Bounds for the number of automorphisms of a compact non-orientable surface

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Abstract

In this paper it is shown that for every positive integer $p > 2$, there exists a compact non-orientable surface of genus p with at least $4p$ automorphisms if p is odd, or at least $8(p-2)$ automorphisms if p is even, with improvements on $4p$ for $p \not\equiv 3 \bmod 12$. Further, these bounds are shown to be sharp (in that no larger group of automorphisms exists with genus p) for infinitely many values of p in each congruence class modulo 12, with the possible (but unlikely) exception of 3 mod 12.

1 Introduction

Some years ago Accola and Maclachlan [1, 13] showed that for every integer $g \geq 2$, there exists a compact Riemann surface of genus g with at least $8g+8$ orientation-preserving automorphisms, and proved this bound is sharp by showing that, for infinitely many values of g, $8g+8$ is the maximum possible number of automorphisms for such a surface of genus g.

In this paper we derive analogous results for compact non-orientable surfaces without boundary. For every positive integer p let $\nu(p)$ denote the largest number of automorphisms of a compact non-orientable surface of genus p. By work of Bujalance [4] on cyclic groups of automorphisms, it is known that $\nu(p) \geq 2(p-1)$, and by Singerman's analogue of Hurwitz's theorem for non-orientable surfaces [16] also $\nu(p) \leq 84(p-2)$, for all $p > 2$.

We refine these results by showing $\nu(p) \geq 4p$ if p is odd, while $\nu(p) \geq 8(p-2)$ if p is even. Indeed for various congruence classes mod 3 and mod 4 we can improve these lower bounds (see Table 1). The non-orientable surfaces admitting these groups of automorphisms are described also in terms of regular or reflexible maps on these surfaces. For each congruence class of p modulo 12 (other than 3) we show that the bounds given in Table 1 cannot be improved for infinitely many values in that class (see Table 3). In the case $p \equiv 3 \bmod 12$, the $4p$ bound can be improved for many values of p (see §3.4), but nonetheless we conjecture that $4p$ is sharp in that case as well.

2 Background on NEC groups

Suppose X is a compact non-orientable surface of genus $p > 2$, with automorphism group G. Then X may be identified with the orbit space D/Λ_p, where D is the upper-half complex plane, and Λ_p is a normal subgroup of finite index in some discrete subgroup Γ of the group $\mathrm{PGL}(2, \mathbb{R})$ of all conformal and anti-conformal homeomorphisms of D, such that Λ_p acts without fixed points on D and the quotient Γ/Λ_p is isomorphic to G.

Any such subgroup Γ with compact orbit space is called a *non-Euclidean crystallographic group* (or NEC group). In the case where Γ lies wholly within the conformal group $\mathrm{PSL}(2, \mathbb{R})$, it is more usually known as a *Fuchsian group*, and gives rise to an orientable surface. In the above case, however, where the orbit space is a non-orientable surface, Γ contains both conformal and anti-conformal homeomorphisms of D, and is known as a *proper* NEC group.

Further, the subgroup Λ_p must have a presentation in terms of p generators d_i (for $1 \leq i \leq p$) subject to the single defining relation $d_1^2 d_2^2 \dots d_p^2 = 1$, corresponding to the fact that X is topologically equivalent to a sphere with p cross-caps. Any such group Λ_p is known as a non-orientable *surface group*.

Note also that Λ_p contains anti-conformal homeomorphisms of D, and so the natural homomorphism $\Gamma \to \Gamma/\Lambda_p$ maps the index 2 conformal subgroup $\Gamma^+ = \Gamma \cap \mathrm{PSL}(2, \mathbb{R})$ onto $\Gamma/\Lambda_p \cong G$. The converse also holds, giving the following theorem which is crucial to our investigations (see [5] for a proof).

Theorem 2.1 *A finite group G is a group of automorphisms of a compact non-orientable surface if and only if there is a proper NEC group Γ and an epimorphism $\theta : \Gamma \to G$ such that the kernel of θ is a non-orientable surface group and θ maps $\Gamma^+ = \Gamma \cap \mathrm{PSL}(2, \mathbb{R})$ onto G.*

In this case, the group G acts on the non-orientable surface $D/\ker\theta$. The genus of the surface depends on the *signature* of Γ, which corresponds to the analytic structure of the associated surface and is determined largely by fixed circles of reflections in Γ and branch points of Γ^+.

Proper NEC groups may be classified according to their signature, and as explained in [5] and [18] for example, there are two types, as follows:

Type $(+)$: Signature $(g; +; [m_1, m_2, \ldots m_\tau]; \{(n_{i1}, n_{i2}, \ldots, n_{is_i}) : 1 \le i \le k\})$

A group Γ with this signature may be presented in terms of generators

- x_i for $1 \le i \le \tau$,
- c_{ij} for $0 \le j \le s_i$, for $1 \le i \le k$,
- e_i for $1 \le i \le k$,
- a_j, b_j for $1 \le j \le g$,

subject to the defining relations below:

- $x_i^{m_i} = 1$ for $1 \le i \le \tau$,
- $c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1}c_{ij})^{n_{ij}} = 1$ for $1 \le j \le s_i$, for $1 \le i \le k$,
- $e_i^{-1} c_{i0} e_i = c_{is_i}$ for $1 \le i \le k$, and
- $x_1 x_2 \ldots x_\tau \, e_1 e_2 \ldots e_k \, [a_1, b_1][a_2, b_2] \ldots [a_g, b_g] = 1$.

Type $(-)$: Signature $(g; -; [m_1, m_2, \ldots, m_\tau]; \{(n_{i1}, n_{i2}, \ldots, n_{is_i}) : 1 \le i \le k\})$

A group Γ with this signature may be presented in the same way as for type $(+)$ above, except with the generators a_j and b_j replaced by generators d_j (for $1 \le j \le g$), and the final relation replaced by

- $x_1 x_2 \ldots x_\tau \, e_1 e_2 \ldots e_k \, d_1^2 d_2^2 \ldots d_g^2 = 1$.

If Λ_p is a non-orientable surface subgroup of finite index in a type $(+)$ group Γ, with quotient group G, then the genus p and the Euler characteristic χ of the associated surface are given by the *Riemann-Hurwitz equation*

$$2 - p = \chi = |G| \left(2 - 2g - k - \sum_{i=1}^{\tau} (1 - 1/m_i) - \sum_{i=1}^{k} \sum_{j=1}^{s_i} (1 - 1/n_{ij})/2 \right).$$

In the case of a type $(-)$ group, the genus and characteristic are given by the same formula, except with the term $2 - 2g - k$ replaced by $2 - g - k$.

