Pseudo-real Riemann surfaces and chiral regular maps

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1 Introduction

A Riemann surface is called *pseudo-real* if it admits anticonformal automorphisms but no anticonformal involution. Pseudo-real Riemann surfaces appear in a natural way in the study of the moduli space $\mathcal{M}_g^K$ of Riemann surfaces considered as Klein surfaces. The moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g is a two-fold branched covering of $\mathcal{M}_g^K$, and the preimage of the branched locus consists of the Riemann surfaces admitting anticonformal automorphisms — which are either real Riemann surfaces (admitting anticonformal involutions) or pseudo-real Riemann surfaces. We study the latter surfaces in this work. Pseudo-real Riemann surfaces are Riemann surfaces that are equivalent to their conjugate but do not have any anticonformal involution. Note that pseudo-real Riemann surfaces have non-trivial conformal automorphisms, hence the locus of pseudo-real Riemann surfaces in $\mathcal{M}_g$ is contained in the singular set of the orbifold $\mathcal{M}_g = \mathbb{T}_g/\mathrm{Mod}_g$.

In Section 3 we study general properties of the automorphism groups of these surfaces and the related uniformizing NEC groups. In particular, we prove that there are pseudo-real Riemann surfaces of every genus $g \geq 2$. In Section 4 we study pseudo-real surfaces of genus 2 and 3.

In recent years a vast literature has been published about real Riemann surfaces — that is, Riemann surfaces admitting anticonformal involutions — but very little is known about pseudo-real surfaces. A study of hyperelliptic pseudo-real Riemann surfaces was made in [S] and [BC], and pseudo-real surfaces with cyclic automorphism group were investigated in [E]. In Section 5 we study pseudo-real cyclic p-gonal Riemann surfaces, generalizing the results obtained in [BC] for hyperelliptic surfaces.

In Section 6 we consider the maximal order of the automorphism group of a pseudo-real Riemann surface (relative to its genus), and we establish a connection between pseudo-real surfaces with maximal automorphism group, and chiral 3-valent regular maps. Finally in Section 7 we show there exist pseudo-real surfaces with automorphism group of maximal order for infinitely many genera, by proving the existence of concrete infinite families of chiral regular maps of type $\{3, k\}$ for $k \geq 7$.

2 Preliminaries on Fuchsian and NEC groups

A *non-Euclidean crystallographic group* (or *NEC group*) is a discrete group of isometries of the hyperbolic plane $\mathbb{D}$. We shall assume that an NEC group has a compact orbit space. If Γ is such a group then its algebraic structure is determined by its signature

$$(h; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}).$$

The orbit space $\mathbb{D}/\Gamma$ is a surface, possibly with boundary. The number h is called the *genus* of Γ and equals the topological genus of $\mathbb{D}/\Gamma$, while k is the number of its boundary components, and the sign is $+$ or $-$ according to whether or not the surface is orientable. The integers $m_i \geq 2$ are called the *proper periods*, and represent the branched indices over interior points of $\mathbb{D}/\Gamma$ in the natural projection $\pi : \mathbb{D} \to \mathbb{D}/\Gamma$. The bracketed expressions $(n_{i1}, \ldots, n_{is_i})$, some or all of which may be empty (with $s_i = 0$), are called the *period cycles* and represent the branchings over the i^{th} hole in the surface, and the numbers $n_{ij} \geq 2$ are the *link periods*.

Associated with each signature [BEGG] there exists a canonical presentation for the group Γ, and a formula for the hyperbolic area of its fundamental domain. If the signature has sign $+$ then Γ has the following generators:

$x_1, \ldots, x_r$ (elliptic elements),

$c_{10}, \ldots, c_{1s_1}, \ldots, c_{k0}, \ldots, c_{ks_k}$ (reflections),

$e_1, \ldots, e_k$ (boundary transformations),

$a_1, b_1, \ldots, a_g, b_g$ (hyperbolic elements);

and these generators satisfy the defining relations

$x_i^{m_i} = 1$ (for $1 \leq i \leq r$),

$c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1}c_{ij})^{n_{ij}} = 1,\ c_{is_i} = e_i^{-1}c_{i0}e_i$ (for $1 \leq i \leq k, 0 \leq j \leq s_i$),

$x_1 \ldots x_r e_1 \ldots e_k a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_h b_h a_h^{-1} b_h^{-1} = 1.$

If the sign is $-$ then we just replace the hyperbolic generators a_i, b_i by glide reflections $d_1, \ldots, d_h$, and the last relation by $x_1 \ldots x_r e_1 \ldots e_k d_1^2 \ldots d_h^2 = 1$.
The hyperbolic area of an arbitrary fundamental region of an NEC group Γ with signature is given by

$$\mu(\Gamma) = 2\pi \left(\varepsilon h - 2 + k + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{s_i} \left(1 - \frac{1}{n_{ij}}\right)\right),$$

where $\varepsilon = 2$ if the sign is $+$, and $\varepsilon = 1$ if the sign is $-$. Furthermore, any discrete group Λ of isometries of $\mathbb{D}$ containing Λ as a subgroup of finite index is also an NEC group, and the hyperbolic area of a fundamental region for Λ is given by the Riemann-Hurwitz formula:

