# Inner reflectors and non-orientable regular maps

Marston Conder
*Department of Mathematics*
*University of Auckland*
*Auckland, New Zealand*

Steve Wilson
*Department of Mathematics*
*Northern Arizona University*
*Flagstaff, Arizona, USA*

**Abstract**: Regular maps on non-orientable surfaces are considered with particular reference to the properties of *inner reflectors*, corresponding to symmetries of the 2-fold smooth orientable covering which project onto local reflections of the map itself. An example is given where no inner reflector is induced by an involution, and the existence of such involutions is related to questions of symmetry of coset diagrams for the symmetry group of the map.

## 1. Introduction:

We begin with a few definitions. A *surface* is a compact, connected 2-manifold without boundary. A *map* is an embedding of a graph (or multigraph) on a surface so that the *faces*—the regions bounded by arcs of the graph—are simply connected. Choose an arbitrary point in the interior of each face to be its centre, and an arbitrary point in the relative interior of each edge to be its midpoint; then a *flag* is one of the triangles formed when each face-centre is joined to each of the edge-centres and vertices around it.

Now consider a flag f in a map M: the flag f has three neighbours, say $fr_0$, $fr_1$ and $fr_2$ as in Figure 1.

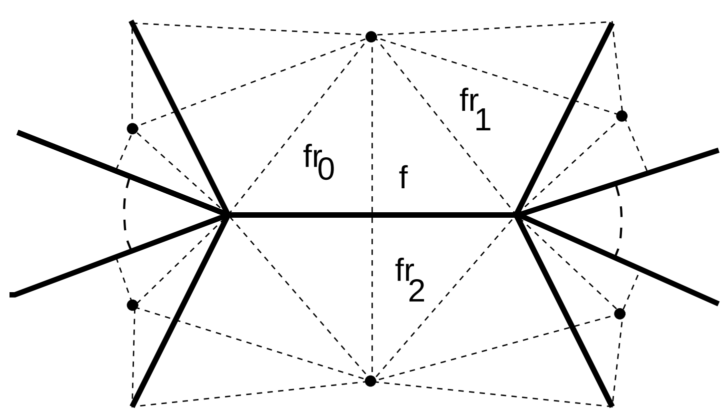


Figure 1: Adjacent flags in a map

If we think of each flag as a "right triangle" with a hypotenuse and two other sides, with one side dark and solid and the other thin and dotted, then f is connected to $fr_0$ along their common thin side, to $fr_1$ along their common hypotenuse, and to $fr_2$ along their common solid side. Notice how these connections are labelled with respect to face, edge and vertex: f and $fr_0$ differ only in vertex (the 0-dimensional part), f and $fr_1$ differ only in edge (the 1-dimensional part, and f and $fr_2$ differ only in face ( the 2-dimensional part).

It is useful to consider the following colouring of flags, as in [**3**]: choose a *root* flag, say I, and colour it red. Note that each flag has three neighbours. Colour all neighbours of I white, and now recursively colour all neighbours of white flags red, and all neighbours of red flags white. Eventually one of two things will happen: if the surface is orientable, then half of the flags will be coloured red and the other half (the neighbours of red flags) will be white, while if the surface is non-orientable, then every flag will be coloured both red and white.

A *symmetry* of a map is a homeomorphism of the surface onto itself which preserves the map structure. We can also take a combinatorial point of view and define a symmetry to be a permutation of the flags which preserves the connections $r_i$. Under composition the symmetries of a map M form a group, which we will denote by G(M). A map is "regular" provided that it has a large group of symmetries. There are two kinds of regular maps, namely *rotary* maps and *reflexible* maps, and we will give two definitions— one topological and one combinatorial— for each kind of regularity:

**Definition**: A map is *rotary* if for some flag I, the group G(M) contains symmetries R and S which act as single-step rotations respectively about the face and vertex incident to I; specifically, $IR = Ir_1r_0$ and $IS = Ir_1r_2$. A rotary map is *reflexible* if it also has a symmetry X which acts as a reflection about the hypotenuse of I: $IX = Ir_1$. These symmetries are illustrated in Figure 2.

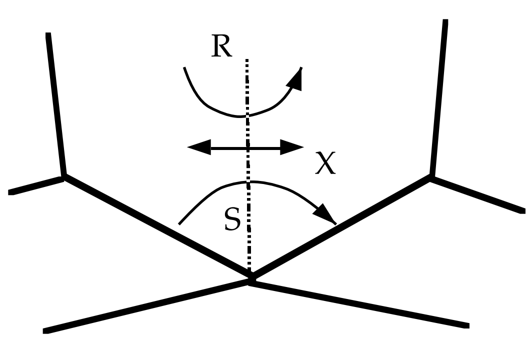


Figure 2: Symmetries in a rotary map

**Definition**: A map M is *rotary* if the group G(M) is transitive on the red flags of M (and therefore transitive on the white flags of M as well). A map M is *reflexible* if the group G(M) is transitive on the set of all flags of M.