As the generators x_i, e_i, a_i, b_i of a proper NEC group Γ represent conformal automorphisms of D, the subgroup $\Gamma^+ = \Gamma \cap \mathrm{PSL}(2, \mathbb{R})$ has index 2 in Γ and contains each of the generators x_i (for $1 \leq i \leq \tau$), e_i (for $1 \leq i \leq k$), a_j and b_j (for $1 \leq j \leq g$) which appear in the presentation of Γ corresponding to its signature, but none of the generators c_{ij} (for $0 \leq j \leq s_i$ and $1 \leq i \leq k$) or d_j (for $1 \leq j \leq g$). Further, if $\theta : \Gamma \to G$ is a homomorphism from Γ to a finite group G such that θ maps Γ^+ onto G, then the kernel of θ is a non-orientable surface group if and only if θ preserves the orders of the generators x_i and c_{ij} and the products $c_{ij-1}c_{ij}$ (for all i and j) in the signature presentation of Γ. When θ has this property, it will be called a *smooth* homomorphism. Also when θ maps Γ onto G, for the purpose of abbreviation we will say that G has the signature of Γ.

3 Non-orientable surfaces having large automorphism groups

In this Section we show how several families of examples of compact non-orientable surfaces of genus p with large automorphism groups may be constructed, with 'large' meaning that the group has order at least $4p$. All such examples can be constructed by a suitable choice of finite group G, NEC group Γ and epimorphism θ satisfying the conditions of Theorem 2.1.

In investigating the existence of regular maps on non-orientable surfaces, some of these families were constructed in [6]. A key step in this process was the construction of a family of finite groups as semi-direct products of cyclic groups of varying order n by a fixed small finite group, and this method can be extended to most of the families considered here.

There are also geometric constructions of regular maps on non-orientable surfaces which give rise to non-orientable surfaces with large automorphism groups, exhibited in [19]. These constructions can also be extended to the families discussed here.

For these reasons we now briefly describe the connection with regular maps; further details may be found in [10, 12, 19] for example.

3.1 Regular Maps

A map M on a surface X is an embedding of a connected graph into X such that the graph separates X into simply-connected regions (discs), called the *faces* of the map. The Euler characteristic of M (or more strictly, of the surface X) is given by $V - E + F$, where V, E and F are the numbers of vertices, edges and faces of the map.

A *symmetry* of a map M is a permutation of the sets of vertices, edges and faces induced by a homeomorphism of the underlying surface X. Indeed, given any M, the surface X can be assigned a conformal or anti-conformal structure such that every symmetry is a structure-preserving automorphism (see [2], and also [12]).

A map M is said to be *rotary* if it satisfies the condition that for any incident vertex-face pair, there are symmetries of M which cyclically permute the edges of the face and which cyclically permute the edges incident at the vertex. Further, a rotary map is said to be *reflexible* provided it has a reflection about some edge.

If M is reflexible, then for any incident vertex-edge-face triple, there are symmetries which act as reflections about the edge, about an axis joining vertex to face-centre and about an axis joining face-centre to edge-midpoint. Reflexible rotary maps are often called *regular*, in the sense that their symmetry groups act regularly (sharply transitively) on vertex-edge-face triples (which are often called *flags* or *blades*) of the map.

Let $\mathrm{Aut}\,M$ denote the group of symmetries of M and, if X is orientable, Aut^+M the subgroup of orientation-preserving symmetries. Note that if M is reflexible, then $\mathrm{Aut}\,M$ has order $4E$, while if M is rotary but not reflexible (or *chiral*), then $\mathrm{Aut}\,M$ has order $2E$. Also if G is a subgroup of $\mathrm{Aut}\,M$ where M is compact and non-orientable, then the anti-conformal structure yields a smooth epimorphism $\theta : \Gamma \to G$ as described in Theorem 2.1.

In particular, if $G = \mathrm{Aut}\,M$ where M is a regular map on a compact non-orientable surface of genus p, then the group G is a smooth factor group of an NEC group whose signature has the form $(0; +; [-]; \{(2, n_1, n_2)\})$, where n_1 and n_2 denote the numbers of edges incident with each vertex and each face respectively. The pair $\{n_1, n_2\}$ is known as the *type* of the map.

A simple counting argument (based on the numbers of incident vertex-edge and edge-face pairs) shows that $2-p = V-E+F = |G|/2n_1 - |G|/4 + |G|/2n_2 \geq 1 - |G|/4 + 1 = 2 - |G|/4$, and so $|G| \geq 4p$. In fact this is a strict inequality, since regular maps with just one vertex and one face have dihedral automorphism groups and are orientable; hence $|\mathrm{Aut}\,M| > 4p$ for every regular map on a compact non-orientable surface of genus p.

3.2 The cross-capping construction

Here we describe a new method of constructing non-orientable surfaces from maps with many automorphisms.

Suppose that M is a rotary map. If we remove a disc from the centre of each face, then $\mathrm{Aut}\, M$ also acts on this (bordered) surface. If we now identify each pair of opposite points on the boundary circles of these discs, then we obtain a new surface, denoted by M^c, on which $\mathrm{Aut}\, M$ still acts as a group of symmetries. This construction amounts to entering a cross-cap into each face at its centre. Adding F cross-caps decreases the value of the Euler characteristic by F, so that M^c has genus $E - V + 2$, where E, V are the number of edges and vertices of M.

In the cases where M is reflexible, a fundamental region for the action of $\mathrm{Aut}\, M$ will be a hyperbolic quadrilateral Q with three right angles, as illustrated in Figure 1.

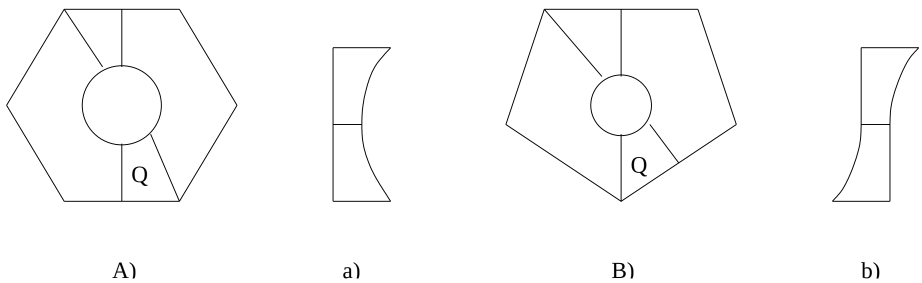

Figure 1: Quadrilateral fundamental regions

By examining two copies of the fundamental region, it can easily be seen that if n_1 is the common degree of the vertices, then the signature of $\mathrm{Aut}\, M$ corresponding to its action on the surface M^c will be $(0; +; [-]; \{(2, 2, 2, n_1)\})$ if the number n_2 of edges around a face of M is even, or $(0; +; [2]; \{(2, n_1)\})$ if n_2 is odd. (This is illustrated in Figure 1, with six edges at A) and a) and five edges at B) and b).)

3.3 Large automorphism groups

In this subsection we establish for each p in one of a number of specific congruence classes, the existence of a compact non-orientable surface of genus

p with a large group of automorphisms.