$$[\Lambda : \Gamma] = \mu(\Gamma)/\mu(\Lambda).$$

For any NEC group Λ, let Λ^+ denote the subgroup of orientation-preserving elements of Λ, called the *canonical Fuchsian subgroup* of Λ. If $\Lambda^+ \neq \Lambda$ then Λ^+ has index 2 in Λ and we say that Λ is a *proper* NEC group.
Let X be a compact Riemann surface of genus $g > 1$. Then there is a Fuchsian surface group Γ (that is, an NEC group with signature $(g; +; [-]; \{-\}))$) such that $X = \mathbb{D}/\Gamma$, and the automorphism group $\mathrm{Aut}(X)$ of X is isomorphic to Δ/Γ, where Δ is an NEC group containing Γ. We denote by $\mathrm{Aut}^+(X)$ the group Δ^+/Γ of all orientation-preserving automorphisms of X .

3 Pseudo-real Riemann surfaces

Definition 1 *An anticonformal involution is an orientation-reversing automorphism of order 2. A Riemann surface is called pseudo-real if it admits anticonformal automorphisms but has no anticonformal involution.*

Proposition 2 *Let X be a pseudo-real Riemann surface, and let G be the group of conformal and anticonformal automorphisms of X. Then 4 divides the order of G.*

Proof. Let g be any anticonformal automorphism of X. If g has order $2^a m$ where m is odd, then g^m is an anticonformal automorphism of order 2^a, and then $a > 1$ because X is pseudo-real. Thus G has an element g^m of order divisible by 4. $\square$

Theorem 3 *Suppose the pseudo-real surface X is conformally equivalent to $\mathbb{D}/\Gamma$, where Γ is a surface Fuchsian group, and Γ is normalized by an NEC group Δ such that $\Delta/\Gamma \cong G = \mathrm{Aut}(X)$. Then the signature of Δ has the form $(p; -; [m_1, ..., m_r])$, and if Δ^+ is the canonical Fuchsian subgroup of Δ, then the signature of Δ^+ is*

$$(p-1; +; [m_1, m_1, m_2, m_2, ..., m_r, m_r]).$$

Proof. Since G has anticonformal automorphisms, the signature of Δ is the signature of a proper NEC group — that is, a group with anticonformal transformations. Thus the signature of Δ has the form

$$(p; -; [m_1, ..., m_r]) \quad \text{or} \quad (p; \pm; [m_1, ..., m_r]; \{(n_{i1}, ..., n_{ik_i})_{i=1,...,l}\}).$$

Note that in the second case, Δ contains reflections. Now let us consider the monodromy epimorphism

$$\theta : \Delta \to \Delta/\Gamma \cong G,$$

which sends anticonformal transformations to anticonformal automorphisms. If Δ contains reflections, that is, if Δ has signature

$$(p; \pm; [m_1, ..., m_r]; \{(n_{i1}, ..., n_{ik_i})_{i=1,...,l}\})$$

where $l > 0$, then the image by θ of a reflection is an anticonformal involution in G, hence the signature of Δ must be of the form $(p; -; [m_1, ..., m_r])$, with no boundary components. The signature of Δ^+ can now be obtained from the signature of Δ using the Riemann-Hurwitz formula and [BEGG]. $\square$

Theorem 4 *For every integer $g \geq 2$, there exist pseudo-real surfaces of genus g.*

Proof. Let Δ be an NEC group with signature $(\delta; -; [2, \overset{g+\varepsilon}{\ldots}, 2])$, where $\delta = \varepsilon = 1$ if g is even, or $\delta = 2$ and $\varepsilon = -1$ if g is odd. Let x_i (for $1 \leq i \leq g + \varepsilon$) and d_j (for $1 \leq j \leq \delta$) be a canonical system of generators of Δ. We may define an epimorphism $\theta : \Delta \to \mathbb{Z}_4 = \langle a : a^4 = 1 \rangle$ by setting

$$\theta(x_i) = a^2 \text{ for } 1 \le i \le g + \varepsilon, \quad \text{and } \theta(d_j) = a \text{ for } 1 \le j \le \delta.$$

Then $X = \mathbb{D}/\ker\theta$ is a Riemann surface such that $\Delta/\ker\theta \cong \mathbb{Z}_4$ is a group of automorphisms of X. Choosing Δ to be maximal (see [BCF]) ensures that the group $\Delta/\ker\theta$ will be the full automorphism group of X, and this will contain anticonformal automorphisms but only one involution, namely a^2, and that involution is conformal. Hence X is pseudo-real. □

4 Pseudo-real surfaces of genus 2 and 3

Theorem 5 *Let X be a pseudo-real Riemann surface of genus 2. Then $Aut(X)$ is isomorphic to $\mathbb{Z}_4$, and if $Aut(X) \cong \Delta/\Gamma$ where $X = \mathbb{D}/\Gamma$, then Δ has signature $(1; -; [2, 2, 2])$.*

Proof. If $G = \Delta/\Gamma$ then $G^+ = \Delta^+/\Gamma$ is the conformal automorphism group of a Riemann surface of genus 2. By Theorem 3 and [Br, Table 4], we know that the only possibilities for G^+ and the signature $s(\Delta^+)$ of Δ^+ are the following (where n^r denotes $n, \overset{r}{\ldots}, n$):

$$\begin{array}{lll} & G^+ \cong \mathbb{Z}_2 \text{ and } s(\Delta^+) = (0; +; [2^6]), \\ \text{or} & G^+ \cong \mathbb{Z}_2 \text{ and } s(\Delta^+) = (1; +; [2^2]), \\ \text{or} & G^+ \cong \mathbb{Z}_4 \text{ and } s(\Delta^+) = (0; +; [2^2, 4^2]). \end{array}$$

