Two-arc closed subsets of graphs

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Abstract

A subset of vertices of a graph is said to be 2-arc closed if it contains every vertex that is adjacent to at least two vertices in the subset. In this paper, 2-arc closed subsets generated by pairs of vertices at distance at most 2 are studied. Several questions are posed about the structure of such subsets and the relationships between two such subsets, and examples are given from the class of partition graphs.

1 Introduction

Let $\Gamma = (V, E)$ be a connected undirected graph with vertex set V and edge set E (identified with a subset of unordered pairs from V). A 1-*arc* (or simply an *arc*) is an ordered pair of adjacent vertices. A 2-*arc* of Γ is a triple (α, β, γ) of vertices such that $\{\alpha, \beta\}$ and $\{\beta, \gamma\}$ are edges and $\alpha \neq \gamma$. A non-empty subset S of V is said to be 2-*arc closed* if for each 2-arc (α, β, γ) with $\alpha, \gamma \in S$, the intermediate vertex β also lies in S. More generally, for each non-empty subset $X \subseteq V$, we define the 2-*arc closure* of X to be

$$\mathcal{C}_2(X) = \bigcap \{S \mid X \subseteq S,\ S \text{ is 2-arc closed}\}.$$

Since V is 2-arc closed, and since the intersection of 2-arc closed subsets is also 2-arc closed, it follows that $\mathcal{C}_2(X)$ is the unique smallest 2-arc closed subset containing X. We say that $\mathcal{C}_2(X)$ is the 2-arc closed subset *generated by* X, and we identify $\mathcal{C}_2(X)$ also with the subgraph of Γ induced on the subset $\mathcal{C}_2(X)$ of V.

The structure of 2-arc closed subsets generated by very small subsets depends on the *girth* $g(\Gamma)$ of Γ, which is defined as the length of the shortest cycle of Γ if Γ contains a cycle, and otherwise is ∞. For $\alpha, \beta \in V$, we write $\mathcal{C}_2(\alpha) = \mathcal{C}_2(\{\alpha\})$ and $\mathcal{C}_2(\alpha, \beta) = \mathcal{C}_2(\{\alpha, \beta\})$. Clearly $\mathcal{C}_2(\alpha) = \{\alpha\} \cong K_1$, and if the distance $d(\alpha, \beta)$ between α and β is greater than 2 then $\mathcal{C}_2(\alpha, \beta) = \{\alpha, \beta\} \cong 2 \cdot K_1$ (where K_n denotes the complete graph on n vertices). Also if $d(\alpha, \beta) = 2$ and $g(\Gamma) \geq 5$, then there is a unique 2-arc (α, γ, β) from α to

β and we have $\mathcal{C}_2(\alpha,\beta) = \{\alpha,\gamma,\beta\} \cong P_2$, a path of length 2. It is interesting therefore to consider the following kinds of 2-arc closed subsets:

(a) $\mathcal{C}_2(\alpha,\beta)$ where $\{\alpha,\beta\}$ is an edge of Γ and $g(\Gamma) = 3$;

(b) $\mathcal{C}_2(\alpha,\beta)$ where $d(\alpha,\beta) = 2$ in Γ and $g(\Gamma) = 3$ or 4;

(c) $\mathcal{C}_2(\alpha,\beta,\gamma)$ where $d(\alpha,\beta) = d(\alpha,\gamma) = 2$, $d(\beta,\gamma) = 1$, and $g(\Gamma) = 5$.

In case (b), if $g(\Gamma) = 4$ and each pair α, β of vertices at distance 2 lies in exactly one cycle of length 4, then $\mathcal{C}_2(\alpha,\beta) \cong C_4$ and Γ is called a *rectagraph*. To our knowledge the first characterisation of a family of rectagraphs was given by Cameron [3, 4]. He proved that the only rectagraphs of valency k admitting a vertex-transitive group of automorphisms G, such that the stabiliser G_α of a vertex α induces A_k or S_k on the k vertices adjacent to α, are the k-cube Q_k and its antipodal quotient $\frac{1}{2} \cdot Q_k$. More generally rectagraphs arise as coset graphs of binary linear codes with minimum weight at least 5; for example, the k-cube Q_k is the coset graph of the zero code of length k. Mulder and Neumaier showed that each rectagraph of valency k has at most 2^k vertices with equality only in the case of the k-cube, see [2, 1.13.1] and the preceding comments. Also Brouwer [2, 4.3.6] studied rectagraphs of valency k for which the 2-arc closure of each 3-claw was a 3-cube, and showed that these are homomorphic images of the k-cube. (A 3-claw is a subgraph with four vertices $\alpha,\beta,\gamma,\delta$ and three edges $\{\alpha,\beta\},\{\alpha,\gamma\},\{\alpha,\delta\}$.) This result was applied to characterise the rectagraphs corresponding to the binary Golay codes of lengths 23 and 24; see [2, 11.3D & 11.3E].

Since in the case of rectagraphs every 2-arc closed subset generated by a pair of vertices at distance 2 is a 4-cycle C_4, sometimes called a quadrangle, such 2-arc closed subsets have sometimes been called *quads* in the literature. The term quad has also been used to denote the 2-arc closed subsets in each of the cases (a) to (c) above. There have been other interesting characterisations of families of graphs by the structure of their quads together with the local action of a vertex-stabiliser. A case of particular interest because of its importance for the class of distance transitive graphs is the family of locally projective graphs (see [10, Chapter 9]). A graph Γ is said to be *locally projective of type* (n, q) if the permutation group induced by a vertex-stabiliser G_α on the set $\Gamma(\alpha)$ of vertices adjacent to α contains a normal subgroup isomorphic to $\mathrm{PSL}_n(q)$ in its natural doubly transitive action on the points of the projective geometry. Locally projective graphs of girth 3 are complete graphs, while those of girth 4 were classified completely by Cameron and Praeger and Ching in [5, 6], and classification of locally projective graphs of girth 5 was begun by Ivanov in [9]. The latter required classification of the flag-transitive P-geometries [10, 12], and was effectively completed in [11]. The final part of the classification involved a detailed study of locally projective graphs of type $(3, 4)$. A key role was played by certain 2-arc closed subgraphs isomorphic to the Armanios-Wells graph on 32 vertices.