In each of the cases 3.3.1 to 3.3.5 (and 3.4.1) below, we use the letters a, b, c, d for generators of the NEC group Γ as described in §2, and letters t, u, v, w for generators of the target finite group G.

3.3.1 A family of examples with $|G| = 4p$ for p odd:

Let Γ have signature $(0, +, [-]; \{(2, 2, 2, p)\})$, and let G denote the dihedral group of order $4p$, so that $G = \langle u, v \mid u^2 = v^{2p} = (uv)^2 = 1\rangle$. Define θ by

$$a \mapsto u, \quad b \mapsto uv^p, \quad c \mapsto v^p, \quad d \mapsto uv^2.$$

This is a homomorphism preserving the orders of the elements of finite order in Γ, and the assumption that p is odd ensures that θ is an epimorphism. Note also that Γ^+ maps onto G. From the Riemann-Hurwitz formula, we find $\chi = 4p(-1/4 + 1/2p) = 2 - p$, and so the genus of the kernel of θ is p.

To otain a geometric construction, we may take the reflexible map $D(\epsilon_p)$ on the sphere, consisting of two vertices (the north and south poles) and p edges which are great semi-circles joining them, and then carry out the cross-capping construction to obtain the non-orientable surface $D(\epsilon_p)^c$ of genus p. As explained in §3.2, the signature of the resulting automorphism group is $(0, +, [-]; \{(2, 2, 2, p)\})$, and its order is still $4p$. Note that there is no restriction on the value of p in this construction, so that a group of order $4p$ exists for all p, although it will not be dihedral in all cases. For even p, however, this bound can be further improved, as we now show.

3.3.2 A family of examples with $|G| = 8(p - 2)$ for p even:

Let Γ have signature $(0; +; [-]; \{(2, 2, 2, 4)\})$. This time let H be the direct product of a dihedral group of order 8 generated by u and v such that $u^2 = v^4 = (uv)^2 = 1$, and a cyclic group of order 2 generated by t, and let K be a cyclic group of order $n = p - 2$ generated by w. Now form the semi-direct product $K \rtimes H$ where each of the generators u, v, t of H conjugates w to its inverse, and define θ by the assignment $a \mapsto tu$, $b \mapsto v^2$, $c \mapsto t$ and $d \mapsto uv$. Let G denote the image of this homomorphism. Since $\theta(da)^2 = v^2 = \theta(b)$ and $\theta(ac)^2 = w^2$, we see w^2 generates a normal cyclic subgroup of G of order $(p - 2)/2$ with quotient isomorphic to $D_4 \times \mathbb{Z}_2$, and so $|G| = 8(p - 2)$. Also Γ^+ maps onto G, and the Riemann-Hurwitz formula shows that the kernel of θ has genus p.

Alternatively the same family of surfaces can be obtained by applying the cross-capping construction to reflexible regular maps on the orientable surfaces of genus g admitting the maximum number $8(g+1)$ of orientation-preserving automorphisms (described in [1, 13]), as follows. The map is oppB$(4,2c)$, a special case of the maps oppB$(2k,2c)$ in [19], and may be described with the help of a schematic diagram on a torus.

On the torus draw a $4 \times 2c$ rectangle of squares, then darken and label every other horizontal edge in each row, the rows alternating as shown (with each row of squares labelled with an arrow pointing right and left alternately). From this schematic diagram, the map may be constructed with the faces corresponding to the horizontal rows of squares and the arrows indicating clockwise directions. Thus each face is a $2c$-gon with edges labelled in clockwise order given by the corresponding arrow. The resulting map is illustrated in Figure 2 for the case $c = 3$.

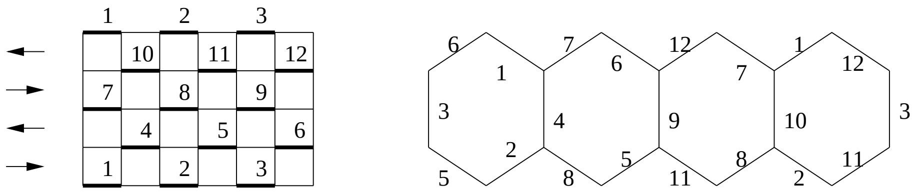

Figure 2: Scheme and Map oppB(4,6)

It is shown in [19] that this map oppB$(4,2c)$ is reflexible and orientable. The vertices correspond to the vertical 'ladders' in the schematic diagram (such as 2 - 4 - 8 - 10 - 2). Thus $E = 4c$, $F = 4$, and $V = 2c$. Note that, if $c = g+1$ then we have $|G| = 16(g+1)$, and from a triangle connecting a face-centre with an adjacent edge-midpoint and vertex we see Aut M has signature $(0;+;[-];\{(2,4,2(g+1))\})$. The group $\mathrm{Aut}^+(M)$ is precisely that constructed by Accola and Maclachlan in [1, 13].

Now carry out the cross-capping construction on M, with $c = (p-2)/2$, to obtain the non-orientable surface M^c of genus p. As before, Aut M has signature $(0;+;[-];\{(2,2,2,4)\})$, and order $16c = 8(p-2)$.

3.3.3 A family of examples with $|G| = 8(p+2)$ for $p \equiv 1 \bmod 3$:

Let Γ have signature $(0;+;[-];\{(2,4,3n)\})$. Take the semi-direct product $K \rtimes H$ of a cyclic group K of order $3n$ generated by w by the group $H \cong S_4$, in such a way that every odd permutation in S_4 conjugates w to w^{-1}. As in 3.3.2 above and described in [6], one can obtain a homomorphism from Γ into this group $K \rtimes H$, with image G of order $24n$ satisfying the conditions of Theorem 2.1 and with kernel of genus $p = 3n - 2$.

Alternatively, this is the automorphism group of the non-orientable map Γ_n illustrated in Figure 3. This map is reflexible (see [19]), and has $6n$ edges, $3n$ four-sided faces, and 4 vertices.

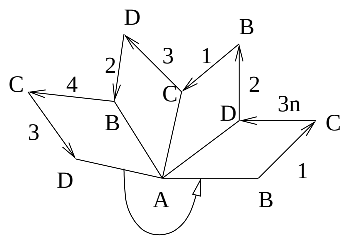

Figure 3: The non-orientable map Γ_n

Restricting the above homomorphism (from Γ to the semi direct product $K \rtimes H$) to the orientation-preserving subgroup Γ^+ shows that $\mathrm{Aut}\,\Gamma_n$ is also isomorphic to the orientation-preserving group of automorphisms of the compact orientable surface of genus $g \equiv 0 \bmod 3$ admitting the maximum possible number of $8(g+3)$ automorphisms for such g, as described in [1, 13].