The second and third of these three cases are ruled out by the analysis undertaken in [BC], so we are left with only the first case. Then, since Δ must contain Δ^+ as a subgroup of index two, the signature of Δ must be $(1; -; [2, 2, 2])$, and since $G^+ \cong \mathbb{Z}_2$, also $G \cong \mathbb{Z}_4$. □

We can easily construct such a pseudo-real Riemann surface X of genus 2, for example as in the proof of Theorem 4 (with $\delta = \varepsilon = 1$ and $g = 2$).

Theorem 6 *Let X be a pseudo-real Riemann surface of genus 3, with automorphism group $G \cong \Delta/\Gamma$, where $X = \mathbb{D}/\Gamma$. Then there are three possible cases, all of which are realisable:*
(a) $G \cong \mathbb{Z}_4$, *and* Δ *has signature* $(2; -; [2, 2])$, *or*
(b) $G \cong D_4$, *and* Δ *has signature* $(1; -; [2, 2, 2])$, *or*
(c) $G \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2$, *and* Δ *has signature* $(1; -; [2, 2, 2])$.

Proof. The index 2 subgroup $G^+ \cong \Delta^+/\Gamma$ of $G \cong \Delta/\Gamma$ is the automorphism group of a Riemann surface of genus 3. By Theorem 3 and [Br, Table 4], and using the results of [BC] and [BCC], we find the only possibilities for G^+ and the signature of Δ^+ are the following (where 2^r denotes $2, \overset{r}{\ldots}, 2$):

$$\begin{array}{lll} & G^+ \cong \mathbb{Z}_2 & \text{and} \ \ s(\Delta^+) = (0; +; [2^8]), \\ \text{or} & G^+ \cong \mathbb{Z}_2 & \text{and} \ \ s(\Delta^+) = (1; +; [2^4]), \\ \text{or} & G^+ \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{and} \ \ s(\Delta^+) = (0; +; [2^6]). \end{array}$$

In the first case G must be $\mathbb{Z}_4$ and Δ must have signature $(1; -; [2^4])$, but then there is no epimorphism $\theta : \Delta \to \mathbb{Z}_4$ with an appropriate kernel Γ, so this case is ruled out. From the other two cases, we deduce that the only possibilities for G, G^+ and Δ are the following:

$$\begin{array}{llll} & G \cong \mathbb{Z}_4, & G^+ \cong \mathbb{Z}_2 & \text{and} \ \ s(\Delta) = (2; -; [2, 2]), \\ \text{or} & G \cong D_4, & G^+ \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{and} \ \ s(\Delta) = (1; -; [2, 2, 2]), \\ \text{or} & G \cong \mathbb{Z}_4 \oplus \mathbb{Z}_2, & G^+ \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{and} \ \ s(\Delta) = (1; -; [2, 2, 2]). \end{array}$$

In each case it is easy to establish an epimorphism $\theta : \Delta \to G$ where Δ is a maximal NEC group with the corresponding signature, so that $G = \Delta/\ker\theta$ is the automorphism group of a pseudo-real Riemann surface of genus 3. $\square$

5 Pseudo-real cyclic p-gonal Riemann surfaces

Definition 7 *A cyclic p-gonal Riemann surface is a Riemann surface X that admits an automorphism h of order p such that the quotient surface $X/\langle h \rangle$ has genus* 0.

Theorem 8 *Let X be a pseudo-real cyclic p-gonal Riemann surface of genus $g \geq (p-1)^2$, where p is prime. Let G be the automorphism group of X, and let $H = \langle h \rangle \cong \mathbb{Z}_p$ be the subgroup of G generated by an automorphism h of p-gonality. Let Δ and Γ be NEC groups such that $X \simeq \mathbb{D}/\Gamma$ and $\Gamma \lhd \Delta$ with $\Delta/\Gamma \cong G$. Then g is even, and one of the following two cases holds:*

(a) $G \cong \mathbb{Z}_n \oplus H$, where 4 divides n and the first factor $\mathbb{Z}_n$ is generated by an anticonformal automorphism, and the NEC group Δ has signature $(1; -; [p, \overset{\frac{2(g+p-1)}{n(p-1)}}{\ldots}, p, \frac{n}{2}])$; or

(b) $G \cong \mathbb{Z}_{pn}$, where 4 divides n, and G is generated by an anticonformal automorphism, and Δ has signature $(1; -; [p, \overset{\frac{2g}{n(p-1)}}{\ldots}, p, \frac{n}{2}p])$.