For most of the characterisations mentioned above the group action was assumed to be 2-arc transitive, so that all quads of Γ were isomorphic to each other. This is not the case in general, and we will give some examples in later sections from the class of partition graphs.

We will restrict our attention to graphs with the following properties:

Hypothesis 1 *The graph* $\Gamma = (V, E)$ *is finite and connected of girth* 3 *or* 4*, and admits a group* G *of automorphisms acting transitively on its arcs. Let the* G*-orbits on unordered pairs of vertices at distance* 2 *be* $P_1, \ldots, P_k$ *(where* $k \geq 1$*), and for each* i *let* $\{\alpha_i, \beta_i\} \in P_i$*. Then up to isomorphism the quad* $\mathcal{C}_2(\alpha_i, \beta_i)$ *is independent of the choice of the pair* $\{\alpha_i, \beta_i\} \in P_i$*, and we will denote it by* Quad_i*. If* $g(\Gamma) = 3$ *then we also let* Quad_0 *denote the quad generated by a pair of adjacent vertices.*

We begin by deriving in Section 2 some general properties of quads. The observations proved in Theorem 1 prompted questions about the possible nature and relationships between the various quads in finite arc-transitive graphs. In the rest of the paper we explore some sub-families of partition graphs, and demonstrate that many of the possibilities highlighted in Section 2 occur in these graphs.

2 General properties of quads

In this section we investigate relationships between the quads of graphs satisfying Hypothesis 1. One property of importance is whether or not the subgraph induced on the set $\Gamma(\alpha)$ of vertices adjacent to α is connected.

For a subset $X \subseteq V$, we denote by $\overline{X}$ the *subgraph of* Γ *induced on* X, that is, the graph with vertex set X such that if $\alpha, \beta \in X$ then $\{\alpha, \beta\}$ is an edge of $\overline{X}$ if and only if it is an edge of Γ. If it is clear from the context however, we will use Quad_i to denote both the set of vertices and also the subgraph induced on it. A *cycle* $C_n = (\alpha_1, \ldots, \alpha_n)$ is a graph with n vertices $\alpha_1, \ldots, \alpha_n$ and n edges $\{\alpha_i, \alpha_{i+1}\}$ for all i, where the subscripts are to be taken modulo n.

Theorem 1 *Suppose that Hypothesis 1 holds.*

(a) *If* $g(\Gamma) = 3$, *then* Quad_0 *is isomorphic to a subgraph of each* Quad_i, *and either* $\mathrm{Quad}_0 \cong \mathrm{Quad}_i$ *for some* $i \geq 1$, *or* Quad_0 *is complete.*

(b) *If* $\overline{\Gamma(\alpha)}$ *is connected, then* $\mathrm{Quad}_0 = \mathrm{Quad}_1 = \ldots = \mathrm{Quad}_k = V$.

(c) *If* Quad_i *contains a pair of vertices* $\{\alpha, \beta\} \in P_j$, *then* $\mathrm{Quad}_j \cong \mathcal{C}_2(\alpha, \beta)$ *and* $\mathcal{C}_2(\alpha, \beta) \subseteq \mathrm{Quad}_i$.

(d) *If* Γ *contains a cycle* $C_4 = (\alpha, \beta, \gamma, \delta)$ *such that* $\mathcal{C}_2(\alpha, \gamma) \cong \mathrm{Quad}_i$ *and* $\mathcal{C}_2(\beta, \delta) \cong \mathrm{Quad}_j$, *then* $\mathrm{Quad}_i \cong \mathrm{Quad}_j$.

(e) *For* $i \geq 0$, *the setwise stabiliser* G_{Quad_i} *of* Quad_i *is transitive on the set of pairs from* P_i *contained in* Quad_i. *Moreover,* G_{Quad_0} *is arc-transitive on* Quad_0.

Proof. (a) For $i \geq 1$, let $\gamma \in \Gamma(\alpha_i) \cap \Gamma(\beta_i)$. Then $\mathrm{Quad}_i = \mathcal{C}_2(\alpha_i, \beta_i)$ contains $\mathcal{C}_2(\alpha_i, \gamma) \cong \mathrm{Quad}_0$. If Quad_0 is not a complete graph, then Quad_0 contains a pair of vertices α, β at distance 2, in which case Quad_0 contains $\mathcal{C}_2(\alpha, \beta)$, but $\mathcal{C}_2(\alpha, \beta)$ is isomorphic to Quad_i where P_i is the orbit on unordered pairs containing $\{\alpha, \beta\}$, hence $\mathrm{Quad}_0 \cong \mathrm{Quad}_i$.

(b) Suppose that $\overline{\Gamma(\alpha)}$ is connected. By (a) it is sufficient to prove that $\mathrm{Quad}_0 = V$. Let $\mathrm{Quad}_0 = \mathcal{C}_2(\alpha, \beta)$ where $\beta \in \Gamma(\alpha)$. First we prove inductively that Quad_0 contains all vertices of $\Gamma(\alpha)$ which are at given distance l from β in $\overline{\Gamma(\alpha)}$. By the definition of Quad_0, this is true for $l = 1$. Suppose it is true for some l less than the diameter of $\overline{\Gamma(\alpha)}$, and let $\gamma \in \Gamma(\alpha)$ be at distance $l + 1$ from β in $\overline{\Gamma(\alpha)}$. Then there is a vertex $\delta \in \Gamma(\alpha) \cap \Gamma(\gamma)$ at distance l from β in $\overline{\Gamma(\alpha)}$, and by the inductive hypothesis Quad_0 contains δ. Hence Quad_0 contains $\mathcal{C}_2(\alpha, \delta)$, and as (α, γ, δ) is a 2-arc, Quad_0 also contains γ. Thus Quad_0 contains every vertex of $\Gamma(\alpha)$, and so by connectivity of Γ it follows that $\mathrm{Quad}_0 = V$.

(c) By definition $Q_j \cong \mathcal{C}_2(\alpha, \beta)$, and $\mathcal{C}_2(\alpha, \beta) \subseteq Q_i$ since $\alpha, \beta \in Q_i$.

(d) This follows immediately from part (c).