3.3.4 A family of examples with $|G| = 8(p-2)$ for $p \equiv 2 \bmod 3$:

In this case, take Γ with signature $(0; +; [2]; \{(2,4)\})$, generated by four involutions a, b, c and x, satisfying the relations $(ab)^2 = (bc)^4 = 1$ and $x^{-1}ax = c$, and let G be the same subgroup G of the direct product $C_{3n} \rtimes S_4$ as described in 3.3.3 above. The homomorphism θ from Γ to G taking $a \mapsto v^2$, $b \mapsto tv$, $c \mapsto tv^2$ and $x \mapsto wut$ has the required properties, and the kernel has genus p when $3n = p-2$. Alternatively, this example can be described in terms of the group $\mathrm{Aut}\,\Gamma_n$ acting on the surface Γ_n^c, where Γ_n is as in Figure 3.

3.3.5 A family of examples with $|G| = 6(p+1)$ for $p \equiv 1 \bmod 4$:

Here we take Γ with signature $(0; +; [-]; \{(2, 6, 2n)\})$, and $G = C_n \rtimes D_6$ a semi-direct product of a cyclic group of odd order n by the dihedral group of order 12. This family is described in [6], and in each case we have G acting on a non-orientable surface of genus p where $2n = p + 1$. Alternatively, for odd c we may view G as the group of a map opp*B$(3, 2c)$, similar to that described in 3.3.2 above; for details, see [19].

3.4 Summary

The following table summarises the information on the families of large automorphism groups of non-orientable compact surfaces constructed in §3.3. In the final column, the dot merely indicates an extension (which is not necessarily split).

Genus	**Signature**	**Order**	**Group**
$p \equiv 1 \bmod 3$	$(0; +; [-]; \{(2, 4, p + 2)\})$	$8(p + 2)$	$\mathbb{Z}_{(p+2)/3} \cdot S_4$
$p \equiv 2 \bmod 3$	$(0; +; [2]; \{(2, 4)\})$	$8(p - 2)$	$\mathbb{Z}_{(p-2)/3} \cdot S_4$
$p \equiv 1 \bmod 4$	$(0; +; [-]; \{(2, 6, p + 1)\})$	$6(p + 1)$	$\mathbb{Z}_{(p+1)/2} \cdot D_6$
$p \equiv 0 \bmod 2$	$(0; +; [-]; \{(2, 2, 2, 4)\})$	$8(p - 2)$	$\mathbb{Z}_{(p-2)/2} \cdot (D_4 \times C_2)$
$p \equiv 1 \bmod 2$	$(0; +; [-]; \{(2, 2, 2, p)\})$	$4p$	D_{2p}

Table 1: Families of large groups

Note that these show that $\nu(p) \geq 8(p+2)$ for $p \equiv 1, 4, 7, 10 \bmod 12$, while $\nu(p) \geq 8(p - 2)$ for $p \equiv 0, 2, 5, 6, 8, 11 \bmod 12$, and $\nu(p) \geq 6(p + 1)$ for $p \equiv 9 \bmod 12$. It will be shown in the next section that these bounds are sharp for infinitely many values of p in each congruence class modulo 12.

This leaves the case $p \equiv 3 \bmod 12$, for which the best lower bound we have is $\nu(p) \geq 4p$. We now show that for *some* values of p in this congruence class, the latter bound can be improved.

3.4.1 A family of examples with $|G| = 6(p - 2)$ for certain $p \equiv 3 \bmod 12$:

Suppose p is an integer such that $p \equiv 3 \bmod 12$ and $p - 2 = p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r} m^2$ where each p_i is a prime congruent to 1 mod 6, each exponent a_i is odd, and $\gcd(p_i, m) = 1$ for $1 \leq i \leq r$. Let Γ have signature $(0; +; [2, 3]; \{(1)\})$, so that Γ has presentation $\langle a, b, c \mid a^2 = b^3 = c^2 = [ab, c] = 1\rangle$. Next let $K_1 = \langle u\rangle$ be cyclic of order $p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r}$, so that K_1 admits an automorphism of order 6 taking $u \mapsto u^\alpha$ where $\alpha^2 - \alpha + 1 \equiv 0 \bmod p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}$, and let $K_2 = \mathbb{Z}_m \times \mathbb{Z}_m$ be generated by v and w (each of order m) so that K_2 also admits an automorphism of order 6, given by $v \mapsto w$ and $w \mapsto v^{-1}w$. Now let H be a cyclic group of order 6 generated by t, and form the semi-direct product $G = (K_1 \times K_2) \rtimes H$ with H acting on $K_1 \times K_2$ in the obvious way. Then $|G| = 6(p-2)$, and the assignment $a \mapsto uvt^3$, $b \mapsto uvt^4$, $c \mapsto t^3$ induces an epimorphism $\theta : \Gamma \to G$ such that θ also maps Γ^+ onto G. By the Riemann-Hurwitz formula, $\ker \theta$ has genus p.

Alternatively, consider the torus map $M = \{6,3\}_{b,c}$, which is always rotary, and is reflexible precisely when $c = 0$ or $b = c$ (see [10]). This map M has D faces, $2D$ vertices and $3D$ edges, where $D = b^2 + bc + c^2$. Thus M^c is a non-orientable surface of genus $D + 2$, and admits a group of automorphisms of order $6D$. Note that a positive integer can be written in the form $b^2 + bc + c^2$ precisely when it is of the form $p_1 p_2 \dots p_r m^2$ or $3 p_1 p_2 \dots p_r m^2$, with each $p_i \equiv 1 \bmod 6$. Further, when $p - 2 = m^2$ or $3m^2$ the surface M^c admits a group of order $12(p-2)$, since M is then also reflexible; in that case, Γ has signature $(0; +; [-]; \{(2,2,2,3)\})$, and G may be obtained as a semi-direct product of $\mathbb{Z}_m \times \mathbb{Z}_m$ or $\mathbb{Z}_{3m} \times \mathbb{Z}_m$ by a dihedral group of order 12.

There are still, however, infinitely many values of $p \equiv 3 \bmod 12$ for which we have no better lower bound than $\nu(p) \geq 4p$.

4 Signatures for large groups

Suppose G is a group of automorphisms of a compact non-orientable surface X of genus $p > 2$. Then G also acts as a group of orientation-preserving automorphisms of the orientable 2-sheeted covering surface $\tilde{X}$ whose genus is $p - 1$ since $\chi(\tilde{X}) = 2\chi(X) = 4 - 2p$.

On the other hand, given a group G of orientation-preserving automorphisms of a compact orientable surface, G may or may not extend to act on a related compact non-orientable surface (if there is one). In particular, the groups of order $8(g+1)$ used in [1, 13] do not extend in this way, while those of order $8(g+3)$ in cases where $g \equiv 0 \bmod 3$ (as discussed in [1, 13]) do so extend; see 3.3.3 above.

In view of the examples in §3.3, let us assume that $|G| > 4p$, so that $0 < -\chi(X)/|G| < 1/4$. The Riemann-Hurwitz formula then severely restricts the possibilities for the signature of the NEC group Γ. In particular, it forces $k \leq 1$ and $\tau \leq 2$, and a straightforward but detailed analysis shows that Γ

must have one of the signatures listed in Table 2.