Proof. Let Λ be the Fuchsian group of genus 0 such that $\Gamma < \Lambda < \Delta$, with $X/H \simeq \mathbb{D}/\Lambda$ and $\Lambda/\Gamma \cong H$. The signatures of Λ and Δ have the form

$$(0; +; [p, \overset{q}{\ldots}, p]) \quad \text{and} \quad (g'; -; [m_1, ..., m_r])$$

respectively. Let $n = [\Delta : \Lambda]$. By [A], we know that if the genus of X satisfies $g \geq (p-1)^2$, then there is a unique p-gonal covering $X \to \widehat{\mathbb{C}} = X/H$. Hence $\Lambda \lhd \Delta$, and so

$$[m_1, ..., m_r] = [ps_1, ..., ps_t, m_{t+1}, ..., m_r] \text{ and } q = \tfrac{n}{s_1} + ... + \tfrac{n}{s_t}.$$

Applying the Riemann-Hurwitz formula, we find

$$-2 + q(1 - \tfrac{1}{p}) = n(g' - 2 + \sum_{i=1}^{t}(1 - \tfrac{1}{s_i p}) + \sum_{i=t+1}^{r}(1 - \tfrac{1}{m_i})). \qquad (*)$$

Since the genus of Λ is 0, the genus of Δ must be 1, and hence the formula (*) is equivalent to

$$2 - n + n\sum_{i=1}^{t}(1 - \tfrac{1}{s_i}) + n\sum_{i=t+1}^{r}(1 - \tfrac{1}{m_i}) = 0.$$

From this formula it is easy to deduce that the only possible signatures for Δ are the following:

$$(1; -; [p, \overset{\frac{2(g+p-1)}{n(p-1)}}{\cdots}, p, \tfrac{n}{2}]) \quad \text{and} \quad (1; -; [p, \overset{\frac{2g}{n(p-1)}}{\cdots}, p, \tfrac{n}{2}p]).$$

Now suppose Δ has one of the above signatures, and define $l = \frac{2(g+p-1)}{n(p-1)}$ for the first signature and $l = \frac{2g}{n(p-1)}$ for the second. We will consider the epimorphism $\theta : \Delta \to \Delta/\Lambda$. Let $d, x_1, ..., x_l, x_{l+1}$ be the generators of a canonical presentation of Δ. From the form of the signature of Λ, we see that $\theta(x_1) = ... = \theta(x_l) = 1$, and that $\theta(x_{l+1}) = b$ is an element of order $\frac{n}{2}$. Also the two elements $a = \theta(d)$ and $b = \theta(x_{l+1})$ generate the image Δ/Λ. Next, from the relation

$$d^2 x_1 ... x_l x_{l+1} = 1$$

we find that $a^2 b^{-1} = 1$, and so Δ/Λ is cyclic of order n. Since the subgroup H generated by the p-gonal automorphism is unique, it is central in G and hence Δ/Γ is a central extension of a cyclic group of order n. But then since $G/Z(G)$ is cyclic, G is abelian (by an easy theorem from group theory). Thus G is isomorphic to either $\mathbb{Z}_n \oplus H$ or $\mathbb{Z}_{pn}$, and the rest follows. $\square$

6 The maximal order of the automorphism group of a pseudo-real Riemann surface

Theorem 9 *If X is a pseudo-real Riemann surface of genus g with automorphism group G, then $|G| \leq 12(g-1)$. Moreover, if $|G| = 12(g-1)$ and $G \cong \Delta/\Gamma$ where $X \simeq \mathbb{D}/\Gamma$, then the signature of Δ is $(1; -; [2,3])$.*

Proof. By Theorem 3, the NEC group Δ has signature $(p; -; [m_1, ..., m_r])$, and then from the Riemann-Hurwitz formula, we find

$$2g-2 = |G|\,(p-2+\sum_{i=1}^{r}(1-\tfrac{1}{m_i})).$$

The minimum positive value of the bracketed expression on the right-hand side is $\frac{1}{6}$, which is attained when $p=1$, $r=2$, $m_1=2$ and $m_2=3$, and in that case $|G| = 12(g-1)$. $\square$

We are interested in the construction of pseudo-real Riemann surfaces with automorphism group of maximal order. In order to find such a pseudo-real surface of genus g with maximal symmetry, we need to find:

(a) a maximal NEC group Δ with signature $(1; -; [2,3])$,
(b) a finite group G of order $12(g-1)$, and
(c) an epimorphism $\theta : \Delta \to G$ such that $\ker\theta$ is a Fuchsian surface group of genus g.

Here we remark that the monodromy epimorphism θ is determined by the image of the canonical generators. If we have a group G of order $12(g-1)$ and a monodromy epimorphism $\theta : \Delta \to G$, then the group Δ is maximal (see [S] and [BEGG]) unless there is another NEC group Δ' with signature $(0; +; [2], \{(2,3)\})$ containing Δ and an epimorphism $\theta' : \Delta' \to G'$, where G' is an index two extension of G and $\theta'|_\Delta = \theta$.