(e) If $\alpha, \beta, \alpha', \beta' \in \mathrm{Quad}_i$ with $\{\alpha, \beta\}, \{\alpha', \beta'\} \in P_i$, then it follows from part (c) that $\mathcal{C}_2(\alpha, \beta) = \mathcal{C}_2(\alpha', \beta') = \mathrm{Quad}_i$. Also by arc-transitivity $\{\alpha, \beta\} = \{\alpha', \beta'\}^g$ for some $g \in G$, so $\mathrm{Quad}_i^g = \mathcal{C}_2(\alpha', \beta')^g = \mathcal{C}_2(\alpha, \beta) = \mathrm{Quad}_i$, and hence G_{Quad_i} is transitive on the pairs from P_i contained in Quad_i. An analogous argument proves that G_{Quad_0} is transitive on the arcs of Quad_0. ■

Theorem 1 suggests some general questions about quads. For example, Quad_0 is a complete graph if Γ is a complete graph; however there are also incomplete graphs for which Quad_0 is a complete graph, and we will give examples from the class of partition graphs in Sections 5, 6 and 7. We find families of examples in which $\mathrm{Quad}_0 \cong K_3$ and K_{10}.

Question 1 *What can be said about incomplete arc-transitive graphs for which* Quad_0 *is a complete graph? In such graphs can* $\mathrm{Quad}_0 \cong K_n$ *for any given* $n > 3$*?*

In addition, all quads equal the full vertex set V if $\overline{\Gamma(\alpha)}$ is connected, but we shall give examples in Section 6 of a class of graphs for which $\overline{\Gamma(\alpha)}$ is disconnected and yet still $\mathrm{Quad}_i = V$ for all $i \geq 1$.

Question 2 *What can be said about arc-transitive graphs* Γ *for which* $\Gamma(\alpha)$ *is disconnected and* $\mathrm{Quad}_i = V$ *for some* i*?*

3 Partition graphs

The partition graphs form a family of arc-transitive graphs admitting a finite symmetric group S_n acting primitively on the vertices. For any composite positive integer n, let $\Omega_n = \{1, 2, \ldots, n\}$ be a set of size n on which $G = S_n$ acts naturally. For each graph $\Gamma = (V, E)$ in this class there is a factorisation $n = mr$ with $1 < r < n$ such that the vertex set V of Γ may be identified with the set of partitions of Ω_n having r parts of size m, hence the nomenclature *partition graphs*. The set of edges is a G-orbit on unordered pairs of these partitions. Originally this class of graphs arose in the problem of classifying distance transitive graphs with automorphism group a finite alternating or symmetric group [8]. In [7] several properties of partition graphs were considered, including their girth, and the local action of the stabiliser of a vertex α on its neighbourhood $\Gamma(\alpha)$.

Investigating the quads in some of the partition graphs of girth 3 drew our attention to some interesting sub-families, and suggested the general properties of quads described in the previous section. For the first sub-family $m = 2$, and for all graphs in this sub-family $\mathrm{Quad}_0 \cong K_3$ and $\overline{\Gamma(\alpha)}$ is a disjoint union of graphs K_2. There are two other quads which arise for graphs in this sub-family, one isomorphic to the Cartesian product graph $C_3 \times C_3$, and the other isomorphic to the complement of the Johnson graph $J(6, 2)$; details are given in Section 5. The second sub-family, which is described in Section 6, has $r = 3$, and again for all graphs in this sub-family $\overline{\Gamma(\alpha)}$ is a disjoint union of graphs K_2 and Quad_0 is the complete graph K_3. This time, however, all the quads Quad_i (for $i \geq 1$) are equal to V. Since in these two rather different sub-families the subgraphs Quad_0 are isomorphic to K_3, we would be interested to know exactly which partition graphs have this property, and also which other complete graphs arise as Quad_0 for partition graphs. In Section 7 we give a third sub-family of partition graphs for which $\mathrm{Quad}_0 \cong K_{10}$.

It was the discovery of these facts experimentally for small graphs in each sub-family, using MAGMA [1], which led us to some of the properties established in Theorem 1. Our findings also suggest the following questions for further investigation:

Question 3 *For which partition graphs* Γ *is the subgraph* $\overline{\Gamma(\alpha)}$ *connected?*

Question 4 *For which partition graphs* Γ *of girth* 3 *is* Quad_0 *a complete graph?*

Question 5 *For which partition graphs* Γ *is some* Quad_i *equal to* V*? When is* $\mathrm{Quad}_i = V$ *for all* $i \geq 1$*?*

4 Partition graphs and matrices

Let $n = mr$ with $1 < r < n$, and let $V(m,r)$ denote the set of partitions of the set $\Omega_n = \{1, 2, \ldots, n\}$ having r parts of size m. The vertex set V of each partition graph corresponding to this factorisation of n may be identified with $V(m,r)$. The edges for a particular graph in this class can be characterised by a class of matrices of order r whose row and column sums are equal to m.

Let $\alpha = \{x_1 \,|\, x_2 \,|\, \ldots \,|\, x_r\}$ and $\beta = \{y_1 \,|\, y_2 \,|\, \ldots \,|\, y_r\}$ be any two partitions of Ω_n with $|x_i| = |y_i| = m$ for $1 \leq i \leq r$. Form the matrix $M = (a_{ij})_{r\times r}$ where

$$a_{ij} = |x_i \cap y_j| \text{ for } 1 \leq i, j \leq r.$$

Clearly the matrix M depends on the orderings of the parts x_i and y_j. If $\alpha = \{x_{1\sigma} \,|\, x_{2\sigma} \,|\, \ldots \,|\, x_{r\sigma}\}$ and $\beta = \{y_{1\tau} \,|\, y_{2\tau} \,|\, \ldots \,|\, y_{r\tau}\}$ are the result of applying the inverses of permutations σ and $\tau \in S_r$ to the parts of α, β respectively, then the corresponding matrix will be $(a_{i\sigma,j\tau})_{r\times r}$, that is, the matrix obtained from M by applying to the rows and columns the permutations σ^{-1} and τ^{-1} respectively. Thus the pair (α, β) is associated with the class $[M]$ of all matrices obtainable from M by permuting the rows and columns. The map

$$(\sigma, \tau) : (a_{ij})_{r\times r} \mapsto (a_{i\sigma,j\tau})_{r\times r}$$

defines an action of $S_r \times S_r$ on the set $\mathcal{M}(m,r)$ of all $r \times r$ matrices with non-negative integer entries and row and column sums equal to m. The class $[M]$ is the orbit containing M under this action, and the set of such orbits labels the partition graphs with vertex set $V(m,r)$. Matrices in the same orbit under this action will be called *equivalent*.