Similarly if we assume $|G| > 6(p+1)$ or $|G| > 8(p-2)$ or $|G| > 8(p+2)$, then the possible signatures for the group Γ will be restricted to a subset of the possibilities given in Table 2.

Case	**Signature**	**Group order**
(a)	$(1;\ -;\ [2,3];\ \{\})$	$6(p-2)$
(b)	$(0;\ +;\ [2,3];\ \{(1)\})$	$6(p-2)$
(c)	$(0;\ +;\ [2];\ \{(n_1,n_2)\}),\ 2 \le n_1 \le n_2 < p$	$\frac{2n_1n_2}{n_1n_2-n_1-n_2}(p-2)$
(d)	$(0;\ +;\ [3];\ \{(2,2)\})$	$6(p-2)$
(e)	$(0;\ +;\ [m];\ \{(n)\}),\ m \ge 3$ and $n \ge 2$	$\frac{2mn}{mn-m-2n}(p-2)$
(f)	$(0;\ +;\ [-];\ \{(2,n_1,n_2)\}),\ 3 \le n_1 \le n_2$	$\frac{4n_1n_2}{n_1n_2-2n_1-2n_2}(p-2)$
(g)	$(0;\ +;\ [-];\ \{(3,n_1,n_2)\}),\ 3 \le n_1 \le n_2$	$\frac{6n_1n_2}{2n_1n_2-3n_1-3n_2}(p-2)$
(h)	$(0;\ +;\ [-];\ \{(4,n_1,n_2)\}),\ 4 \le n_1 \le n_2 < p$	$\frac{8n_1n_2}{3n_1n_2-4n_1-4n_2}(p-2)$
(i)	$(0;\ +;\ [-];\ \{(5,n_1,n_2)\}),\ 5 \le n_1 \le n_2 \le 9$	$\frac{10n_1n_2}{4n_1n_2-5n_1-5n_2}(p-2)$
(j)	$(0;\ +;\ [-];\ \{(2,2,2,n)\}),\ 3 \le n < p$	$\frac{4n}{n-2}(p-2)$
(k)	$(0;\ +;\ [-];\ \{(2,2,3,n)\}),\ 3 \le n \le 5$	$\frac{6n}{2n-3}(p-2)$

Table 2: Signatures for large group orders

To show that the lower bounds given in §3.4 are sharp (with the possible exception of the case $p \equiv 3 \bmod 12$), we will produce an infinite family of values of p in each residue class of integers mod 12 (other than 3 mod 12), for which no group of order exceeding the bound acts on a compact non-orientable surface of genus p. This means that for every NEC group Γ with signature in the appropriate subset of Table 2, there is no epimorphism $\theta : \Gamma \to G$ satisfying Theorem 2.1 for any value of p in the chosen family.

Remark: This type of problem has been solved for certain restricted classes of groups. For cyclic groups, the bounds are $2p$ if p is odd and $2(p-1)$ if p is even (see [4]). For dihedral groups, Example 3.3.1 and a similar argument for all even p show that the bounds are $4p$ if p is odd and $4(p-1)$ if p is even. For abelian groups, the bounds are $2p$ if $p \neq 6$ and 16 if $p = 6$; these can be deduced from the results in [11], or proved directly using the same sorts of arguments as in [4] and this paper.

5 Sharpness of bounds

In this section we show that for each of the eleven residue classes mod 12 covered by Table 1 in §3.4, the bound is sharp. The general methodology is more or less the same in each case, although the chosen values of p which exhibit the sharpness differ from case to case. Accordingly we have elected to give details for only one case — indeed one of the more difficult cases, where $p \equiv 9 \bmod 12$ and the bound is $\nu(p) \geq 6(p+1)$. Complete details of the other cases may be found in the third author's PhD thesis [17].

The Schur-Zassenhaus Theorem (see [14] for example) is used extensively, as are its following two consequences, proofs of which can be found in [13] and [9] respectively.

Theorem 5.1 (Schur-Zassenhaus) *Let N be a normal subgroup of a finite group G. If the order $|N|$ and the index $m = |G : N|$ of N in G are relatively prime, then G contains at least one subgroup of order m, and any two such subgroups are conjugate in G.*

Lemma 5.2 [13] *Let H be a cyclic subgroup of order q and index m in a finite group G. If s is the largest integer dividing q such that* $\gcd(s, m) = 1$ *and* $\gcd(s, t-1) = 1$ *for every divisor t of m such that $t > 1$, then H contains a cyclic subgroup of order s which is normal in G.*

Lemma 5.3 [9] *Suppose p, q and d are positive integers such that* $\gcd(p, q) = 1$*. Then there exist only finitely many finite groups which can be generated by two elements x and y of orders p and q respectively, such that their product xy generates a subgroup of index at most d; indeed $|G| \leq p \cdot q \cdot d \cdot d!$ for any such group G.*

The following result, which is a further application of Theorem 5.1, is used extensively to exclude various signatures. The proof varies slightly for each of the signatures given, however it will suffice to give the details in just one case to illustrate the principal idea used in all cases.

Lemma 5.4 *Suppose G is a finite group acting on a compact non-orientable surface, with one of the signatures given below:*

(i) $(0; +; [2]; \{(2,3)\})$,

(ii) $(0; +; [m]; \{(n)\})$,

(iii) $(0:+;[-];\{(2,n_1,n_2)\})$,

(iv) $(0;+;[-];\{(3,n_1,n_2)\})$,

(v) $(0;+;[-]:\{(2,2,2,n)\})$, *where n is odd.*

Also suppose $|G| = \xi q$ where q is a prime, $\gcd(\xi,q) = 1$, and q is relatively prime to any integer appearing in the signature. If a Sylow q-subgroup Q of G is normal in G, then $\gcd(\xi, q-1) > 2$.

Proof. If G has signature $(0;+;[2];\{(2,3)\})$, it has a partial presentation of the form $\langle x,a,b,c \mid x^2 = a^2 = b^2 = c^2 = 1, (ab)^2 = (bc)^3 = 1, xax = c, \ldots.\rangle$.

Suppose that $Q = \langle w \rangle$ is a cyclic normal subgroup of G of order q. Then conjugation by elements of G yields a homomorphism $\phi\colon G \to \mathrm{Aut}(Q)$ whose kernel is $C_G(Q)$, and it follows that $|G/C_G(Q)|$ divides both ξ and $q-1 = |\mathrm{Aut}(Q)|$.

Now assume that $\gcd(\xi, q-1) \leq 2$, so that $G/C_G(Q)$ has order 1 or 2.

If $G/C_G(Q)$ has order 1, then $C_G(Q) = G$ and then by Theorem 5.1, there is a normal subgroup L of G such that $G/L \cong Q$. But G/L (like G) is generated by involutions, while Q is clearly not, hence this case cannot arise.