Proposition 10 *Let Δ be an NEC group with signature $(1; -; [2,3])$, let d, x_1 and x_2 be the generators of a canonical presentation for Δ, and let $\theta : \Delta \to G$ be an epimorphism such that $\theta(d_1) = a$ and $\theta(x_1) = b$. Then θ can be extended to an epimorphism $\theta' : \Delta' \to G'$, where Δ' is an NEC group containing Δ as a subgroup of index 2 and G' is a group containing G as a subgroup of index 2, if and only if G admits an automorphism of order 2 such that $\alpha(a) = a^{-1}$ and $\alpha(b) = b^{-1}$.*

Proof. If G admits such an automorphism α, then we can construct the semidirect product $G' = G \rtimes_\alpha \mathbb{Z}_2$, which is generated by $G = \langle a, b\rangle$ and an involution c, conjugation by which induces the automorphism α on G. Also we can let Δ' be an NEC group with signature $(0; +; [2], \{(2,3)\})$ and having canonical generators x'_1 c'_1, c'_2, c'_3, and then define an epimorphism $\theta' : \Delta' \to G' = G \rtimes_\alpha \mathbb{Z}_2$ by setting

$$\theta'(x'_1) = ac, \quad \theta'(c'_1) = c, \quad \theta'(c'_2) = cb, \quad \text{and} \quad \theta'(c'_3) = a^2c.$$

Conversely, if such an extension $\theta' : \Delta' \to G'$ of θ exists, then by [S] and [BEGG], Δ' must have signature $(0; +; [2], \{(2,3)\})$ with canonical generators x'_1 c'_1, c'_2, c'_3, and without loss of generality the embedding of Δ in Δ' is given by

$$d_1 \mapsto x'_1c'_1, \quad x_1 \mapsto c'_1c'_2, \quad x_2 \mapsto c'_2c'_3;$$

hence if c is the involution $\theta'(c'_1)$, then

$$cac = \theta'(c'_1d_1c'_1) = \theta'(c'_1x'_1) = \theta(d_1)^{-1} = a^{-1}$$

and

$$cbc = \theta'(c'_1x_1c'_1) = \theta'(c'_2c'_1) = \theta(x_1)^{-1} = b^{-1},$$

so conjugation by c gives the required automorphism. □

The last proposition and a theorem to follow provide a link with the theory of 3-valent regular maps.

Definition 11 *An orientably-regular map M is a 2-cell embedding of a connected graph into an orientable surface, such that the group $Aut^+(M)$ of all orientation-preserving automorphisms of the surface that preserve the embedding has a single orbit on the arcs (directed edges) of the graph. The map is called reflexible if there exist orientation-reversing automorphisms that preserve the embedding, and otherwise it is said to be chiral.*

More details may be found in [CD], where all orientably-regular maps of genus 2 to 15 were determined. If M is an orientably-regular map of type $\{m, n\}$ (with vertices of valence m and faces of size n), then $\mathrm{Aut}^+(M)$ is generated by two elements R and S satisfying $R^m = S^n = (RS)^2 = 1$, and M is reflexible if and only if there is an automorphism τ of $G = \langle R, S\rangle$ such that $\tau(R) = R^{-1}$ and $\tau(S) = S^{-1}$ (or equivalently, an automorphism inverting any one of the pairs (R, S), (R, RS) or (S, RS)).

Theorem 12 *For each chiral regular map M of type $\{3, n\}$, where n is odd, if M has automorphism group G, then there exists a pseudo-real Riemann surface X with automorphism group of maximal order and isomorphic to $G \times \mathbb{Z}_4$.*

Proof. Let Δ be the $(2,3,n)$ triangle group, with signature $(0;[2,3,n])$ and canonical presentation

$$\langle\, x_1, x_2, x_3 : x_1x_2x_3 = 1,\; x_1^2 = x_2^3 = x_3^n = 1\,\rangle.$$

Also let $\theta : \Delta \to G$ be the epimorphism that corresponds to a chiral regular map of type $\{3,n\}$, taking (say) x_1, x_2 and x_3 to the automorphisms RS, R and S of M, so that $\ker\theta$ is a surface group, and let b be an element of G such that $b^2 = \theta(x_3)$, which is known to exist because n is odd.
Now let Λ be an NEC group with signature $(1;-;[2,3])$ and canonical presentation

$$\langle\, d, y_1, y_2 : y_1y_2d^2 = 1,\; y_1^2 = y_2^3 = 1\,\rangle,$$

and define an epimorphism $\omega : \Lambda \to G \times \mathbb{Z}_4 = G \times \langle a : a^4 = 1\rangle$ by setting

$$\omega(d) = (b,a), \quad \omega(y_1) = (\theta(x_1), a^2) \quad \text{and} \quad \omega(y_2) = (\theta(x_2), 1).$$

Then $\Gamma = \ker\omega$ is a surface group, and since the image of the subgroup $\langle x_1, x_2, dx_1d, dx_2d\rangle$ of index 2 in Λ is the subgroup $G \times \langle a^2\rangle$ of index 2 in $G \times \langle a\rangle$, the surface $X = \mathbb{D}/\Gamma$ (with automorphism group $\Lambda/\Gamma \cong G \times \langle a\rangle$) is orientable; see [BEGG, Theorem 2.1.3 (2)]. Moreover, every element of $G \times \langle a\rangle$ lying outside the orientation-preserving subgroup $G \times \langle a^2\rangle$ is of the form $(u, a^{\pm 1})$ for some $u \in G$, and it follow that every anticonformal automorphism of $X = \mathbb{D}/\Gamma$ has order divisible by 4. Hence the surface X is pseudo-real. □

In the following section, we prove that for every integer $k \geq 7$, there exist chiral regular maps of type $\{3,k\}$ on orientable surfaces of infinitely many genera. Using Theorem 12, we therefore obtain the following theorem:

Theorem 13 *There exist pseudo-real surfaces with automorphism group of maximal order, for infinitely many genera. In particular there are infinitely many pseudo-real Riemann surfaces with maximal automorphism group.*

7 Chiral 3-valent regular maps

In the previous Section we proved that from every chiral regular map of type $\{3,n\}$ for n odd, we can construct a pseudo-real Riemann surface with maximal symmetry. In this Section we shall find explicit families of chiral 3-valent regular maps that produce such pseudo-real Riemann surfaces.