If M is a rotary map, then the group $G^+(M)$ generated by the symmetries R and S is called the *rotation group* of M. Note that these generators satisfy the relation $(RS^{-1})^2 = 1$. Conversely, any group G generated by two elements R and S satisfying $(RS^{-1})^2 = 1$ can be shown to be the rotation group of some orientable rotary map M; see Lemma 1 of [**3**] for an explicit construction of the map M from the group G.

Also note that for a reflexible map, the symmetries X and R together generate a dihedral group, the stabilizer of the central face in Figure 2. In this dihedral group, the conjugate of R by X is $R^{-1}$, and similarly, the conjugate of S by X is $S^{-1}$. It follows that if M is reflexible, then conjugation by X is an automorphism $\mu$ of $G^+(M)$ such that $\mu(R) = R^{-1}$ and $\mu(S) = S^{-1}$. We will call such an automorphism $\mu$ a *reflector*. Observe that the converse holds too: if $G^+(M)$ has a reflector, then M is reflexible.

**2. Inner reflectors**:

In this paper, we are particularly interested in non-orientable maps, and two items about the symmetry groups of non-orientable maps are to be noted here:

(a) An obvious consequence of the definitions and the comments about orientability is that a non-orientable rotary map M must be reflexible, and that for any such map, G(M) is the same as $G^+(M)$. Moreover, since X is an element of $G^+(M)$, we find that for any non-orientable map M, each reflector is an inner automorphism of $G^+(M)$.

(b) Consider the relationship between the non-orientable regular map M and its two-fold smooth orientable covering, 2M. It was shown in [**5**] that 2M is reflexible, and it follows from the proof of this fact that $G^+(2M) = G^+(M) = G(M)$. Hence if $\pi$ is the projection of 2M onto M, then $\pi$ can also be considered as a homomorphism from G(2M) onto G(M), such that the restriction of $\pi$ to $G^+(2M)$ is an isomorphism.

Now let W be the unique element of $G^+(2M)$ such that $\pi(W) = X$. Then W has order 2, and conjugation by W is a reflector in $G^+(2M)$. Our main result is that the existence of such an involution, inducing an inner reflector, is characteristic of two-fold smooth orientable covers of non-orientable maps:

**Theorem**: *Let* N *be an orientable rotary map of negative characteristic. Then there exists a non-orientable regular map* M *such that* N = 2M *if and only if the group* $G^+(N)$ *contains an involution* W, *conjugation by which inverts both the generators* R *and* S *of* $G^+(N)$.

*Proof of Theorem.* One part is easy: our comments above show that if N = 2M, then there exists an inner reflector induced by an involution W in $G^+(N)$.

To prove the converse, suppose $G^+(N)$ has an inner reflector which is induced by an element W of order 2. Then since $G^+(N)$ has a reflector, the map N is reflexible, and so G(N) contains an element X as described in the definition of reflexible maps. Now let U = WX. Then

$$R^U = R^{WX} = (R^{-1})^X = R \quad \text{and similarly} \quad S^U = S^{WX} = (S^{-1})^X = S$$

so U commutes with all of $G^+(N)$. In particular, U commutes with W, and so U commutes with X = UW. Thus U centralizes all of G(N). But further,

$$1 = X^2 = (UW)^2 = U^2W^2 = U^2$$

so U is an involution. Next, since W is in $G^+(N)$, it preserves orientation in N, while clearly X reverses orientation, hence U = WX also reverses orientation.

Thus U is an orientation-reversing central involution in G(N). It follows that the map M = N/<U> derived from N by identifying each flag with its image under U is a non-orientable regular map with symmetry group G(M) = G(N)/<U>. It remains only to show that the projection is smooth, which amounts to showing that U is in neither <X, R> nor <X, S>. But N is of negative charactistic, so the symmetries R and S both have order at least 3. Accordingly no orientation-reversing element of the dihedral group <X, R> commutes with R, so U is not in <X, R>, and as a similar argument holds for S, this completes the proof.

Let us call an orientable map N *antipodal* if there exists at least one non-orientable map M such that N = 2M. We can then restate the theorem as follows: *an orientable rotary map is antipodal if and only if its rotation group has an inner reflector induced by an element of order* 2.

### 3. **Non-involutory reflectors**:

Suppose $G^+(N)$ has an inner reflector induced by an element W which is not an involution; might or must N still be antipodal? If the order of W is 2k where k is odd, then the kth power of W satisfies the hypothesis of the theorem, and so such an M exists. But if on the other hand W has order divisible by 4, then this is is not necessarily the case, as is shown by the example below.

