), za, b, c)$, where z is a generator of C_t (where possibly $z = 1$ if $t = 1$), and $(\mathrm{PGL}(2,q), a, b, c)$ is a regular map group. If $t > 1$, then $a \notin \mathrm{PSL}(2,q)$ and at least one of b, c is not in $\mathrm{PSL}(2,q)$.

Assume first that the case (B) occurs. Since $\mathcal{M}$ has negative odd characteristic, $\mathcal{M}$ is non-orientable, implying that $\langle zab, zac\rangle = C_t \rtimes \mathrm{PGL}(2,q)$. In particular, at least one of the elements ab and ac lies outside of $\mathrm{PSL}(2,q)$, and if $t > 1$, then exactly one of ab and ac belongs to $\mathrm{PSL}(2,q)$.

Observe that for $x \in \mathrm{PGL}(2,q)$, the order $|zx|$ of $zx \in C_t \rtimes \mathrm{PGL}(2,q)$ is either $t|x|$ (if $x \in \mathrm{PSL}(2,q)$) or $|x|$ (if $x \notin \mathrm{PSL}(2,q)$). Let $\{k,\ell\}$ and $\{m,n\}$ denote the types of the maps $\mathcal{M}' = \mathcal{M}(\mathrm{PGL}(2,q), a, b, c)$ and $\mathcal{M}$, respectively. Then $k = |ab|$, $m = |zab|$, $\ell = |ac|$, and $n = |zac|$. Hence if ab belongs to $\mathrm{PSL}(2,q)$, then $m = tk$; and if ab lies outside $\mathrm{PSL}(2,q)$, then $m = k$. Similarly n is either $t\ell$ or ℓ, depending on whether $ac \in \mathrm{PSL}(2,q)$. We may thus conclude that $\{m,n\} = \{tk,\ell\}$ or $\{k,t\ell\}$ (possibly both, if $t = 1$).

By non-orientabilty of $\mathcal{M}$, it follows that at least one of m, n (or equivalently k, ℓ) is even. Also, if either of k, ℓ were 2, then one of b or c would belong to the centre of $\mathrm{PGL}(2,q)$, and hence $\mathrm{PGL}(2,q)$ would be a 2-extension of a dihedral group. Therefore $m, n, k, \ell \geq 3$.

By Euler's formula and the fact that $mn = tk\ell$, we deduce that

$$\alpha q(q^2-1) = 2k\ell p^2, \qquad \text{where } \alpha = \frac{mn-2m-2n}{2} = \frac{(m-2)(n-2)-4}{2}. \tag{6}$$

Since q^2-1 is always divisible by 8, we see that either both k, ℓ are even, or one is odd, but then the other is divisible by 4. Moreover, if both are even, but neither is divisible by 4, then α is even, and so the left-hand side of the equality is divisible by 16, while the right-hand side is not. Hence always at least one of k, ℓ is divisible by 4.

It is well known that the order of an element in $\mathrm{PGL}(2,q)$ is either q, or a divisor of $q-1$, or a divisor of $q+1$. Moreover, the elements of order $q \pm 1$ are not contained in $\mathrm{PSL}(2,q)$. If $t > 1$, then $a \notin \mathrm{PSL}(2,q)$ and at least one of b, c is not in $\mathrm{PSL}(2,q)$, implying that at least one of ab, ac is in $\mathrm{PSL}(2,q)$, and in turn that (when $t > 1$) at least one of k, ℓ is either q or a divisor of $(q \pm 1)/2$.

We shall consider two cases, depending on whether one of k, ℓ equals q. Suppose first that neither of k, ℓ equals q. Then both k and ℓ are coprime to q. In view of (6), this implies that $q = p$, and so $\alpha(q^2-1) = 2k\ell q$. In particular, q divides α, and thus

$$\beta(q^2-1) = 2k\ell, \qquad \text{where } \beta = \frac{(m-2)(n-2)-4}{2q} \in \mathbb{Z}. \tag{7}$$

If both k, ℓ divide $q \pm 1$ for some choice of the sign, then $rs\beta(q \mp 1) = 2(q \pm 1)$ for some integers r, s, which easily implies that $q \leq 5$. Hence we may assume that $rk = (q \pm 1)$ and $s\ell = (q \mp 1)$ for some integers r, s, and thus $rs\beta = 2$.

If $\beta = 2$, then $r = s = 1$, and so $k = q \pm 1$ and $\ell = q \mp 1$. In particular, both ab, ac lie outside $\mathrm{PSL}(2,q)$, implying that $t = 1$, and so $m = k$, $n = \ell$. By (7) we see that $4q + 4 = (m-2)(n-2) = q^2 - 4q + 3$, which has no integer solutions.