Theorem 14 *For every prime p congruent to $1, 2$ or 4 mod 7, there exists a normal subgroup K_p of index $168p^3$ in the ordinary triangle group $\Delta = \Delta(2,3,7)$ such that Δ/K_p is isomorphic to an extension by $PSL(2,7)$ of the 3-generator abelian group $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ of order p^3 and exponent p. Moreover, the subgroup K_p is not normal in the extended triangle group $\Delta^*(2,3,7)$, so Δ/K_p has no automorphism that inverts the images of the two generators x and y of $\Delta = \Delta(2,3,7)$.*

Note: here Δ has signature $(0;+;[2,3,7];\{-\})$, while the extended triangle group Δ^* has signature $(1;+;[-];\{(2,3,7)\})$, with $(\Delta^*)^+ \cong \Delta$.

Corollary 15 *There exist chiral regular maps of type $\{3,7\}$ on orientable surfaces of infinitely many genera.*

Proof of Theorem. Most of this follows from observations made by Leech in [L] and pursued by Cohen in [Ch], and explained also in [Cn2, Cn3]. First, the extended triangle group $\Delta^* = \Delta^*(2,3,7)$ has a normal subgroup N of index 336, generated by $a_0 = [y,x]^4$ and its conjugates, such that Δ^*/N is isomorphic to PGL$(2,7)$. By observations made by Leech [L], this normal subgroup N has a nice presentation in terms of six generators and a single relation (in which each of the generators appears twice, with exponents ± 1). Now for each prime p as given in the statement of the theorem, let N_p denote the normal subgroup of Δ^* generated by the derived subgroup $N' = [N,N]$ of N and the set N^p of all pth powers of elements of N. Then $N_p = N'N^p$ has index p^6 in N, and is normal in Δ^*, with quotient N/N_p elementary abelian of order p^6. Moreover, by observations made by Cohen [Ch] about the action of PSL$(2,7)$ on N/N_p induced by conjugation of N by elements of $\Delta = \Delta(2,3,7)$, there exist intermediate subgroups L_1 and L_2 of N containing N_p, such that each L_i is normal in Δ, and $N = L_1L_2$ with $L_1 \cap L_2 = N_p$, and with N/L_i elementary abelian of order p^3 for $i \in \{1,2\}$. On the other hand, L_1 and L_2 are not normal in the extended triangle group $\Delta^* = \Delta^*(2,3,7)$; indeed every element of $\Delta^* \backslash \Delta$ conjugates L_1 to L_2 and vice versa. Hence we can take $K_p = L_1$ or L_2, to give the required result. $\square$

Theorem 16 *For every integer $k \geq 7$, all but finitely many of the alternating groups A_n can be generated by two elements x and y such that x, y and xy have orders 2, 3 and k respectively, and that there exists no automorphism of $\langle x,y \rangle = A_n$ taking x and y to x^{-1} and y^{-1} respectively.*

Corollary 17 *For each integer $k \geq 7$, there exist chiral regular maps of type $\{3,k\}$ on orientable surfaces of infinitely many genera.*

Proof of Theorem. In all cases our argument relies heavily on a construction used by the second author in [Cn1] to prove that (for every $k \geq 7$) all but finitely many A_n are homomorphic images of the extended triangle group

$$\Delta^*(2,3,k) = \langle\, x, y, t \mid x^2 = y^3 = (xy)^k = t^2 = (xt)^2 = (yt)^2 = 1\,\rangle,$$

a group with signature $(1; +; [-]; \{(2,3,k)\})$. We refer the reader to [Cn1] for important details. In that construction, permutation representations of $\Delta^*(2,3,k)$ are depicted by Schreier coset diagrams, and specially chosen examples of such diagrams are linked together to form representations of arbitrarily large degree n, in a way that makes the resulting permutations generate A_n or S_n. We will amend that construction by adding one more small diagram that depicts a permutation representation of the ordinary triangle group $\Delta = (\Delta^*)^+$, but not depict one of the extended triangle group $\Delta^*(2,3,k)$ itself. Note that $\Delta = \Delta(2,3,k)$ is the index 2 subgroup of $\Delta^*(2,3,k)$ generated by x and y.

We do this first for the case $k = 7$, and then explain in less detail how the theorem can be proved for larger k using the same method.