If $G/C_G(Q)$ has order 2, then $C_G(Q)$ is one of up to three possible subgroups of G of index 2, namely $\langle a, xax, b, xbx\rangle$, $\langle ax, bx\rangle$ or $\langle x, ba, ca\rangle$. If $C_G(Q) = \langle a, xax, b, xbx\rangle$, then $C_G(Q)$ is generated by four involutions, and so cannot map onto Q. Similarly if $C_G(Q) = \langle x, ba, ca\rangle$, then we find $C_G(Q)$ can be generated by the three involutions x, ba and xca (by noting that $(xca)^2 = xcaxca = xcxa = 1$), so cannot map onto Q.

Hence the only possibility is $C_G(Q) = \langle u, v\rangle$ where $u = ax$ and $v = bx$, so that $(uv^{-1})^2 = (vu)^3 = 1$. In this case, the epimorphism $C_G(Q) \to Q$ provided by Theorem 5.1 maps both uv^{-1} and vu to the identity, so that if the images of u and v are w^α and w^β respectively, then $\alpha \equiv \beta \bmod q$ and $\alpha \equiv -\beta \bmod q$, but together these imply $\alpha \equiv \beta \equiv 0 \bmod q$, a contradiction.

Thus $\gcd(\xi, q-1) > 2$ in this case. The general methodology for the groups with other signatures is similar. For full details, see [17].

The theorem below establishes the sharpness of the bound $\nu(p) \geq 6(p+1)$ for one congruence class of genera modulo 12:

Theorem 5.5 *For $p \equiv 9$ mod 12, there are infinitely many values of p such that no group of order greater than $6(p+1)$ is a group of automorphisms of a compact non-orientable surface of genus p.*

Proof. We choose p to belong to the following set of integers, for reasons which should become clear during the course of the argument:

$$\mathcal{P} = \{\ 149q+2 \mid q \text{ is prime},\ q \equiv 11 \bmod 288,\ q \equiv 2 \bmod 5^2 \cdot 7^2,\ q > 3000!, \text{ and } q \equiv 3 \bmod r \text{ for every prime } r \text{ in the range } 11 \leq r < 400\ \}.$$

Note that, by Dirichlet's Theorem, there are infinitely many primes q which satisfy these conditions. Also for simplicity of expression, we have indulged in a certain amount of overkill in restricting the choice of q. A much smaller subcollection of primes in the range $11 \leq r < 400$ would suffice (see [17]).

Now if G is a group of automorphisms of a compact non-orientable surface X of genus p and $|G| > 6(p+1)$, then the signature of the corresponding NEC group Γ is one of those described in Table 2. Since Table 2 gives all signatures for which $|G| > 4p$, some cases can immediately be eliminated, namely (a), (b), (d), (i) and (k). We treat each of the remaining cases in turn, using the same labelling as in Table 2. The proof consists of determining the possible values of the parameters (among n_1, n_2, m and n) in the given signature such that $|G| > 6(p+1)$, and then eliminating each of these by reaching a suitable contradiction, usually based on Theorem 5.1 or one of Lemmas 5.2 to 5.4.

Type (c): Here the NEC group Γ has signature $(0; +; [2]; \{(n_1, n_2)\})$ and the order of G is given by $|G| = \frac{2n_1n_2}{n_1n_2-n_1-n_2}(p-2)$. Clearly $n_1 = 2$, $n_2 \in \{3, 4, 5\}$ and $|G| \in \{\ 12 \cdot 149q,\ 8 \cdot 149q,\ 20 \cdot 149q/3\ \}$.

The case $n_2 = 5$ can be easily be eliminated as $|G|$ needs to be an integer. When $n_2 = 3$, we have $|G| = \xi q$ where $\xi = 12 \cdot 149$, but our choice of q ensures that $\gcd(\xi, q-1) = 2$, in contradiction to Lemma 5.4. This leaves only the case $n_2 = 4$ to consider.

In this case a Sylow q-subgroup Q of G is normal in G since q is large, and then also the factor group G/Q has a normal subgroup of order 149, so that G possesses a normal subgroup of odd order $149q$ and index 8. It follows that G is a smooth homomorphic image of the group with presentation $\langle x, a, b \mid x^2 = a^2 = b^2 = 1,\ (ab)^2 = (bxax)^4 = 1\rangle$. Every normal subgroup of index 8 in this group, however, can be shown to contain the element $(bxax)^2$ of order 2, and hence there is no such smooth image.

(Remark: The final observation in the above case can be verified by using Reidemeister-Schreier theory, or more conveniently, by using one of the group-theoretic computational systems GAP [15] or MAGMA [3]. Alternatively, the low index subgroups procedure available in these packages has been adapted in [7] for finding only normal subgroups, and in this and similar cases it is even more convenient to use this adaptation.)

Type (e): Here Γ has signature $(0;+;[m];\{(n)\})$, so $|G| = \frac{2mn}{mn-m-2n}(p-2)$. Clearly $m \in \{3,4,5\}$ or $n=2$. The possibilities when $m=4$ or 5 or $n=2$ can all be eliminated by Lemma 5.4 (or the condition that $|G|$ be an integer).

If $m=3$, then $n \geq 4$ and $|G| = \frac{6 \cdot 149nq}{n-3}$. We consider here the possibility that q does not divide $n-3$. In this case $n-3$ divides $6 \cdot 149$ and so we have $n = 4,5,6,9,152,301,450$ or 897, and correspondingly $|G| = 3576q, 2235q, 1788q, 1341q, 912q, 903q, 900q$ or $897q$. As G contains elements of order 2, its order must be even, so $n = 4,6,152$ or 450. In each of these cases, however, $|G| = \xi q$ where the largest prime factor of ξ is 149, and hence we obtain a contradiction by Lemma 5.4 and our choice of q.

Next suppose that q does divide $n-3$, so that $n-3 = sq$ where $s \mid 6 \cdot 149$. Again those values of s which make $|G|$ odd can be eliminated, leaving the following possibilities:

s	n	$\lvert G\rvert$
1	$q+3$	$6 \cdot 149(q+3)$
3	$3q+3$	$2 \cdot 149(3q+3)$
149	$149q+3$	$6(149q+3)$
447	$447q+3$	$2(447q+3)$.

When $s=1$, we see that G has a cyclic subgroup H of order $n=q+3$ and index $6 \cdot 149$. By Lemma 5.2 and our choice of q, it follows that G has a normal cyclic subgroup K of order $(q+3)/10$. Likewise when $s = 3, 149$ or 447, we find G has a cyclic normal subgroup K of order $|K| = (q+1)/12$, $(149q+3)/2$ or $(447q+3)/8$ respectively.