We may therefore assume that $\beta = 1$ and $rs = 2$. Then $\{k,\ell\} = \{\frac{q\pm1}{2}, q \mp 1\}$, and by (7), we have $(m-2)(n-2) = 2q + 4$. Hence

$$4q + 8 = 2(m-2)(n-2) \geq 2(k-2)(\ell-2) = q^2 - 6q + 7 \pm 2.$$

This inequality implies that $q < 11$, and so $q = 7$, $(m-2)(n-2) = 18$, and $\{k, \ell\} = \{4, 6\}$ or $\{k, \ell\} = \{3, 8\}$. If $t = 1$, then the above inequality is in fact equality, which, however, has no integer solutions. Therefore $t > 1$, and since t is comprime to $q(q^2 - 1)$, also $t \geq 5$. But then (in all possible cases) $(m - 2)(n - 2) > 18$, a contradiction.

We may thus assume that one of k, ℓ equals q. Then the other is divisible by 4, and thus a divisor of $q \pm 1$. Without loss of generality we may assume that $k = q$ and $q \pm 1 = r\ell$ for some integer r. Then by formula (6) we obtain

$$\alpha r \frac{q \mp 1}{2} = p^2. \tag{8}$$

Since $q \geq 5$, it follows that $\frac{q \mp 1}{2} \neq 1$, and so p divides $\frac{q \mp 1}{2}$. If $\alpha = 1$, then $(m-2)(n-2) = 6$, and since $m \in \{q, tq\}$ is odd, we have either $q = m = 3$ (which contradicts our assumption) or $q = m = 5$, and so $\frac{q \mp 1}{2} \in \{2, 3\}$, which is not divisible by p. This shows that $\alpha \neq 1$, implying that

$$r = 1, \qquad \alpha = \frac{q \mp 1}{2} = p. \tag{9}$$

In particular, $\ell = q \pm 1$, and so $ac \notin \mathrm{PSL}(2, q)$, impling that $n = \ell$ and $m = tq$. By (6) it follows that

$$q \mp 1 = 2\alpha = tq(q \pm 1) - 2tq - 2(q \pm 1).$$

Considering this equality modulo q, we see that q divides $\mp 2 \pm 1$, which is a clear contradiction. This completes the proof in case of (B).

Suppose now that (A) occurs, that is, $G \cong \mathrm{PSL}(2, q)$ for some prime $q \geq 5$. The analysis in this case will be similar to that of case (B).

First, if $q \leq 7$, then the size of $|G|$ is at most $6 \cdot 7 \cdot 4 = 168$, and then the result follows from [14] where all regular maps on groups of size at most 512 were determined. We may therefore assume that $q \geq 11$.

Let $\mathcal{M} = \mathcal{M}(G, a, b, c)$. Then the type of $\mathcal{M}$ is $\{m, n\}$ where m and n are the orders of ab and ac, respectively. Hence each of m and n either equals q or it is a divisor of $\frac{q \pm 1}{2}$. Moreover, in view of (2), it follows that

$$(mn - 2m - 2n) q \frac{q^2 - 1}{4} = 2mnp^2. \tag{10}$$

Suppose that neither of m and n equals q. Then both m and n are coprime to q, and so $q = p$. Also, since q is coprime to $\frac{q^2 - 1}{4}$, it follows from (10) that $q = p$ divides $(mn - 2m - 2n)$, and so

$$\gamma \frac{q^2 - 1}{4} = 2mn, \quad \text{where } \gamma = \frac{mn - 2m - 2n}{q} \in \mathbb{Z}. \tag{11}$$

If both m and n divide the same element $\frac{q \pm 1}{2}$, then they are coprime to $\frac{q \mp 1}{2}$, and by (11), we find that $\frac{q \mp 1}{2} = 1$ or 2, which contradicts the assumption that $q \geq 11$. We may therefore assume that $\frac{q \pm 1}{2} = \alpha m$ and $\frac{q \mp 1}{2} = \beta n$ for some integers α and β, and so $\gamma \alpha \beta = 2$. If $\alpha = \beta = 1$ and $\gamma = 2$, then $m = \frac{q \pm 1}{2}$, $n = \frac{q \mp 1}{2}$, and $mn - 2m - 2n = 2q$, which implies that $q^2 - 16q - 1 = 0$, a clear contradiction. Hence $\alpha\beta = 2$ and $\gamma = 1$, and so $mn - 2m - 2n = q$ where $\{m, n\} = \{\frac{q\pm1}{4}, \frac{q\mp1}{2}\}$. This implies that $q^2 - 20q \pm 4 - 1 = 0$, which again contradicts our assumptions on q. This concludes the case where neither of m and n equals q.

We may now assume without loss of generality that $m = q$. By (10) we see that

$$(qn - 2q - 2n)\frac{q^2 - 1}{4} = 2np^2. \tag{12}$$

If $n = q$, then (12) together with the assumption $q \geq 11$ gives the equation $q^2 - 8q + 31 = 0$, which has no integer solutions. Hence we may assume that $\frac{q\pm1}{2} = \alpha n$ for some integer α, and then

$$(qn - 2q - 2n)\alpha\frac{q \mp 1}{2} = 2p^2. \tag{13}$$

Since $\frac{q\mp1}{2} \geq 3$, we see that p divides $\frac{q\mp1}{2}$, and is therefore coprime to $\frac{q\pm1}{2}$, and thus also to α. In particular, $\alpha = 1$ or 2.