(Incidentally, this example reveals a small error in the proof of Theorem 2 in [**4**]: not every inner automorphism of order 2 is induced by an element of order 2; however with minor rewording the proof holds and the theorem remains valid.)

**Example**: In the group GL(4,3) of all invertible 4×4 matrices with integer entries modulo 3, let G be the the subgroup generated by the following two matrices:

$$R = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix}.$$

It is easy to check that $RS^{-1}$ has order 2, so G is the rotation group of some orientable rotary map N. To find all W in GL(4,3) such that $W^{-1}RW = R^{-1}$ and $W^{-1}SW = S^{-1}$ is not as difficult as may appear, since these equations may be rewritten in the form $RWR - W = 0$ and $SWS - W = 0$, which are linear in the matrix W.

Solving them (mod 3) gives the only solutions as $W_1, W_2 = \pm \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}$.

Each solution has order 4, its square being $-I_4$. Also multiplication in GL(4,3) yields $W_1 = (S^2R^3)^2$, so conjugation by $W_1$ is an *inner* reflector.

No such inner reflector in G, is induced by an element of order 2, however, and so our Theorem shows the map N is not antipodal.

(Still, the map N is reflexible, with symmetry group having twice the order of G, and is a 4-fold covering of the non-orientable map $L = N/\langle W \rangle$; also $2L = N/\langle -I_4 \rangle$.)

## 4. Mirror symmetries of coset diagrams:

Our observations above have an interesting connection with the coset diagrams associated with regular maps. A coset diagram is a graphical representation of the action of a transitive permutation group on some set; see [**2**].

In the proof of the Theorem in Section 2, let $T = RS^{-1}$, so that T has order 2, and $R = TS$ and $S = TR$. A coset diagram for any transitive permutation representation of the group $G^+(N) = \langle R,S \rangle = \langle T,R \rangle$ may be constructed in the following way: draw a solid-line polygon oriented clockwise for each cycle of R, and join the points of each 2-cycle of T by a dotted line.

For instance, the transitive action $R \longrightarrow (1,2,3)(4,5,6)$, $T \longrightarrow (3,4)(6,7)$ gives rise to the diagram in Figure 3.

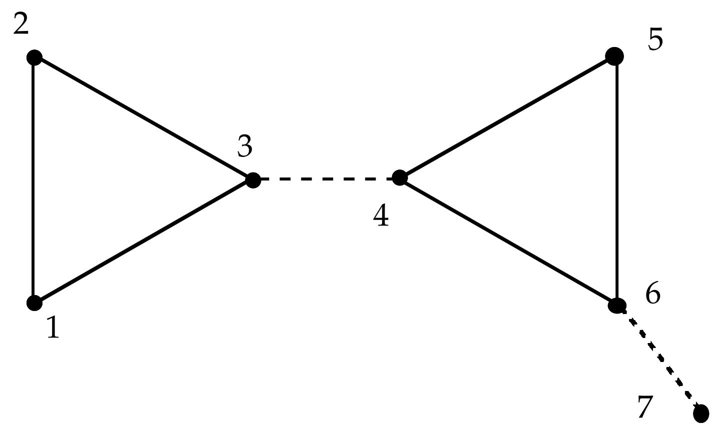


Figure 3: A coset diagram

The coset diagram has interesting relationships with the map. For example, if the action is faithful and the diagram itself is reflexible — that is, if it has a mirror symmetry — then the map N is reflexible. The converse is definitely not true, however; for instance, the diagram above has no mirror symmetry, yet it is a coset diagram for the rotation group of the map $\{3,7\}_8$, which is known to be reflexible.

To make these relationships more clear, let us say the diagram is *reversible* if it has an automorphism which reverses the direction of each cycle of the permutation induced by R, and *reflexible* if the diagram can be arranged with respect to some axis so that the reversing automorphism acts like reflection in this axis. Essentially, the latter requires the reversal to have order 2.

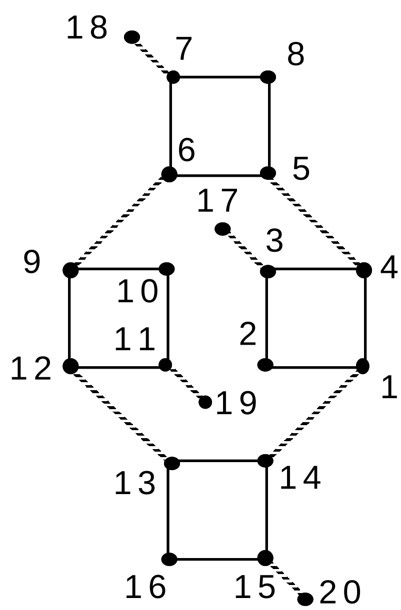


Figure 4: A reversible diagram

For example, the diagram in Figure 4 is reversible but not reflexible, because the only reversing automorphism corresponds to a motion of order 4 which moves each square to the next in line around the circle.