The partition graph $\Gamma_{[M]}$ corresponding to $[M]$ is the directed graph with vertex set $V(m,r)$ such that (α, β) is a directed edge if and only if the matrix formed from α, β in the manner described above lies in $[M]$. Each such graph admits S_n acting arc-transitively. Also the graph obtained by reversing the orientation of each directed edge of $\Gamma_{[M]}$ is the partition graph associated with the orbit consisting of the transposes of the matrices in $[M]$. In particular if $[M]$ is closed under transposes, then $\Gamma_{[M]}$ can be regarded as an undirected graph. In the *trivial* case $M = mI_r$ (where I_r denotes the $r \times r$ identity matrix), we have $[M] = \{mI_r\}$ and the graph $\Gamma_{[M]}$ is degenerate, consisting of a loop on each vertex. The families of partition graphs which we will examine in this paper will all consist of undirected graphs corresponding to non-trivial $[M]$. We note that $|V(m,r)| = \frac{n!}{(m!)^r r!}$.

5 Some partition graphs with $m = 2$

Suppose that $n = mr$. The first partition graphs we consider are those for the case $m = 2$, corresponding to the matrices $N_r \in \mathcal{M}(2,r)$ given by

$$N_r = \begin{pmatrix} J_2 & 0 \\ 0 & 2I_{r-2} \end{pmatrix} \quad \text{for each integer } r \geq 4,$$

where J_s denotes the $s \times s$ matrix with all entries equal to 1. We determine the basic parameters and the quads for these graphs. We will order the quads Quad_i (for $1 \leq i \leq k$) according to their size.

For integers m, n satisfying $1 \leq m \leq n/2$, the *Johnson graph* $J(n,m)$ is the graph with vertices the m-element subsets of $\{1, 2, \ldots, n\}$, such that two vertices u and v are adjacent whenever $|u \cap v| = m - 1$. The *complement* Γ^c

of a graph $\Gamma = (V, E)$ is the graph with vertex set V such that a 2-element subset $\{\alpha, \beta\}$ of V is an edge of Γ^c if and only if $\{\alpha, \beta\} \notin E$. For graphs $\Gamma = (V, E)$ and $\Gamma' = (V', E')$, we define the *Cartesian product* graph $\Gamma \times \Gamma'$ as the graph with vertex set $V \times V'$ such that (α, α') is adjacent to (β, β') if and only if either $\alpha = \beta$ and $\{\alpha', \beta'\} \in E'$, or $\alpha' = \beta'$ and $\{\alpha, \beta\} \in E$.

Proposition 2 *Let $\Gamma = \Gamma_{[N_r]}$ with $r \geq 4$. Then Γ is an undirected graph having girth 3 and valency $r(r-1)$, and $\overline{\Gamma(\alpha)} \cong \binom{r}{2} K_2$. The group S_n has two orbits on unordered pairs of vertices at distance 2, and the quads satisfy $\mathrm{Quad}_0 \cong K_3$, $\mathrm{Quad}_1 \cong C_3 \times C_3$, and $\mathrm{Quad}_2 \cong J(6,2)^c$.*

PROOF. Since N_r is equal to its transpose it follows that Γ is undirected. Let $\alpha = \{x_1 \mid \ldots \mid x_r\} \in V(2, r)$. Then each vertex $\beta \in \Gamma(\alpha)$ is a partition of Ω_n with $r-2$ parts in common with α. Suppose that the parts $x_i = \{a_1, a_2\}$ and $x_j = \{b_1, b_2\}$ are not parts of β. Then either $\{a_1, b_1\}$ and $\{a_2, b_2\}$, or $\{a_1, b_2\}$ and $\{a_2, b_1\}$, are parts of β. Thus we have two vertices in $\Gamma(\alpha)$ corresponding to the parts x_i and x_j, say $\beta_{i,j}$ and $\beta'_{i,j}$, and these are adjacent in Γ. For example if $(i, j) = (1, 2)$ these two vertices are

$$\beta_{1,2} = \{a_1, b_1 \mid a_2, b_2 \mid x_3 \mid \ldots \mid x_r\} \quad \text{and} \quad \beta'_{1,2} = \{a_1, b_2 \mid a_2, b_1 \mid x_3 \mid \ldots \mid x_r\},$$

and we notice that $\{\beta_{1,2}, \beta'_{1,2}\}$ is also an edge of Γ. Moreover there are no other edges between vertices of $\Gamma(\alpha)$, as any other vertex in $\Gamma(\alpha)$ would differ in at least three parts from each of $\beta_{i,j}$ and $\beta'_{i,j}$. This proves the assertions in the second sentence of the proposition, and also shows that $\mathrm{Quad}_0 \cong K_3$.

If (α, β, γ) is a 2-arc in Γ such that α, γ are at distance 2, then α, γ must differ in either 3 or 4 of their parts. Moreover $G_\alpha = S_2 \wr S_r$ has two orbits on vertices at distance 2 from α, one for each of these two types (depending on whether the vertes differs from α in 3 or 4 parts), and we therefore have two quads generated by pairs of vertices at distance 2 to determine. Suppose first that α, γ differ in four parts. Without loss of generality we may suppose that $\beta = \beta_{1,2}$, and that α, γ differ in the first four parts. Then the matrix

M_1 corresponding to the pair α, γ is

$$M_1 = \begin{pmatrix} J_2 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & 2I_{r-4} \end{pmatrix}.$$

Writing

$$\alpha = \{a_1, a_2 \mid b_1, b_2 \mid c_1, c_2 \mid d_1, d_2 \mid \mathbf{x}\},$$

where $\mathbf{x}$ denotes the remaining $r-4$ parts of α, we may take

$$\gamma = \{a_1, b_1 \mid a_2, b_2 \mid c_1, d_1 \mid c_2, d_2 \mid \mathbf{x}\}.$$