By considering (13) modulo q, we see that q divides $4p^2 - 1$ and so q divides either $2p - 1$ or $2p + 1$. On the other hand, p divides $\frac{q\mp1}{2}$, implying that $q = 2p \pm 1$ (with the choice of the sign consistent with that in the definition of α). Hence, by (13), we see that $(qn - 2q - 2n)\alpha = 2p = q \mp 1$. If $\alpha = 1$, then $n = \frac{q\pm1}{2}$, and the latter equality gives $q \pm 1 = 8$, which contradicts our assumption that $q \geq 11$. On the other hand, if $\alpha = 2$, then $n = \frac{q\pm1}{4}$, and we obtain $q \pm 1 = 12$, and since $q \mp 1 = 2p$, we see that $m = q = 13$, $p = 7$, and $n = 3$. It can be seen that there exists exactly one pair of mutually dual regular maps of type $\{3, 13\}$ with automorphism group isomorphic to PSL(2, 13), and that these two maps are precisely the maps mentioned in the statement of the theorem. ∎

Acknowledgements. The research for this paper was carried out while the second and third authors were affiliated with the University of Auckland. They gratefully acknowledge the support from the NZIMA programme on Combinatorics and its Applications. Also the first and third authors wish to acknowledge partial support from the Marsden Fund, and the third author would like to acknowledge partial support from APVV Grants No. 20-000704 and 0040-06.

References

[1] A. B. D'Azevedo, R. Nedela, and J. Širáň, Classification of regular maps of negative prime Euler characteristic, *Trans. Amer. Math. Soc.* **357** (2005), 4175–4190.

[2] W. Bosma, C. Cannon, and C. Playoust, The MAGMA algebra system I: The user language, *J. Symbolic Comput.* **24** (1997), 235–265.

[3] M. Conder, Maximal automorphism groups of symmetric Riemann surfaces with small genus, *J. Algebra* **114** (1988), 16–28.

[4] M. Conder and P. Dobcsányi, Determination of all regular maps of small genus, *J. Combin. Theory Ser. B* **81** (2001), 224–242.

[5] M. Conder, P. Potočnik, and J. Širáň, Regular hypermaps over projective linear groups, submitted for publication.

[6] H. S. M. Coxeter and W. O. J. Moser, *Generators and relations for discrete groups*, Springer-Verlag, Berlin (1957).

[7] S.-F. Du, J. H. Kwak, and R. Nedela, A classification of regular embeddings of graphs of order a product of two primes, *J. Algebraic Combin.* **19** (2004), 123–141.

[8] A. L. Edmonds, J. H. Ewing, and R. S. Kulkarni, Regular tessellations of surfaces and $(p, q, 2)$-triangle groups, *Ann. Math.* **116** (1982), 113–132.

[9] A. Gardiner, R. Nedela, J. Širáň, and M. Škoviera, Characterisation of graphs which underlie regular maps on closed surfaces, *J. London Math. Soc. (2)* **59** (1999), 100–108.

[10] W. Gaschütz, Zur Erweiterungstheorie der endlichen Gruppen, *J. Reine Angew. Math.* **190** (1952), 93–107.

[11] D. Gorenstein and J. H. Walter, The characterization of finite groups with dihedral Sylow 2-subgroups II *J. Algebra* **2** (1965) 218–270.

[12] T. L. Heath, *The Thirteen Books of the Elements, 2nd ed., Vol. 2: Books III-IX*, Dover Publications, Inc., New York, (1956).

[13] G. A. Jones and J. S. Thornton, *Operations on maps, and outer automorphisms*, *J. Combin. Theory* Ser. B, **35** (1983), 93–103.

[14] A. Orbanić, "Edge-transitive maps", PhD Thesis, University of Ljubljana (2006).

[15] D. J. S. Robinson, *A Course in the Theory of Groups, 2nd ed.*, Graduate Texts in Mathematics **80**, Springer-Verlag (1996), New York.

[16] J. J. Rotman, *An Introduction to the Theory of Groups, 4th ed.*, Graduate Texts in Mathematics **148**, Springer Verlag (1995), New York.

[17] D. Singerman, Automorphisms of compact non-orientable Riemann surfaces, *Glasgow Math. J.* **12** (1971), 50-59.

[18] M. Suzuki, On finite groups with cyclic sylow subgroups for all odd primes, *Amer. J. Math.* **77** (1955), 657–691.

[19] T. W. Tucker, Finite groups acting on surfaces and the genus of a group, *J. Combin. Th. Ser. B* **34** (1983) 82–98.

[20] S. Wilson, Families of regular graphs in regular maps, *J. Combin. Theory Ser. B* **85** (2002), 269–289.

[21] J. A. Wolf, *Spaces of Constant Curvature*, McGraw–Hill (1967), New York.

[22] W. J. Wong, On Finite Groups with Semi-Dihedral Sylow 2-Subgroups, *J. Algebra* **4** (1966), 52–63.

[23] H. Zassenhaus, Über endliche Fastkörper, *Abh. Math. Sem. Hamburg Univ.* **11** (1936), 187–220.