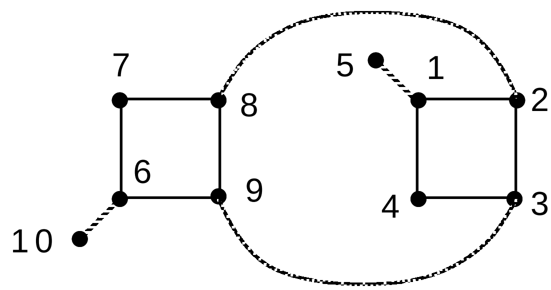


Figure 5: Another reversible diagram

On the other hand, the diagram in Figure 5 does not look reflexible, but it is, as it can be re-drawn as in Figure 6. In this particular example T —> (1,5)(2,8)(3,9)(6,10) while R —> (1,2,3,4)(6,7,8,9), so that S —> (1,5,2,9,4)(3,6,10,7,8).

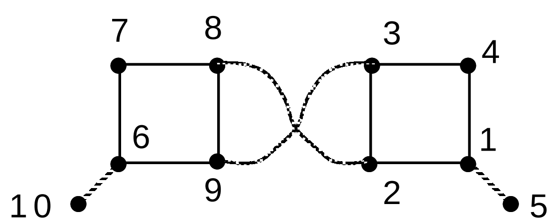


Figure 6: A reflexible drawing of the diagram in Fig. 5

At this stage we may introduce an alternate diagram, formed from the standard one by turning each cycle of R back one half-step, so that the dotted lines for the action of T come to the centre of each edge. For the last example above, the modified diagram is drawn in Figure 7.

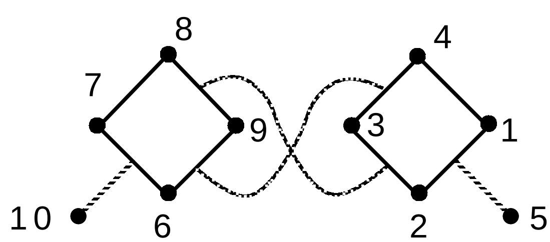


Figure 7: A modified diagram of the action in Fig. 6

In the reflexible case this modified diagram has the pleasant feature of allowing both the permutation induced by S and the permutation induced by a reflector to be read directly from it. In the above example, conjugation by the reflective symmetry (1,7)(2,6)(3,9)(4,8)(5,10) inverts the permutations induced by R and S, and hence corresponds to a reflector, while the reversing symmetry $(1,6)(2,9)(3,8)(4,7)(5,10)$ of the standard diagram does not have this property.

Clearly there may be several diagrams possible for each group, indeed one for each equivalence class of (faithful) transitive permutation representations, but this relationship is quite general: the coset diagram is reversible if and only if there exists a permutation of its vertices which conjugates the permutations induced by each of R and S to their inverses, and if this is true, then of course the associated map must be reflexible. Further, the diagram is reflexible if and only if this conjugating permutation is an involution.

For antipodal maps, we can say even more: since the reflector is *always* inner, then no matter which permutation representation of the group is depicted, there will always be an element of order 2 conjugation by which will invert R and S, and so we have the following result.

**Theorem**: *Every coset diagram for an antipodal map has mirror symmetry.*

The converse is again false, and the subgroup of GL(4,3) studied earlier provides a counter-example. To see this, note that every coset diagram on, say, k vertices corresponds to a conjugacy class of subgroups of index k in the group, and for any such subgroup H, the effect of the generators R and S on these vertices is equivalent to the action of R and S by (right) multiplication on right cosets of H. In the GL(4,3) example, representatives of conjugacy classes of all subgroups of G can easily be determined — with the help of the MAGMA computer system [**1**] for instance — and it can be checked that every diagram for G does indeed have a mirror symmetry. As noted earlier, however, the associated map N is not antipodal.

Still, many questions relating antipodality, inner reflectors and diagrams remain. For example: given a single coset diagram for $G^+(N)$ which *has* a mirror symmetry, how can we decide if N is antipodal?

## References


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[**2**] H.S.M. Coxeter & W.O.J. Moser, *Generators and Relations for Discrete Groups*, 4th ed. (Springer Verlag), 1980.

[**3**] A. Gray & S.E. Wilson, A More Elementary Proof of Grünbaum's Conjecture, *Congressus Numerantium* **72** (1990), 25–32.

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[**5**] S.E. Wilson, Non-orientable regular maps, *Ars Combinatoria* **5** (1978), 213–218.