Now $\mathrm{Quad}_1 = \mathcal{C}_2(\alpha, \gamma)$ contains both the vertices

$$\beta_{1,2} = \{a_1, b_1 \mid a_2, b_2 \mid c_1, c_2 \mid d_1, d_2 \mid \mathbf{x}\}, \quad \beta_{3,4} = \{a_1, a_2 \mid b_1, b_2 \mid c_1, d_1 \mid c_2, d_2 \mid \mathbf{x}\}$$

in $\Gamma(\alpha) \cap \Gamma(\gamma)$, and hence also contains the four vertices

$$\beta'_{1,2} = \{a_1, b_2 \mid a_2, b_1 \mid c_1, c_2 \mid d_1, d_2 \mid \mathbf{x}\}, \quad \beta'_{3,4} = \{a_1, a_2 \mid b_1, b_2 \mid c_1, d_2 \mid c_2, d_1 \mid \mathbf{x}\},$$

$$\gamma' = \{a_1, b_1 \mid a_2, b_2 \mid c_1, d_2 \mid c_2, d_1 \mid \mathbf{x}\}, \quad \delta = \{a_1, b_2 \mid a_2, b_1 \mid c_1, d_1 \mid c_2, d_2 \mid \mathbf{x}\},$$

which form triangles on the four edges of the 4-cycle $(\alpha, \beta_{1,2}, \gamma, \beta_{3,4})$. It also contains the vertex

$$\delta' = \{a_1, b_2 \mid a_2, b_1 \mid c_1, d_2 \mid c_2, d_1 \mid \mathbf{x}\}$$

which forms a triangle on the edge $\{\beta'_{1,2}, \delta\}$ (and one on the edge $\{\beta'_{3,4}, \gamma'\}$). Thus $\mathcal{C}_2(\alpha, \gamma)$ contains the set $X := \{\alpha, \beta_{1,2}, \beta'_{1,2}, \beta_{3,4}, \beta'_{3,4}, \gamma, \gamma', \delta, \delta'\}$ of nine vertices. It is not difficult to see that the induced subgraph $\overline{X}$ is isomorphic to $C_3 \times C_3$. Let $H = S_4 \wr S_2$ be the subgroup of S_8 which leaves invariant the partition $\{a_1, a_2, b_1, b_2 \mid c_1, c_2, d_1, d_2\}$ of the set $\Omega_8 := \{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2\}$, and consider H as a subgroup of G leaving $\Omega_n \setminus \Omega_8$ fixed pointwise. Then X is an orbit of H in $V(2, r)$. We claim that there are no vertices of $V(2, r) \setminus X$ which are adjacent to two vertices of X. Since H is transitive on X it is sufficient to prove that no vertex $\beta \in \Gamma(\alpha) \setminus X$ is joined to a vertex of $X \setminus \{\alpha\}$.

As $\overline{\Gamma(\alpha)} \cong \binom{r}{2} K_2$, we see that β is not joined to any of the four vertices $\beta_{1,2}, \beta'_{1,2}, \beta_{3,4}, \beta'_{3,4}$ in $\Gamma(\alpha)$, and as each of $\gamma, \gamma', \delta, \delta'$ must be joined to exactly two vertices of $\Gamma(\alpha)$, and these vertices are all in X, the claim follows. Thus we have $\mathrm{Quad}_1 = \mathcal{C}_2(\alpha, \gamma) = X \cong C_3 \times C_3$.

Now we turn to the case of the quad $\mathrm{Quad}_2 = \mathcal{C}_2(\alpha, \gamma)$, where this time α, γ differ in exactly three parts. This time we write

$$\alpha = \{a_1, a_2 \mid b_1, b_2 \mid c_1, c_2 \mid \mathbf{x}\},$$

where $\mathbf{x}$ denotes the remaining $r-3$ parts of α, and take

$$\gamma = \{a_1, b_2 \mid b_1, c_2 \mid c_1, a_2 \mid \mathbf{x}\}.$$

Then the matrix M_2 corresponding to the pair α, γ is

$$M_2 = \begin{pmatrix} Y_3 & 0 \\ 0 & 2I_{r-3} \end{pmatrix} \quad \text{where} \quad Y_3 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

The quad $\mathrm{Quad}_2 = \mathcal{C}_2(\alpha, \gamma)$ contains the three vertices

$$\begin{aligned} \beta'_{1,2} &= \{a_1, b_2 \mid b_1, a_2 \mid c_1, c_2 \mid \mathbf{x}\}, \\ \beta'_{2,3} &= \{a_1, a_2 \mid b_1, c_2 \mid c_1, b_2 \mid \mathbf{x}\}, \\ \beta'_{1,3} &= \{a_1, c_2 \mid b_1, b_2 \mid c_1, a_2 \mid \mathbf{x}\} \end{aligned}$$

of $\Gamma(\alpha) \cap \Gamma(\gamma)$, and hence it also contains the six vertices

$$\begin{aligned} \beta_{1,2} &= \{a_1, b_1 \mid a_2, b_2 \mid c_1, c_2 \mid \mathbf{x}\}, \\ \beta_{2,3} &= \{a_1, a_2 \mid b_1, c_1 \mid b_2, c_2 \mid \mathbf{x}\}, \\ \beta_{1,3} &= \{a_1, c_1 \mid b_1, b_2 \mid a_2, c_2 \mid \mathbf{x}\}, \\ \gamma'_{1,2} &= \{a_1, b_2 \mid b_1, c_1 \mid c_2, a_2 \mid \mathbf{x}\}, \\ \gamma'_{2,3} &= \{a_1, c_1 \mid b_1, c_2 \mid a_2, b_2 \mid \mathbf{x}\}, \\ \gamma'_{1,3} &= \{a_1, b_1 \mid b_2, c_2 \mid a_2, c_1 \mid \mathbf{x}\}, \end{aligned}$$

which form triangles on the six edges which make up the three 2-arcs from α to γ. It also contains the vertices