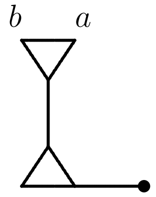

Figure 1: Additional coset diagram $R(7,0)$ with 7 vertices

When $k = 7$, consider the permutation representation of $\Delta(2,3,7)$ on 7 points given by the diagram $R(7,0)$ in Figure 1.Like the diagrams $S(7,0)$, $T(7,0)$, $U(7,0)$ and $V(7,0)$ in [Cn1], this has a (1)-*handle* $[a,b]_1$, consisting of two points a and b such that x fixes both a and b, and y takes a to b. Note that the point a is fixed by the commutator $xyxy^{-1}$, while b lies in a 2-cycle of $xyxy^{-1}$, and the other four points lie in a 4-cycle. Similarly, if $[a',b']_1$ is a (1)-handle of the diagram $S(7,0)$, then a' is fixed by $xyxy^{-1}$, while b' lies in a 13-cycle of $xyxy^{-1}$, consisting of the 13 points of the cycle of xyt in the representation of $\Delta(2,3,7)$ that it depicts. Indeed it follows from the relations for the extended triangle group $\Delta^*(2,3,7)$ that $(xyt)^2 = xytxyt = xyxtyt = xyxy^{-1}$, and hence the cycle structure of $xyxy^{-1}$ can be derived easily from that of xyt on the points of the diagram $S(7,0)$.

Next suppose that a single copy of the diagram $R(7,0)$ is linked together with a single copy of the diagram $S(7,0)$, by adding the transpositions (a,a') and

(b, b') to the permutation induced by x (while not altering the permutation induced by y). Then the resulting diagram is easily seen to be a coset diagram for the ordinary triangle group $\Delta(2,3,7)$, by the same argument as in [Cn1]. Also in the corresponding permutation representation, the two points a and a' are still both fixed by $xyxy^{-1}$, while the cycles containing b and b' and the other four points of the diagram $R(7,0)$ are joined together to form a new cycle of $xyxy^{-1}$, of length 19. (This is easily verified, either by writing out the permutations, or by chasing points around the combined diagrams.)
The construction in [Cn1] explains how a transitive permutation representation of $\Delta^*(2,3,7)$ on $n = 42f + 71g + 36$ points (when $f > g \geq 0$) can be formed by linking together f copies of diagram $S(7,0)$ and then adjoining g copies of $T(7,0)$ and a single copy of $U(7,0)$, by composition of (1)-handles. In the resulting representation, the element xyt has cycle structure

$$1^{f+1-g}2^{f+g}5^{1}6^{g}8^{1}11^{1}13^{f+1-g}15^{g}20^{g}24^{1}26^{f-1}42^{g},$$

so the commutator $xyxy^{-1}$ has cycle structure

$$1^{3f+1}3^{2g}4^{2}5^{1}11^{1}13^{3f-1-g}10^{2g}12^{2}15^{g}21^{2g}.$$

The unique 11-cycle here comes from the single copy of $U(7,0)$, and this can be used (with the help of Jordan's theorem from [W]) to prove that the permutations induced by x, y and t generate S_n, while those induced by x and y generate A_n.
Now suppose that a single copy of the diagram $R(7,0)$ is linked to one of the copies of $S(7,0)$ still having a free (1)-handle in this representation. Then we have a new transitive permutation representation of $\Delta(2,3,7)$ on $n+7$ points, in which $xyxy^{-1}$ has cycle structure

$$1^{3f+2}3^{2g}4^{2}5^{1}11^{1}13^{3f-2-g}10^{2g}12^{2}15^{g}19^{1}21^{2g}.$$

Again the unique 11-cycle here comes from the single copy of $U(7,0)$, and can be used to prove that the permutations induced by x and y generate A_{n+7}. An important difference this time, however, is that because the point fixed by y in the single copy of $R(7,0)$ is the only point fixed by y that lies close to a fixed point of $xyxy^{-1}$ or $xy^{-1}xy$ in the resulting coset diagram (on $n+7$ points), this diagram has no axis of reflectional symmetry. Thus we have a homomorphism from $\Delta(2,3,7)$ to A_{n+7} that does not extend to a representation of the extended triangle group $\Delta^*(2,3,7)$, as claimed.

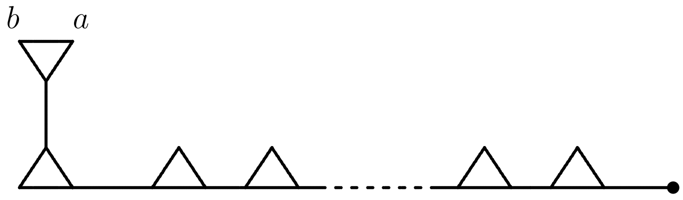

Figure 2: Additional coset diagram $R(7,d)$ with $7+6d$ vertices

When $k = 7 + 6d$ for some positive integer d, we can apply the same construction using $S(7,d)$, $T(7,d)$ and $U(7,d)$ from [Cn1], and add a single copy of the new coset diagram $R(7,d)$ for $\Delta(2,3,7+6d)$ on $7+6d$ points given in Figure 2.

In the permutation representation of $\Delta(2,3,7+6d)$ depicted by $R(7,d)$, the commutator $xyxy^{-1}$ fixes the point a, and has two 2-cycles, two 4-cycles, and $2(d-1)$ 3-cycles. Linking a single copy of $R(7,d)$ to a copy of $S(7,d)$ by their free (1)-handles gives rise to a new permutation representation of $\Delta(2,3,7{+}6d)$ in which one of the 2-cycles and one of the 4-cycles from $R(7,d)$ are combined together with two of the cycles from $S(7,d)$, to form a 7-cycle and a 10-cycle when $d = 1$, or a 6-cycle and an 8-cycle when $d \geq 2$.