We treat these four cases together. Since K is normal in G, so is $C_G(K)$, and further, $\bar{G} = G/C_G(K)$ is isomorphic to a subgroup of $\mathrm{Aut}(K)$ which is abelian. As Γ has presentation $\langle x, a \mid x^3 = a^2 = (axax^{-1})^n = 1\rangle$, we see that its abelian quotient $\bar{G}$ has to be a factor group of $\mathbb{Z}_6$. On the other hand, by non-orientability we require that $\ker\theta$ contains a word in the generators a and x of Γ such that the number of occurrences of a is odd. The image of this word in $\bar{G}$ must have the form $\bar{a}^k\bar{x}^l$ where k is odd. This implies $\bar{a} = 1$, so we find $a \in C_G(K)$, and hence $\bar{G} = G/C_G(K)$ has order 1 or 3. But also $C_G(H) = L \times K$ for some L (by Theorem 5.1), and so there must be an epimorphism onto K from some subgroup of of Γ of index dividing 3, which is easily shown to be impossible (in all four cases).

Type (f): Here Γ has signature $(0;+;[-];\{(2,n_1,n_2)\})$, and presentation $\langle a,b,c \mid a^2 = b^2 = c^2 = 1,\ (ab)^2 = (bc)^{n_1} = (ca)^{n_2} = 1\rangle$. By abuse of notation we will also use a, b, c to denote the images in G of the generators of Γ.

First consider the cases where $n_1 = 3$, so that $n_2 \geq 7$ and $|G| = \frac{12 \cdot 149 n_2 q}{n_2 - 6}$. If q does not divide $n_2 - 6$, then $n_2 - 6 \mid 12 \cdot 149$, and $|G| = \xi q$ where n_2 and ξ are given by the following table:

n_2	7	8	9	10	12	18
ξ	$84 \cdot 149$	$48 \cdot 149$	$36 \cdot 149$	$30 \cdot 149$	$24 \cdot 149$	$18 \cdot 149$
n_2	155	304	453	602	900	1794
ξ	1860	1827	1812	1806	1800	1794

In all of these cases, G has a normal cyclic Sylow q-subgroup Q, and further, $\gcd(\xi, q-1) = 2$ by our choice of q. This contradicts Lemma 5.4. On the other hand, if $n_2 - 6 = sq$ for some s, then $s \mid 12 \cdot 149$ and $|G| = 12 \cdot 149 n_2 / s$, and all such cases are eliminated by Lemma 5.3 since $q > 3000!$.

Next suppose $n_1 = 4$, so that $n_2 \geq 5$ and $|G| = \frac{8 \cdot 149 n_2 q}{n_2 - 4}$. If $n_2 - 4$ is not divisible by q, then $n_2 - 4 \mid 8 \cdot 149$, and all possibilities for the pair (n_2, ξ) can be calculated as above and eliminated by Lemma 5.4. On the other hand, if $n_2 - 4 = sq$ where $s \mid 8 \cdot 149$, then we have the following possibilities, each of which requires a separate argument:

s	1	2	4	8
n_2	$q+4$	$2q+4$	$4q+4$	$8q+4$
$\|G\|$	$8 \cdot 149(q+4)$	$4 \cdot 149(2q+4)$	$2 \cdot 149(4q+4)$	$149(8q+4)$
s	149	298	596	1192
n_2	$149q+4$	$298q+4$	$596q+4$	$1192q+4$
$\|G\|$	$8(149q+4)$	$4(298q+4)$	$2(596q+4)$	$1192q+4$

- If $s = 1192$, then G is cyclic, which is clearly impossible.
- If $s = 596$, then $|G| = 2n_2$, but its subgroup $\langle a, c \rangle$ is dihedral of order $2n_2$, so that $G = \langle a, c \rangle$ is dihedral. In particular, as bc has order 4, it follows that bc lies in the maximal cyclic subgroup $H = \langle ac \rangle$ of G. Then since G is the image of Γ^+, we find $G = \langle ac, bc \rangle = H$, which is a contradiction.
- If $s = 298$, then G has a cyclic normal subgroup K of order $(149q+2)/3$, by Lemma 5.2. The quotient G/K has order 24 and is a smooth image of the group $\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^2 = (bc)^4 = (ca)^6 = 1 \rangle$. The latter group, however, has no smooth image of order 24 in which the image of the rotation subgroup has index 1, so this case is impossible.

• If $s = 149$, then G has a normal cyclic subgroup K of order $149q + 4$, by Lemma 5.2. The quotient G/K has order 8 and must be a factor group of $\langle a, b, c \mid (ab)^2 = (bc)^4 = ac = 1\rangle$, but the latter group has order 4.
• If $s = 8$, then $|G| = 149(8q + 4)$ which is divisible by 4 but not by 8, contradicting the fact that G has a dihedral subgroup $\langle b, c\rangle$ of order $2n_1 = 8$.
• If $s = 4$, then G has a cyclic normal subgroup K of order $(q+1)/12$, such that $\bar{G} = G/K$ has order $96 \cdot 149$, and $\bar{G}$ in turn has a normal cyclic subgroup $\bar{P}$ of order 149 and index 96. The quotient $\bar{G}/\bar{P}$ must then be a smooth image of the group $\langle a, b, c \mid a^2 = b^2 = c^2 = 1,\ (ab)^2 = (bc)^4 = (ca)^{48} = 1\rangle$, and hence coincides with the image of the dihedral subgroup $\langle a, c\rangle$. In this case, however, the image of the element bc (of order 4) must lie in the maximal cyclic subgroup of order 48. It follows that the image of the rotation subgroup $\langle ac, bc\rangle$ is cyclic of order 48 and therefore of index 2, contradiction.
• If $s = 2$, then G has a cyclic normal subgroup K of order $q + 2$ with quotient G/K of order $8 \cdot 149$, but then also G/K is a smooth image of the group $\langle x, y \mid x^2 = y^4 = (xy)^2 = 1\rangle$, which has order 8, so this is impossible.
• If $s = 1$, then G has a cyclic normal subgroup of order $(q+4)/3$ and index $24 \cdot 149$, but the corresponding quotient has to be a smooth image of the group $\langle x, y \mid x^2 = y^4 = (xy)^3 = 1\rangle$, which has order 24, contradiction.

When $n_1 = 5$, all possibilities can be eliminated using Lemmas 5.3 and 5.4, precisely as in the cases where $n_1 = 3$ above.