$$\begin{aligned} \gamma_{1,2} &= \{a_1, b_1 \mid a_2, c_2 \mid b_2, c_1 \mid \mathbf{x}\}, \\ \gamma_{2,3} &= \{a_1, c_2 \mid b_1, c_1 \mid a_2, b_2 \mid \mathbf{x}\}, \\ \gamma_{1,3} &= \{a_1, c_1 \mid b_1, a_2 \mid b_2, c_2 \mid \mathbf{x}\}, \end{aligned}$$

which form triangles on the edges $\{\beta_{1,2}, \gamma'_{1,3}\}, \{\beta_{2,3}, \gamma'_{1,2}\}, \{\beta_{1,3}, \gamma'_{2,3}\}$ respectively, and finally it contains

$$\gamma' = \{a_1, c_2 \mid b_1, a_2 \mid c_1, b_2 \mid \mathbf{x}\},$$

which is adjacent to all three of $\gamma_{1,2}, \gamma_{2,3}, \gamma_{1,3}$. Thus $\mathcal{C}_2(\alpha, \gamma)$ contains the set X consisting of all the 15 vertices of Γ which have $x_4, \ldots, x_r$ as parts. In this case the setwise stabiliser H in G of $\Omega_6 := \{a_1, a_2, b_1, b_2, c_1, c_2\}$ has X as an orbit in its action on $V(2,r)$. We claim that there are no vertices of $V(2,r) \setminus X$ which are adjacent to two vertices of X. Since H is transitive on X it is sufficient to prove that no vertex $\beta \in \Gamma(\alpha) \setminus X$ is joined to a vertex of $X \setminus \{\alpha\}$. As $\overline{\Gamma(\alpha)} \cong \binom{r}{2} K_2$, we see that β is not joined to any of the six vertices $\beta_{1,2}, \beta'_{1,2}, \beta_{2,3}, \beta'_{2,3}, \beta_{1,3}, \beta'_{1,3}$ in $\Gamma(\alpha)$, and as each of the vertices $\gamma, \gamma', \gamma_{i,j}$ and $\gamma'_{i,j}$ (for $1 \leq i < j \leq 3$) must be joined to exactly three vertices of $\Gamma(\alpha)$, and all of these vertices are known to lie in X, the claim follows. Thus we have $\mathrm{Quad}_2 = \mathcal{C}_2(\alpha, \gamma) = X$.

We have seen that the vertex set X of Quad_2 may be identified with the set of 15 partitions of the 6-set Ω_6 into 3 parts of size 2; such partitions are called *synthemes* of Ω_6. Moreover under this identification we have seen that two synthemes in X are adjacent in Quad_2 if and only if they have one part in common. Now Quad_2 admits the group H acting arc-transitively as $H^X \cong S_6$ on synthemes. To identify Quad_2 with the complement of the Johnson graph $J(6,2)$, we apply an outer automorphism σ of S_6. Such a map σ interchanges synthemes and unordered pairs from Ω_6, and maps two synthemes with a common part to two unordered pairs with no common element. Thus σ induces an isomorphism from Quad_2 to $J(6,2)^c$. ■

This result prompts one to ask whether these partition graphs might be the only arc-transitive graphs for which the quads are precisely $K_3, C_3 \times C_3$ and $J(6,2)^c$. One might begin to understand the situation by asking this question in the case where an arc-transitive automorphism group induces the same local action as for the partition graphs $\Gamma_{[N_r]}$.

Question 6 *If Γ is a connected G-arc-transitive graph of girth 3 and valency $r(r-1)$, for some $r \geq 4$, such that each quad is $K_3, C_3 \times C_3$ or $J(6,2)^c$, and such that G_α induces the natural action of $S_2 \wr S_r$ on $\Gamma(\alpha) = \binom{r}{2} K_2$, then is Γ isomorphic to the partition graph $\Gamma_{[N_r]}$?*

Remark 3 The proof of Proposition 2 shows that $\Gamma = \Gamma_{[N_3]}$ is isomorphic to $J(6,2)^c$, and that in this case $\Gamma(\alpha) = 3 \cdot K_2$, $\mathrm{Quad}_0 = K_3$, and there is only one orbit on pairs of vertices α, β at distance 2, giving $\mathrm{Quad}_1 = \mathcal{C}_2(\alpha,\beta) = V$.

6 Some partition graphs with $r = 3$

In this section we investigate the partition graphs $\Gamma_{[M_m]}$, where

$$M_m = \begin{pmatrix} m-1 & 1 & 0 \\ 0 & m-1 & 1 \\ 1 & 0 & m-1 \end{pmatrix} \quad \text{for each integer } m \geq 2.$$

For $m = 2$ this graph is the complement of $\Gamma_{[N_3]}$, and hence by Remark 3, $\Gamma_{[M_2]} \cong J(6,2)$. This graph has just two quads, namely $\mathrm{Quad}_0 = K_3$ and $\mathrm{Quad}_1 = \Gamma_{[M_2]}$. We note that $\Gamma_{[M_2]}$ is somewhat degenerate within this sub-family since its valency is $8 = 2^3$, whereas the valency of $\Gamma_{[M_m]}$ turns out to be $2m^3$ for all $m \geq 3$. A computational investigation of the quads of $\Gamma_{[M_m]}$ for small values of m using MAGMA suggested to us that certain properties for graphs in this sub-family might hold for all m, leading to the following result. Its proof will be followed by further observations about the degenerate cases $m = 2$ and 3.

Proposition 4 *Let $\Gamma = \Gamma_{[M_m]}$ where $m \geq 4$. Then Γ is an undirected graph having girth 3, valency $2m^3$, and $\overline{\Gamma(\alpha)} \cong m^3 \cdot K_2$, so $\mathrm{Quad}_0 \cong K_3$. The group S_n has five orbits on unordered pairs of vertices at distance 2, and $\mathrm{Quad}_i = \Gamma$ for $1 \leq i \leq 5$.*

PROOF. First we observe that M_m is equivalent to its transpose (which can be seen by interchanging columns 1 and 3, and then interchanging rows 1 and 3), and so Γ is an undirected graph. Let $\alpha = \{A \mid B \mid C\} \in V(m,3)$ where $|A| = |B| = |C| = m$. For each of the m^3 choices of $(a,b,c) \in A \times B \times C$, the two vertices

$$\begin{aligned} \beta_{abc} &= \{A \setminus \{a\} \cup \{b\} \mid B \setminus \{b\} \cup \{c\} \mid C \setminus \{c\} \cup \{a\}\} \\ \beta'_{abc} &= \{A \setminus \{a\} \cup \{c\} \mid B \setminus \{b\} \cup \{a\} \mid C \setminus \{c\} \cup \{b\}\} \end{aligned}$$

lie in $\Gamma(\alpha)$, and β_{abc} is adjacent to β'_{abc}. As $m \geq 3$, these $2m^3$ vertices are pairwise distinct, and we conclude that Γ has girth 3 and valency $2m^3$.