Thus again we can form transitive permutation representations of $\Delta(2,3,7+6d)$ of arbitrarily large degree, and use the unique 11-cycle from the single copy of $U(7,d)$ to prove that the resulting permutations generate an alternating group, and the single copy of $R(7,d)$ to eliminate the possibility of a reflectional symmetry.

The proof for other cases (with k in different congruence classes mod 6) is analogous to the above, using the additional diagrams given in Figures 3 to 7 below.

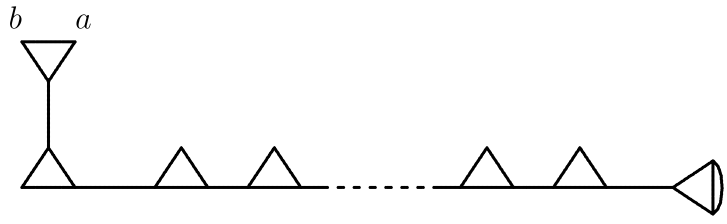

Figure 3: Additional coset diagram $R(8,d)$ with $8+6d$ vertices

For the case $k = 8 + 6d$, joining a single copy of diagram $R(8,d)$ replaces cycles of $xyxy^{-1}$ by one cycle of length 15 if $d = 0$, or cycles of length 7 and

9 if $d = 1$, or cycles of length 6 and 8 if $d \geq 2$, leaving a unique 11-cycle from the single copy of diagram $U(8, d)$ for application of Jordan's theorem.

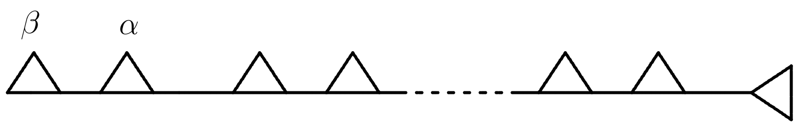

Figure 4: Additional coset diagram $R(9, d)$ with $9 + 6d$ vertices

For the case $k = 9 + 6d$, diagrams are composed using (2)-handles $[\alpha, \beta]_2$, consisting of fixed points α and β such that y^2 takes α to β. Joining a single copy of diagram $R(9, d)$ replaces cycles of $xyxy^{-1}$ by cycles of length 12 and 14 if $d = 0$, or cycles of length 3, 5, 9 and 10 if $d = 1$, or cycles of length 3, 5, 6 and 7 if $d \geq 2$, leaving a unique 13-cycle from the single copy of diagram $U(9, d)$ for application of Jordan's theorem.

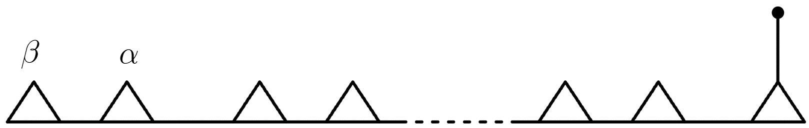

Figure 5: Additional coset diagram $R(10, d)$ with $10 + 6d$ vertices

For the case $k = 10 + 6d$, joining a single copy of diagram $R(10, d)$ replaces cycles of $xyxy^{-1}$ by cycles of length 6, 7, 7 and 10 if $d = 0$, or cycles of length 5, 5, 6 and 7 if $d = 1$, or cycles of length 4, 5, 6 and 6 if $d \geq 2$, leaving a unique 13-cycle from the single copy of diagram $U(10, d)$ for application of Jordan's theorem.

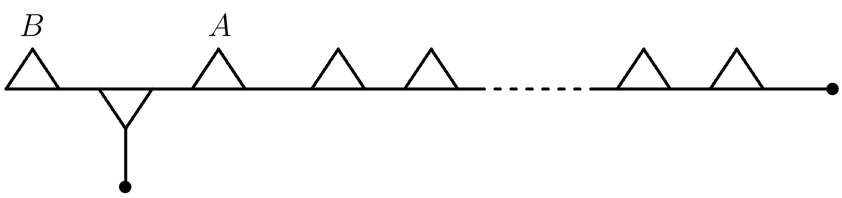

Figure 6: Additional coset diagram $R(11, d)$ with $11 + 6d$ vertices

For the case $k = 11 + 6d$, joining a single copy of diagram $R(11, d)$ replaces cycles of $xyxy^{-1}$ by cycles of length 9 and 19 if $d = 0$, or cycles of length 6, 8 and 9 if $d \geq 1$, leaving a unique 11-cycle from adjoining the single copy of diagram $U(11,d)$ for application of Jordan's theorem.

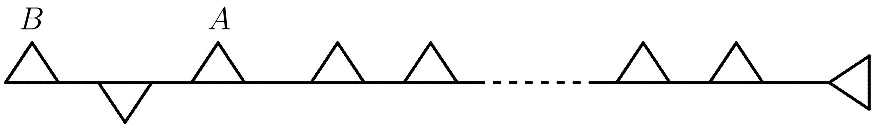

Figure 7: Additional coset diagram $R(12,d)$ with $12+6d$ vertices

For the case $k = 12 + 6d$, joining a single copy of diagram $R(12,d)$ replaces cycles of $xyxy^{-1}$ by cycles of length 9 and 10 if $d = 0$, or cycles of length 5, 7 and 9 if $d = 1$, or cycles of length 4, 6 and 9 if $d \geq 2$, leaving a unique 13-cycle from adjoining the single copy of diagram $U(12,d)$ for application of Jordan's theorem. $\square$

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