When $n_1 = 6$, we have also $n_2 \geq 6$, and $|G| = \frac{6 \cdot 149 n_2 q}{n_2 - 3}$. All cases where $n_2 - 3$ is not divisible by q can be eliminated by Lemma 5.4, so we may suppose that $n_2 - 3 = sq$ where $s \mid 6 \cdot 149$. Also since $|G|$ must be even, s must be odd, and as $|G| > 6(p + 1) = 6(149q + 3)$ we find $n_2 < 149q + 3$ and so $s < 149$. This leaves only the possibilities $s = 1$ and 3, with $|G| = 6 \cdot 149(q + 3)$ and $2 \cdot 149(3q + 3)$ respectively. Again we deal with each case separately:

• If $s = 1$, then G has a normal cyclic subgroup K of order $(q + 3)/10$ by Lemma 5.2, and the quotient $\bar{G}$ of order $60 \cdot 149$ has a normal cyclic subgroup $\bar{P}$ of order 149 and index 60. The centraliser $C_{\bar{G}}(\bar{P})$ is normal in $\bar{G}$ and its quotient is abelian, and is a factor group of $\langle x, y \mid x^2 = y^6 = (xy)^{10} = 1\rangle$, so its order divides 4. Hence $C_{\bar{G}}(\bar{P})$ has index 1, 2 or 4 in $\bar{G}$. Also by Theorem 5.1 we know $C_{\bar{G}}(\bar{P}) \cong \bar{L} \times \bar{P}$ for some $\bar{L}$, and so $C_{\bar{G}}(\bar{P})$ has a cyclic factor group of order 149. It is easily checked, however, that no subgroup of index dividing 4 in G admits an epimorphism onto $\mathbb{Z}_{149}$.
• If $s = 3$, we can apply a similar argument (as above) to a normal cyclic subgroup K of order $(q+1)/12$, noting that since $\bar{G}/C_{\bar{G}}(\bar{P})$ is isomorphic to

a subgroup of $\mathrm{Aut}(\mathbb{Z}_{149})$, its order divides 4.

In all remaining cases, namely $n_1 = 7, 8, 9, 10$ or 11, either $|G|$ fails to be an even integer, or $n_1 = n_2 = 8$, and this can be eliminated by Lemma 5.4.

Types (g), (h) and (j): All of these can be eliminated using arguments similar to those above, requiring no new methods, and so we omit the details.

This completes the proof of Theorem 5.5.

For other congruence classes of genera modulo 12, different infinite sets of values of p need to be chosen to prove sharpness. We list these in the table below, again defining more restrictive sets of values than are necessary in order to simplify the description. In this table, q and r are primes.

	p	Conditions
0	$2q+2$	$q \equiv 11 \bmod 72$; $q \equiv 2 \bmod 35$; $q \geq 25!$
1	$q+2$	$q \equiv 11 \bmod 12$; $q \equiv 2 \bmod 35$; $q \geq 13!$
2	$12q+2$	$q \equiv 11 \bmod 144$; $q \equiv 2 \bmod 5^2 \cdot 7^2$; $q \equiv 3 \bmod r$ for $11 \leq r < 50$; $q \geq 145!$
4	$2 \cdot 89q+2$	$q \equiv 11 \bmod 144$; $q \equiv 2 \bmod 25$; $q \equiv 4 \bmod 7$; $q \geq 2200!$; $q \equiv 3 \bmod r$ for $11 \leq r \leq 89$ and $r = 179, 181, 1423$
5	$3q+2$	$q \equiv 5 \bmod 48$; $q \equiv 2 \bmod 25$; $q \equiv 4 \bmod 49$; $q \equiv 3 \bmod 11$; $q \equiv 2 \bmod 23$; $q \geq 40!$
6	$4 \cdot 149q+2$	$q \equiv 11 \bmod 432$; $q \equiv 3 \bmod 25$; $q \equiv 4 \bmod 49$; $q \geq 7200!$; $q \equiv 4 \bmod r$ for $11 \leq r < 400$ and $r = 599, 1193, 2383$
7	$q+2$	$q \equiv 5 \bmod 24$; $q \equiv 2 \bmod 35$; $q \geq 13!$
8	$6q+2$	$q \equiv 11 \bmod 144$; $q \equiv 2 \bmod 5^2 \cdot 7^2$; $q \equiv 3 \bmod r$ for $r = 11, 13, 23, 47$; $q \geq 75!$
9	$149q+2$	see Theorem 5.5
10	$4q+2$	$q \equiv 11 \bmod 72$; $q \equiv 2 \bmod 35$; $q \equiv 2 \bmod 31$; $q \geq 50!$
11	$3q+2$	$q \equiv 11 \bmod 72$; $q \equiv 2 \bmod 5^2 \cdot 7^2$; $q \equiv 3 \bmod r$ for $r = 11, 23$; $q \geq 40!$

Table 3: Sharpness values of p

Remarks: Almost all the possibilities that arise in the cases listed in Table 3 can be eliminated using variations of those arguments described above for the case of genus $p \equiv 9 \bmod 12$. There are however, a couple of additional arguments which are necessary but are not included above. We now illustrate these in three cases where $p \equiv 5 \bmod 12$ and Γ has signature $(0;+;[-];\{(2,4,n_2)\})$ with $n_2 - 4 = sq$ for some s. For full details, see [17].

When $s = 3$, we find G has a normal cyclic subgroup K of order $(3q+4)/5$ and index 40, by Lemma 5.2, and the corresponding factor group G/K has signature $(0;+;[-];\{(2,4,5)\})$, which gives a reflexible regular map of type $\{4,5\}$ on a non-orientable surface of genus 3. Likewise when $s = 6$, we obtain a reflexible regular map of type $\{4,14\}$ and genus 7. Such regular maps, however, do not exist (see [8]).

When $s = 4$, the group G has signature $(0;+;[-];\{(2,4,4(q+1)\})$ and has order $24(q+1)$. This cannot happen, however, as it can be shown more generally that if G has signature $(0;+;[-];\{(2,4,\lambda)\})$ where $\lambda > 36$, then G cannot have order 6λ. One proof of the latter assertion is an adaptation of a result in [1], and again may be found in [17].

Finally, as noted earlier, we have not yet resolved the case where the genus $p \equiv 3$ modulo 12. The bound of $\nu(p) \geq 4p$ in this case is sharp for $p = 3$, but not for $p = 15, 27, 39, 51, 63$ or 75 as these are values of the genus for which 3.4.1 applies. We have been able to prove the bound is sharp again for $p = 87$, however our proof requires a detailed analysis of dozens of sub-cases, and even a summary would be too long to provide here.

We conjecture that the bound $\nu(p) \geq 4p$ is sharp for infinitely many values of p of the form $83q + 2$ where q comes from an infinite set of large primes satisfying conditions like those given in Table 3.

In particular, for one such infinite set we have been able to eliminate all possibilities for the signature from Table 2, with the exception of type (f), namely signature $(0;+;[-];\{(2,n_1,n_2)\})$, for which the order of the group G is given by $|G| = \frac{4n_1n_2}{n_1n_2-2n_1-2n_2}(p-2)$. The problem with this signature type is that both of the parameters n_1 and n_2 are unbounded, making it very difficult to apply the sort of arguments described above (except for specific choices of n_1 or n_2). Of course the difficulty here is related to the question of the precise spectrum of genera of regular maps on non-orientable surfaces, which is also unresolved, and our conjecture should be seen in the context of this question and the results of [6, 8].

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