We claim that there are no edges in $\Gamma(\alpha)$ apart from the edges between the pairs of vertices $\beta_{abc}, \beta'_{abc}$. Since G acts transitively on arcs of Γ, it is sufficient to fix a triple (a,b,c) and prove that β_{abc} is not adjacent to any vertex $\beta_{a'b'c'}$ or $\beta'_{a'b'c'}$ with $(a',b',c') \neq (a,b,c)$. Without loss of generality we may assume that $a' \neq a$. Then the first parts of β_{abc} and $\beta'_{a'b'c'}$ intersect in a subset of size $m-2$ and so these vertices are not adjacent. For the same reason β_{abc} and $\beta_{a'b'c'}$ are not adjacent if $b \neq b'$, so suppose that $b = b'$. If also $c = c'$ then the second parts of β_{abc} and $\beta_{a'b'c'}$ will be equal and so these vertices will not be adjacent, while if $c \neq c'$, then the third parts of β_{abc} and $\beta_{a'b'c'}$ have exactly $m-2$ points in common, so again they are not adjacent. Thus the claim is proved, and we have established that $\overline{\Gamma(\alpha)} \cong m^3 \cdot K_2$ and that $\mathrm{Quad}_0 \cong K_3$.

Next we note that the five orbits of S_n on unordered pairs of vertices at distance 2 correspond to the following matrices $M_m^{(i)}$, for $1 \leq i \leq 5$:

$$M_m^{(1)} = \begin{pmatrix} m-2 & 1 & 1 \\ 1 & m-2 & 1 \\ 1 & 1 & m-2 \end{pmatrix}, \quad M_m^{(2)} = \begin{pmatrix} m-2 & 2 & 0 \\ 0 & m-2 & 2 \\ 2 & 0 & m-2 \end{pmatrix},$$

$$M_m^{(3)} = \begin{pmatrix} m-1 & 0 & 1 \\ 0 & m & 0 \\ 1 & 0 & m-1 \end{pmatrix}, \quad M_m^{(4)} = \begin{pmatrix} m-2 & 1 & 1 \\ 1 & m-1 & 0 \\ 1 & 0 & m-1 \end{pmatrix},$$

$$M_m^{(5)} = \begin{pmatrix} m-2 & 2 & 0 \\ 1 & m-2 & 1 \\ 1 & 0 & m-1 \end{pmatrix}.$$

As $m \geq 4$, clearly these five matrices are pairwise inequivalent.

Now write $A = \overline{A} \cup \{a, a'\}, B = \overline{B} \cup \{b, b'\}$, and $C = \overline{C} \cup \{c, c'\}$. Then there is a path of length 2 from α to each of the following five vertices:

$$\begin{aligned}
\gamma_1 &= \{\overline{A} \cup \{b\} \cup \{c\} \mid \overline{B} \cup \{a\} \cup \{c'\} \mid \overline{C} \cup \{a'\} \cup \{b'\}\},\\
\gamma_2 &= \{\overline{A} \cup \{b\} \cup \{b'\} \mid \overline{B} \cup \{c\} \cup \{c'\} \mid \overline{C} \cup \{a\} \cup \{a'\}\},\\
\gamma_3 &= \{\overline{A} \cup \{a'\} \cup \{c\} \mid \overline{B} \cup \{b\} \cup \{b'\} \mid \overline{C} \cup \{a\} \cup \{c'\}\},\\
\gamma_4 &= \{\overline{A} \cup \{b\} \cup \{c\} \mid \overline{B} \cup \{a\} \cup \{b'\} \mid \overline{C} \cup \{a'\} \cup \{c'\}\},\\
\gamma_5 &= \{\overline{A} \cup \{b\} \cup \{b'\} \mid \overline{B} \cup \{a\} \cup \{c\} \mid \overline{C} \cup \{a'\} \cup \{c'\}\}.
\end{aligned}$$

Moreover, for each i, the matrix corresponding to the pair (γ_i, α) is $M_m^{(i)}$, and it follows that the γ_i lie in pairwise distinct G_α-orbits of vertices at distance 2 from α, for $1 \leq i \leq 5$. To see that these are the only G_α-orbits of vertices at distance 2 from α, consider an arbitrary such vertex γ. Because of the definition of adjacency in Γ we may order the parts of γ so that, for $j \in \{1, 2, 3\}$, the j^{th} parts of α and γ have at least $m - 2$ common points. The only matrices in $\mathcal{M}(m, 3)$ with each diagonal entry at least $m - 2$ are equivalent to one of the matrices $M_m, M_m^{(1)}, \ldots, M_m^{(5)}$ or to

$$M_m^{(6)} := \begin{pmatrix} m-2 & 2 & 0 \\ 2 & m-2 & 0 \\ 0 & 0 & m \end{pmatrix}.$$

If however (γ, α) corresponds to $M_m^{(6)}$, then it is not difficult to show that there is no path of length 2 joining γ to α.

It remains to prove that the quad Quad_i generated by $\{\alpha, \gamma_i\}$ is equal to Γ, for $1 \leq i \leq 5$. We first apply the general result Theorem 1. There are eight 2-arcs from α to γ_2, the intermediate vertices being β_{xyz} for each choice of $x \in \{a, a'\}, y \in \{b, b'\}, z \in \{c, c'\}$. The matrices corresponding to the pairs (β, β_{abc}), for $\beta = \beta_{a'b'c'}, \beta_{a'bc}$ and $\beta_{a'b'c}$, are $M_m^{(1)}, M_m^{(3)}$ and $M_m^{(4)}$ respectively. Hence, by part (d) of Theorem 1, Quad_2 is isomorphic to each of $\mathrm{Quad}_1, \mathrm{Quad}_3$ and Quad_4. Similarly, considering the 4-cycle $(\alpha, \beta'_{abc}, \gamma, \beta'_{ab'c})$ where $\gamma = \{\overline{A} \cup \{a\} \cup \{c'\} \mid \overline{B} \cup \{a\} \cup \{c\} \mid \overline{C} \cup \{b\} \cup \{b'\}\}$, we see that the matrices corresponding to the pairs (γ, α) and $(\beta'_{ab'c}, \beta'_{abc})$ are equivalent to

$M_m^{(5)}$ and $M_m^{(3)}$ respectively, and hence $\mathrm{Quad}_3 \cong \mathrm{Quad}_5$. Thus all five Quad_i (for $1 \le i \le 5$) are isomorphic to each other.

Finally we use some group theory to help prove that all these quads are equal to Γ. Set $Q := \mathcal{C}_2(\alpha, \gamma_2)$, and let G_Q denote the setwise stabiliser of Q in G. Then G_Q contains $G_{\alpha\gamma_2} = (S_{m-2} \times S_2) \wr A_3$, with orbits $\overline{A} \cup \overline{B} \cup \overline{C}$ and $\{a, a', b, b', c, c'\}$ in Ω_n. In the previous paragraph we saw that Q contains β_{xyz} for each choice of $x \in \{a, a'\}, y \in \{b, b'\}, z \in \{c, c'\}$, and the argument there shows in particular that $Q = \mathcal{C}_2(\beta_{abc}, \beta_{abc'})$. Hence G_Q contains $G_{\alpha, \beta_{abc}, \beta_{abc'}}$, which contains $\mathrm{Sym}(\overline{A} \cup \{a'\}) \times \mathrm{Sym}(\overline{B} \cup \{b'\})$. It follows that $G_Q \cap G_\alpha$ contains $S_m \wr A_3$. Similarly $Q = \mathcal{C}_2(\beta_{abc}, \beta_{a'b'c'})$, and hence G_Q contains any permutation of Ω_n which interchanges β_{abc} and $\beta_{a'b'c'}$. Thus G_Q contains any permutation which interchanges $\overline{A}$ and $\overline{B}$, and acts on $\{a, a', b, b', c, c'\}$ as $(a, c)(a', c')$. It follows that $G_Q = G$, and hence that $Q = \Gamma$. ■

For $m = 2$ and $m = 3$ the situation is as follows. We use the same notation $M_m^{(i)}$ as in the proof of Proposition 4 for matrices corresponding to unordered pairs of vertices at distance 2.

Remark 5 As we noted earlier, for $m = 2$ the graph $\Gamma_{[M_2]}$ is isomorphic to $J(6, 2)$, and for this case we find that $[M_2^{(1)}] = [M_2^{(4)}] = [M_2]$, while $[M_2^{(2)}] = [2I_3]$ is trivial, and $[M_2^{(3)}] = [M_2^{(5)}]$, giving $\mathrm{Quad}_1 = \Gamma_{[M_2]}$.

Remark 6 For $m = 3$, the graph $\Gamma_{[M_3]}$ has 280 vertices. Here we find that $[M_3^{(4)}] = [M_3^{(5)}]$, giving four rather than five equivalence classes of matrices, corresponding to four G-orbits on unordered pairs of vertices at distance 2. Computation using MAGMA showed that each of the four quads corresponding to these four orbits is equal to the whole graph $\Gamma_{[M_3]}$, and that $\overline{\Gamma_{[M_3]}(\alpha)}$ is a connected, bipartite, regular graph (of order 54), with valency 9, diameter 3 and girth 4. In particular, if $\alpha = \{A \mid B \mid C\}$ where $|A| = |B| = |C| = 3$, and for each of the 27 choices of $(a, b, c) \in A \times B \times C$, we define β_{abc} and β'_{abc} as in the proof of Proposition 4, then the 27 edges $\{\beta_{abc}, \beta'_{abc}\}$ form a perfect matching in $\overline{\Gamma_{[M_3]}(\alpha)}$, and the 27-vertex quotient of $\overline{\Gamma_{[M_3]}(\alpha)}$ formed by removing the edges of this perfect matching and identifying each vertex

β_{abc} with its partner β'_{abc}, is isomorphic to the 8-regular graph on vertex-set $\mathrm{GF}(3)^3$ with two vertices (x, y, z) and (u, v, w) adjacent if and only if they differ in all three coordinates.

7 A family of examples where Quad_0 is K_{10}

We complete this paper by giving a family of examples of partition graphs for which Quad_0 is a complete graph on 10 vertices (see Question1). This family consists of the partition graphs $\Gamma_{[H_r]}$, where

$$H_r = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3I_{r-2} \end{pmatrix} \quad \text{for each integer } r \geq 3.$$

By arc-transitivity, the vertices of any arc (α, β) in $\Gamma_{[H_m]}$ are of the form

$$\alpha = \{a_1, a_2, a_3 \mid b_1, b_2, b_3 \mid \mathbf{x}\} \quad \text{and} \quad \beta = \{a_1, a_2, b_3 \mid b_1, b_2, a_3 \mid \mathbf{x}\},$$

where $\mathbf{x}$ denotes the remaining $r-2$ parts of α. Now if γ is any vertex in $\Gamma(\alpha) \cap \Gamma(\beta)$, then γ must have exactly $r-2$ parts in common with each of α and β. It follows that the $r-2$ parts of $\mathbf{x}$ are also parts of γ, and that every such vertex γ is of the form $\{\mathbf{u} \mid \mathbf{v} \mid \mathbf{x}\}$ where $\{\mathbf{u} \mid \mathbf{v}\}$ is a partition of $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ into two subsets of size 3. There are $\binom{5}{2} = 10$ vertices of this form (including both α and β), and moreover, any two of them are adjacent in $\Gamma_{[H_r]}$. It follows that $\mathrm{Quad}_0 \cong C_2(\alpha, \beta)$ is a complete graph on 10 vertices.

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